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RESEARCH ARTICLE

On disjoint union of M-graphs

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Abstract. Given a pair (X, σ) consisting of a finite tree X and its vertex self-map σ one can construct the corresponding Markov graph $\Gamma(X, \sigma)$ which is a digraph that encodes σ-covering relation between edges in X. M-graphs are Markov graphs up to isomorphism. We obtain several sufficient conditions for the disjoint union of M-graphs to be an M-graph and prove that each weak component of M-graph is an M-graph itself.

Introduction

In 1964 Sharkovsky proved the following remarkable theorem.

Theorem 1. [9] *If the continuous map $f : [0, 1] \to [0, 1]$ has a periodic point of period $n \in \mathbb{N}$, then it also has a periodic point of period $m \in \mathbb{N}$ for all $m \lhd n$, where*

$$1 \lhd 2 \lhd 2^2 \lhd \cdots \lhd 2^n \lhd \cdots \lhd 7 \cdot 2^n \lhd 5 \cdot 2^n \lhd 3 \cdot 2^n \lhd \cdots \lhd 7 \cdot 2 \lhd 5 \cdot 2 \lhd 3 \cdot 2 \lhd \cdots \lhd 7 \lhd 5 \lhd 3$$

is Sharkovsky's ordering *of $\mathbb{N}$. Moreover, for every number $m \in \mathbb{N}$ there exists a continuous map that has a periodic point of period m but does not have periodic points of periods $n \in \mathbb{N}$, where $m \lhd n$.*

In [10] Straffin proposed a strategy on how to prove Sharkovsky's theorem using some elegant combinatorial arguments. The cornerstone of his idea is to use directed graphs which naturally arise from orbits of periodic

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points. Namely, let $x \in [0,1]$ be n-periodic point of a continuous map $f : [0,1] \to [0,1]$. Consider the orbit $\mathrm{orb}_f(x) = \{x, f(x), \ldots, f^{n-1}(x)\} = \{x_1 < \cdots < x_n\}$ and its natural ordering inherited from the interval. *Periodic graph* $G_f(x)$ has the vertex set $\{1, \ldots, n-1\}$ and the arc set $\{(i,j) : \min\{f(x_i), f(x_{i+1})\} \leqslant x_j < \max\{f(x_i), f(x_{i+1})\}\}$. Here each $1 \leqslant i \leqslant n-1$ represents the minimal interval $[x_i, x_{i+1}]$ and there is an arc $i \to j$ in $G_f(x)$ if $[x_i, x_{i+1}]$ "covers" $[x_j, x_{j+1}]$ under f. Periodic graphs are useful in combinatorial dynamics because of the following result known as *Itinerary lemma.*

Lemma 1. [10] *Let $x \in [0,1]$ be some periodic point of a continuous map $f : [0,1] \to [0,1]$. Suppose that there is a closed walk $W = \{i_0 \to \cdots \to i_{m-1} \to i_0\}$ of length m in $G_f(x)$. Then there exists a periodic point $y \in [0,1]$ such that $f^m(y) = y$ and $f^k(y) \in [x_{i_k}, x_{i_k+1}]$ for all $0 \leqslant k \leqslant m-1$. Moreover, if W is primitive, then the period of y equals m.*

Here the closed walk is called *primitive* if it is not entirely consists of a smaller walk traced several times. Note that Lemma 1 admits a converse statement. Namely, for any periodic point $x \in [0,1]$ of f we can consider its *linearization* $L_x(f) : [0,1] \to [0,1]$ which is a "connect-the-dots" map with respect to the orbit $\mathrm{orb}_f(x)$. Then each m-periodic point of $L_x(f)$ corresponds to some primitive closed walk of length m in $G_{L_x(f)}(x) = G_f(x)$.

Full proof of Sharkovsky's theorem using periodic graphs can be found in [2]. Graph-theoretic properties of periodic graphs were studied in [6–8]. These are include calculation of the number of non-isomorphic periodic graphs with given number of vertices [6] and obtaining graph-theoretic criteria for periodic graphs [7] and their induced subgraphs [8].

Similar approach can be used for dynamics of continuous maps on finite topological trees (see [1] for the Sharkovsky-type result in this case). The

defined for combinatorial trees and their vertex maps. Thus, periodic graphs appear as a particular case of Markov graphs where underlying trees are paths and maps are cyclic permutations. M-graphs then defined as Markov graphs up to isomorphism.

In [3] maps on trees were characterized for several classes of M-graphs including complete digraphs, complete bipartite digraphs, disjoint unions of cycles and digraphs in which each arc is a loop. It is also shown [4] that M-graphs satisfy Seymour's Second Neighbourhood Conjecture as well as Caccetta–Häggkvist Conjecture. Various transformations including deletion and addition of vertices, doubling and reverse doubling of vertices and taking disjoint unions of M-graphs are studied in [5]. Also, it is proved that there exist exactly 11 pairwise non-isomorphic M-graphs which are tournaments as well as 86 pairwise non-isomorphic 3-vertex M-graphs (again, see [5]).

In this paper we obtain several sufficient conditions for the disjoint union of M-graphs to be an M-graph and prove that each weak component of M-graph is an M-graph itself.

1. Definitions and preliminary results

In what follows *map* is just a function. For any given map σ by $\operatorname{Im}\sigma$ and $\operatorname{fix}\sigma$ we denote its image and the set of its fixed points, respectively.

A *graph* G is a pair (V, E), where $V = V(G)$ is the set of its vertices and $E = E(G)$ the set of its edges. By $E_G(u)$ we denote the set of all edges incident to the vertex u in G. A vertex u is called *isolated* if $|E_G(u)| = 0$. Similarly, u is a *leaf vertex* provided $|E_G(u)| = 1$. The unique edge incident to a leaf vertex is called a *leaf edge*. The set of all leaf vertices in G is denoted by $L(G)$. For the set of vertices $A \subset V(G)$ we put $E(A) = \{uv \in E(G) : u, v \in A\}$ and $\partial_G A = \{u \in A : E_G(u) - E(A) \neq \varnothing\}$. By $G[A]$ and $G[E']$ we denote the subgraphs of G induced by $A \subset V(G)$ and $E' \subset E(G)$, respectively.

A graph G is called *connected* if for every pair of its vertices $u, v \in V(G)$ there exists a path joining them. The minimum number of edges in such a path is called the *distance* $d_G(u, v)$ between u and v in G. The set of vertices $A \subset V(G)$ is *connected* if the induced subgraph $G[A]$ is connected. Similarly, $E' \subset E(G)$ is *connected* if so is $G[E']$.

The *eccentricity* of a vertex u in a connected graph G is the value $\operatorname{ecc}_G(u) = \max_{v \in V(G)} d_G(u, v)$. For the pair of vertices $u, v \in V(G)$ in a connected graph G we put $[u, v]_G = \{w \in V(G) : d_G(u, w) + d_G(w, v) = d_G(u, v)\}$. The set $A \subset V(G)$ is *convex* provided $[u, v]_G \subset A$ for all $u, v \in A$. The *convex hull* $\operatorname{Conv}_G(A)$ of A is defined as the smallest convex set containing A.

Put $d_G(u, A) = \min_{v \in A} d_G(u, v)$ and $d_G(A, B) = \min_{u \in B} d_G(u, A)$ for all vertex sets $\{u\}, A, B \subset V(G)$ in a connected graph G. The set $A \subset V(G)$ is called *Chebyshev* if for every vertex $u \in V(G)$ there exists a unique $v \in A$ with $d_G(u, v) = d_G(u, A)$. The corresponding map $\operatorname{pr}_A : V(G) \to V(G)$, where $\operatorname{pr}_A(u) = v$ is called the *projection* on a Chebyshev set A.

A *tree* is a connected acyclic graph. It should be noted that in a tree each connected set of vertices is Chebyshev.

A *directed graph* or *digraph* Γ is a pair (V, A), where $V = V(\Gamma)$ is the set of its vertices and $A = A(\Gamma) \subset V \times V$ is the set of its arcs. If $(u, v) \in A(\Gamma)$, then we write $u \to v$ in Γ. The arc of the form $u \to u$ is called a *loop*. For the vertex $u \in V(\Gamma)$ we put $N^+_\Gamma(u) = \{v \in V(\Gamma) : u \to v \text{ in } \Gamma\}$ and $N^-_\Gamma(u) = \{v \in V(\Gamma) : v \to u \text{ in } \Gamma\}$. The cardinalities $d^+_\Gamma(u) = |N^+_\Gamma(u)|$ and $d^-_\Gamma(u) = |N^-_\Gamma(u)|$ are called the *outdegree* and the *indegree* of u, respectively.

A digraph Γ is called *complete* provided $A(\Gamma) = V(\Gamma) \times V(\Gamma)$. Similarly, Γ is *empty* if $A(\Gamma) = \varnothing$. By K_n and $\overline{K}_n$ we denote the complete and the empty digraph with n vertices, respectively.

A digraph is called *weakly connected* if its underlying graph (which is obtained by "forgetting" orientation of the edges and ignoring loops) is connected. *Weak component* of a digraph is its maximal weakly connected subgraph. By $\Gamma_1 \sqcup \Gamma_2$ we denote the disjoint union of digraphs Γ_1 and Γ_2.

A pair (X, u_0) consisting of a tree X and its distinguished vertex $u_0 \in V(X)$ is called a *rooted tree*. The digraph Γ which is obtained from the rooted tree (X, u_0) by orienting the edges of X towards u_0 is called an *in-tree*. The vertex u_0 is the *center* of an in-tree Γ. It is easy to see that for an in-tree its center is the unique vertex with zero outdegree.

For every map $f : X \to X$ one can define its *functional graph* as a digraph with the vertex set X and the arc set $\{(x, y) : f(x) = y\}$. A digraph is called *functional* if it isomorphic to a functional graph for some map. It is easy to see that Γ is functional digraph if and only if $d^+_\Gamma(v) = 1$ for all $v \in V(\Gamma)$. Similarly, Γ is called *partial functional* if $d^+_\Gamma(v) \leqslant 1$ for all $v \in V(\Gamma)$. Each partial functional digraph Γ corresponds to some partial map of the form $f : V(\Gamma) \to V(\Gamma)$.

Definition 1. Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. The *Markov graph* $\Gamma = \Gamma(X, \sigma)$ has the vertex set $V(\Gamma) = E(X)$ and there is an arc $e_1 \to e_2$ in Γ if $u_2, v_2 \in [\sigma(u_1), \sigma(v_1)]_X$ for $e_i = u_i v_i \in V(\Gamma), i = 1, 2$. In other words, $N^+_\Gamma(uv) = E([\sigma(u), \sigma(v)]_X)$ for all edges $uv \in E(X)$.

Example 1. Consider the tree X with $V(X) = \{1, \dots, 7\}$, $E(X) = \{12, 23, 34, 45, 26, 37\}$ and its map $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 3 & 6 & 2 & 4 & 2 \end{pmatrix}$ which are shown in Figure 1. Then the corresponding Markov graph $\Gamma(X, \sigma)$ is shown in Figure 2.

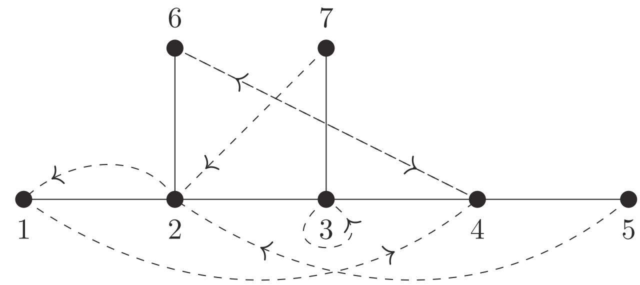

FIGURE 1. The pair (X, σ) from Example 1 (dashed arrows denote σ).

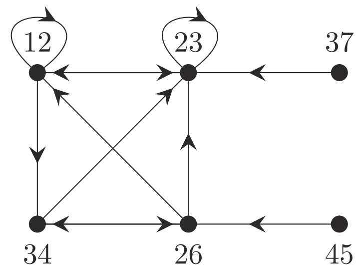

FIGURE 2. Markov graph $\Gamma(X, \sigma)$ for the pair (X, σ) from Example 1.

A digraph Γ is called an M-*graph* if there exists a pair (X, σ) such that $\Gamma \simeq \Gamma(X, \sigma)$. Each such a pair is called the *realization* of Γ.

Lemma 2. [3] *Let X be a tree and $\sigma : V(X) \to V(X)$ be a map. Then for every pair of vertices $u, v \in V(X)$ and an edge $xy \in E([\sigma(u), \sigma(v)]_X)$ there exists an edge $wz \in E([u, v]_X)$ with $wz \to xy$ in $\Gamma(X, \sigma)$. In particular,*

$$[\sigma(u), \sigma(v)]_X \subset \bigcup_{wz \in E([u,v]_X)} [\sigma(w), \sigma(z)]_X.$$

Lemma 3. [5] *Let X be a tree, $A \subset V(X)$ be some connected set of vertices, $\sigma : V(X) \to V(X)$ be a map and $\Gamma = \Gamma(X, \sigma)$. Then $\Gamma(X[A], \mathrm{pr}_A \circ \sigma) = \Gamma[E(A)]$.*

Proposition 1. *Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. Put $E(\sigma) = \{e \in E(X) : d^-_\Gamma(e) \geqslant 1\}$. Then $E(\sigma) = E(\mathrm{Conv}_X(\mathrm{Im}\,\sigma))$. In particular, $X[E(\sigma)]$ is the connected subgraph of X.*

Proof. Let $V_1 = V(E(\sigma))$ and $V_2 = \mathrm{Conv}_X(\mathrm{Im}\,\sigma)$. If $u \in V_1$, then there exists an edge $e = uv \in E(\sigma)$. By definition, $d^-_\Gamma(e) \geqslant 1$. This means that there is an edge $e' = u'v' \in E(X)$ with $e' \to e$ in $\Gamma(X, \sigma)$, i.e. $u, v \in [\sigma(u'), \sigma(v')]_X$. Therefore, $u \in V_2$.

Conversely, suppose $u \in V_2$. Then there exists a pair of vertices $x, y \in V(X)$ such that $u \in [\sigma(x), \sigma(y)]_X$. At first, suppose that $\sigma(x) \neq \sigma(y)$. Then we can fix an edge $e = uv \in E([\sigma(x), \sigma(y)]_X)$. From Lemma 2 it follows that there is an edge $e' \in E([x, y]_X)$ with $e' \to e$ in Γ. Thus $d^-_\Gamma(e) \geqslant 1$ and $u \in V_1$. Otherwise, let $\sigma(x) = \sigma(y)$. Then $u \in \operatorname{Im}\sigma$. If σ is a constant map, then $E(\sigma) = E(\operatorname{Conv}_X(\operatorname{Im}\sigma)) = \varnothing$. Thus, suppose that σ is non-constant. This means that there exists a vertex $v \in \operatorname{Im}\sigma - \{u\}$. Let $\sigma(z) = v$. Since $u \neq v$, we can fix an edge $e = uw \in E([u, v]_X)$. Again, by Lemma 2, $d^-_\Gamma(e) \geqslant 1$ which implies $u \in V_1$. □

2. Main results

From Lemma 3 it strictly follows that each nontrivial M-graph Γ contains a vertex $v \in V(\Gamma)$ such that $\Gamma - \{v\}$ is also an M-graph. In [5] it was proved that any digraph obtained from an M-graph by deletion of a vertex with zero outdegree (in particular, an isolated vertex) is an M-graph itself. We generalize this result using the following theorem.

Theorem 2. *Let X be a tree and $\sigma : V(X) \to V(X)$ be some map. Suppose that we have a collection $A_i \subset V(X)$, $1 \leqslant i \leqslant m$ of pairwise disjoint connected sets such that for every $1 \leqslant i \leqslant m$ either $|\sigma(\partial_X A_i)| = 1$ or there exists $1 \leqslant j \leqslant m$ with $\sigma(\partial_X A_i) \subset A_j$. Then $\Gamma(X, \sigma) - \bigcup_{i=1}^m E(A_i)$ is an* M*-graph.*

Proof. Consider the set of indices $I_1 = \{1 \leqslant i \leqslant m : |\sigma(\partial_X A_i)| = 1\}$ and the corresponding map $g : I_1 \to V(X)$, where $\sigma(\partial_X A_i) = \{g(i)\}$ for all $i \in I_1$. Similarly, the set of indices $I_2 = \{1, \dots, m\} - I_1$ defines the map $f : I_2 \to \{1, \dots, m\}$, where $\sigma(\partial_X A_i) \subset A_{f(i)}$ for all $i \in I_2$.

Take a graph $X - \bigcup_{i=1}^m A_i$ and add to it m new vertices z_i for each $1 \leqslant i \leqslant m$ with new edges $z_i y_i$ for all $y_i \in \partial_X(V(X) - A_i)$ to obtain a new graph X'. It is easy to see that X' is a tree (one can think of X' as of tree which is obtained from X by "contracting" sets A_i into points). Put

$$\sigma'(x) = \begin{cases} z_i & \text{if } \sigma(x) \in A_i, \\ g(i) & \text{if } x = z_i \text{ and } i \in I_1, \\ z_{f(i)} & \text{if } x = z_i \text{ and } i \in I_2, \\ \sigma(x) & \text{otherwise,} \end{cases}$$

for all $x \in V(X')$. Then $\Gamma(X, \sigma) - \bigcup_{i=1}^m E(A_i) \simeq \Gamma(X', \sigma')$. □

Corollary 1. *Let* Γ *be an* M*-graph and* $v \in V(\Gamma)$ *be its vertex with* $N^+_\Gamma(v) \subset \{v\}$*. Then* $\Gamma - \{v\}$ *is an* M*-graph. Moreover, if* $N^+_\Gamma(v) = \{v\}$*, then there exists a realization* (X, σ) *of* $\Gamma - \{v\}$ *such that* $\operatorname{fix}\sigma \neq \varnothing$*.*

Proof. Fix some realization (X, σ) of Γ. Let the edge $e = ux \in E(X)$ corresponds to the vertex $v \in V(\Gamma)$. If $N^+_\Gamma(v) = \varnothing$, then $\sigma(u) = \sigma(x)$. In this case for the connected set of vertices $A = \{u, x\}$ we have $|\sigma(\partial_X A)| = 1$. By Theorem 2, $\Gamma - \{v\}$ is an M-graph.

Otherwise, let $N^+_\Gamma(v) = \{v\}$. Then $\sigma(u) = u$ and $\sigma(x) = x$, or $\sigma(u) = x$ and $\sigma(x) = u$. In both cases $\sigma(\partial_X A) \subset A$. Again, by Theorem 2, $\Gamma - \{v\}$ is an M-graph. Moreover, with the notation of Theorem 2, $\sigma'(z_1) = z_1$ (here $A = A_1$). Therefore, in this case $\operatorname{fix}\sigma' \neq \varnothing$. □

Example 2. Consider the tree X with $V(X) = \{1, \ldots, 7\}$, $E(X) = \{12, 23, 34, 16, 25, 67\}$ and its map $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 5 & 4 & 5 & 2 & 3 \end{pmatrix}$. Then $d^-_{\Gamma(X,\sigma)}(16) = 0$, however $\Gamma(X, \sigma) - \{16\}$ is not an M-graph (see Figure 3).

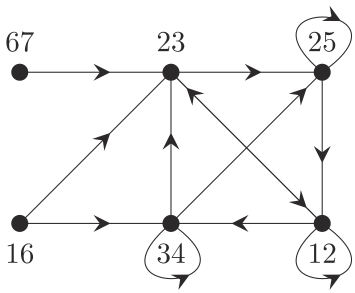

FIGURE 3. Markov graph $\Gamma(X, \sigma)$ for which $\Gamma(X, \sigma) - \{16\}$ is not an M-graph.

Denote by $V_0(\Gamma) = \{v \in V(\Gamma) : d^-_\Gamma(v) = 0\}$ the set of vertices with zero indegree in Γ.

Proposition 2. *For every* M*-graph* Γ *and* $0 \leqslant k \leqslant |V_0(\Gamma)|$ *there exists* $V' \subset V_0(\Gamma)$ *with* $|V'| = k$ *such that* $\Gamma - V'$ *is an* M*-graph. In particular,* $\Gamma - V_0(\Gamma)$ *is an* M*-graph.*

Proof. Fix a realization (X, σ) of Γ. Let the edge set $E' \subset E(X)$ corresponds to $V_0(\Gamma)$. By Proposition 1, the set $E(X) - E' = E(\sigma)$ is connected. Since X is a connected graph, for any $0 \leqslant k \leqslant |V_0(\Gamma)|$ there exists a connected set of edges $E'' \subset E(X)$ with $E(X) - E' \subset E''$

and $|E'| = |E(X)| - k$. Let $V' \subset V(\Gamma)$ corresponds to $E(X) - E''$. Then $|V'| = k$ and by Lemma 3, $\Gamma - V' \simeq \Gamma(X,\sigma) - (E(X) - E'') = \Gamma(X,\sigma)[E''] = \Gamma(X[E''], \mathrm{pr}_{V(E'')} \circ \sigma)$ is an M-graph. □

Note that any digraph obtained from an M-graph by addition of an isolated vertex is also an M-graph. Using this fact one can conclude that Γ is an M-graph if and only if so is $\Gamma \sqcup \overline{K}_1$. However, not every disjoint union of two M-graphs is an M-graph itself.

Example 3. Suppose that Γ is obtained from the complete digraph with two vertices K_2 by deletion of a loop. Then Γ is an M-graph, but $\Gamma \sqcup K_1$ is not (see Figure 4).

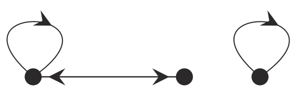

Figure 4. Disjoint union of two M-graphs which is not an M-graph.

Remark 1. [5] If we have a pair of trees $X_i, i = 1, 2$ and a pair of their maps $\sigma_i : V(X_i) \to V(X_i)$ with $\mathrm{fix}\,\sigma_i \neq \varnothing, i = 1, 2$, then the disjoint union $\Gamma(X_1, \sigma_1) \sqcup \Gamma(X_2, \sigma_2)$ is an M-graph. Indeed, "gluing" realizations (X_1, σ_1) and (X_2, σ_2) together along some pair of fixed vertices we obtain the realization of $\Gamma(X_1, \sigma_1) \sqcup \Gamma(X_2, \sigma_2)$.

As a corollary of the construction in Remark 1 one can obtain a sufficient condition for the disjoint union of two M-graphs to be an M-graph.

Corollary 2. [5] *Let Γ_1 and Γ_2 be a pair of* M*-graphs with even numbers of loops in each. Then $\Gamma_1 \sqcup \Gamma_2$ is an* M*-graph. In particular, any disjoint union of two* M*-graphs without loops is an* M*-graph itself.*

It turns out that for any given M-graph we can provide a graph-theoretic criterion for the existence of its realization (X, σ) with $\mathrm{fix}\,\sigma \neq \varnothing$.

Proposition 3. *Let Γ be a digraph. Then $\Gamma \sqcup K_1$ is an* M*-graph if and only if Γ is an* M*-graph and there exists its realization (X, σ) with $\mathrm{fix}\,\sigma \neq \varnothing$.*

Proof. Sufficiency of this condition follows from Remark 1, since for K_1 there obviously exists its realization (X, σ) with $\mathrm{fix}\,\sigma \neq \varnothing$. Thus, we must prove only the necessity of this condition. To do so fix a realization (X', σ') of $\Gamma \sqcup K_1$. Let the vertex $v \in V(\Gamma \sqcup K_1)$ corresponds to a unique vertex from K_1. Then $N^+_{\Gamma \sqcup K_1}(v) = \{v\}$ implying that by Corollary 1, Γ is an M-graph and there exists its realization (X, σ) with $\mathrm{fix}\,\sigma \neq \varnothing$. □

Corollary 3. *If* $\Gamma \sqcup K_1$ *is an* M-*graph, then there exists its realization* (X, σ) *with* $\operatorname{fix} \sigma \neq \varnothing$.

Combining Remark 1 and Proposition 3, we obtain the following result.

Proposition 4. *If for a pair of digraphs* Γ_1 *and* Γ_2 *the digraphs* $\Gamma_1 \sqcup K_1$ *and* $\Gamma_2 \sqcup K_1$ *are* M-*graphs, then* $\Gamma_1 \sqcup \Gamma_2$ *is an* M-*graph.*

Theorem 3. *Let* Γ_1 *be an* M*-graph and* Γ_2 *be acyclic partial functional digraph. Then* $\Gamma_1 \sqcup \Gamma_2$ *is also an* M*-graph.*

Proof. Without loss of generality, we can assume that Γ_2 is weakly connected. Since Γ_2 is acyclic and partially functional, Γ_2 is an in-tree. Let $x_0 \in V(\Gamma_2)$ be its center (thus, $d^+_{\Gamma_2}(x_0) = 0$). Denote by X' the underlying tree of Γ_2. For every $0 \leqslant i \leqslant \operatorname{ecc}_{X'}(x_0)$ put $a_i = |N^i_{X'}(x_0)|$ for the cardinality of the sphere with radius i centered at x_0 in X'.

Now fix a realization (X, σ_0) of Γ_1. Since $V(X)$ is finite, σ_0 has a periodic point $u_0 \in V(X)$ with period $m \geqslant 1$. Consider the restriction $\sigma = \sigma_0|_{\operatorname{orb}_{\sigma_0}(u_0)}$ of σ_0 to $\operatorname{orb}_{\sigma_0}(u_0)$. Clearly, σ is a cyclic permutation of $\operatorname{orb}_{\sigma_0}(u_0)$.

For every $0 \leqslant i \leqslant \operatorname{ecc}_{X'}(x_0)$ add a_i new vertices $y^i_1, \dots y^i_{a_i}$ to X with the new edges $y^i_j\sigma^{-i \bmod m}(u_0)$ for all $1 \leqslant j \leqslant a_i$ (of course, $\sigma^0(u_0) = u_0$). Denote the obtained tree as X''. For all $u \in V(X'')$ put

$$\sigma'(u) = \begin{cases} \sigma(u) & \text{if } u \in V(X), \\ y^{i-1}_k & \text{if } u = y^i_j, i \geqslant 1 \text{ and } N^+_{\Gamma_2}(x^i_j) = \{x^{i-1}_k\}, \\ \sigma(u_0) & \text{if } u = y^0_1, \end{cases}$$

where $N^i_{X'}(x_0) = \{x^i_j : 1 \leqslant j \leqslant a_i\}$ (for example, $N^0_{X'}(x_0) = \{x^0_1\} = \{x_0\}$). Then $\Gamma(X'', \sigma') \simeq \Gamma_1 \sqcup \Gamma_2$ (the edges from $E(X)$ correspond to the vertices of Γ_1 and edges of the form $y^i_j\sigma^{-i \bmod m}(u_0)$ correspond to the vertices x^i_j). □

Note that the acyclicity condition in Theorem 3 is essential as can be seen from the digraph in Example 3.

Example 4. Consider the pair (X, σ) from Example 1 and the corresponding Markov graph $\Gamma_1 = \Gamma(X, \sigma)$. Also, let Γ_2 be the in-tree depicted in Figure 5 (the vertices of Γ_2 are labeled according to the notation in the proof of Theorem 3). Thus, x_0 is the center of Γ_2, $\operatorname{ecc}_{X'}(x_0) = 2$, $N^1_{X'}(x_0) = \{x^1_1, x^1_2\}$, $N^2_{X'}(x_0) = \{x^2_1, x^2_2\}$ and $a_1 = a_2 = 2$. Put $u_0 = 4$.

Then $\mathrm{orb}_\sigma(u_0) = \{4, 6\}$ and therefore $m = 2$. The corresponding tree X'' is shown in Figure 6. We also have $\sigma'(y_1^0) = 6, \sigma'(y_1^1) = \sigma'(y_2^1) = y_1^0$ and $\sigma'(y_1^2) = \sigma'(y_2^2) = y_2^1$.

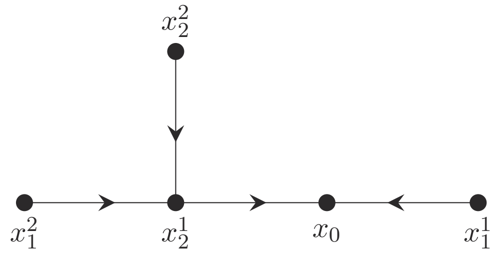

Figure 5. The in-tree Γ_2.

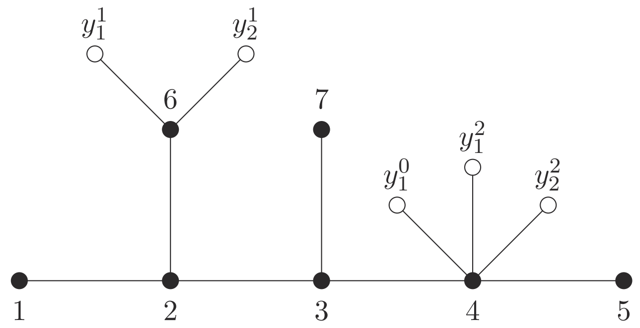

Figure 6. The tree X'' from Example 4.

Theorem 4. *The disjoint union of any collection of weak components (in particular, each weak component) in an* M*-graph is an* M*-graph itself.*

Proof. It is sufficient to prove that for any M-graph Γ and its weak component Γ' the digraph $\Gamma - V(\Gamma')$ is an M-graph. To do so fix a realization (X, σ) of Γ. Let the set of edges $E' \subset E(X)$ corresponds to the vertex set of Γ'. Consider the components $X_1, \dots, X_m$ of the induced subgraph $X[E']$ in X and put $A_i = V(X_i)$ for all $1 \leqslant i \leqslant m$.

Suppose that for some $1 \leqslant i \leqslant m$ there exists a vertex $x \in A_i$ with $\sigma(x) \notin \cup_{j=1}^m A_j = V(X[E'])$. Then $E_X(\sigma(x)) \cap E' = \varnothing$. If for some $y \in A_i$ we have $\sigma(x) \neq \sigma(y)$, then $d_X(\sigma(x), \sigma(y)) \geqslant 2$. This means that there is a vertex $u \in [\sigma(x), \sigma(y)]_X$ such that $e = u\sigma(x) \in E(X)$. By Lemma 2, there exists an edge $e' \in E([x, y]_X) \subset E(A_i)$ with $e' \to e$ in $\Gamma(X, \sigma)$. Since

$e \notin E'$, Γ' is not a weak component of Γ. The obtained contradiction implies that in this case we have $\sigma(x) = \sigma(y)$ for every $y \in A_i$. In other words, $|\sigma(A_i)| = 1$.

Now let $1 \leqslant i \leqslant m$ is fixed and for every vertex $x \in A_i$ there exists $1 \leqslant j_x \leqslant m$ with $\sigma(x) \in A_{j_x}$. We want to prove that in this case $j_x = j_y$ for each pair of vertices $x, y \in A_i$. To the contrary, suppose $j_x \neq j_y$ for some $x, y \in A_i$. Then $d_X(A_{j_x}, A_{j_y}) \geqslant 1$. This implies the existence of an edge $e \in E([\sigma(x), \sigma(y)]_X) - \cup_{k=1}^m E(A_k)$. Again, from Lemma 2 it follows that there exists $e' \in E([x, y]_X) \subset E(A_i)$ with $e' \to e$ in $\Gamma(X, \sigma)$ which is a contradiction. Thus, in this case $\sigma(A_i) \subset A_j$ for some $1 \leqslant j \leqslant m$. Theorem 2 now asserts that $\Gamma - V(\Gamma')$ is an M-graph. □

Corollary 4. *If for a pair of digraphs Γ_1 and Γ_2 their disjoint union $\Gamma_1 \sqcup \Gamma_2$ is an* M*-graph, then both Γ_1 and Γ_2 are* M*-graphs.*

Proof. Clearly, Γ_1 and Γ_2 are both disjoint unions of weak components in $\Gamma_1 \sqcup \Gamma_2$. □

Proposition 5. *Let Γ_1 and Γ_2 be a pair of nontrivial digraphs having loops at each of their vertices. Then $\Gamma_1 \sqcup \Gamma_2$ is an* M*-graph if and only if $\Gamma_1 \sqcup K_1$ and $\Gamma_2 \sqcup K_1$ are both* M*-graphs.*

Proof. The sufficiency of this condition strictly follows from Proposition 4. To prove its necessity fix a realization (X, σ) of $\Gamma = \Gamma_1 \sqcup \Gamma_2$. Let Γ' be a weak component in Γ and $E' \subset E(X)$ be the corresponding set of edges in X. We want to prove that E' is connected. To the contrary, suppose that there is a partition $E' = E_1 \sqcup E_2$ with $d_X(V(E_1), V(E_2)) \geqslant 1$.

Since Γ' is weakly connected, there is a pair of edges $e_i = u_i v_i \in E_i, i = 1, 2$ with $e_1 \to e_2$ or $e_2 \to e_1$ in Γ'. Without loss of generality, assume $e_1 \to e_2$ in Γ'. We have $e_1, e_2 \in N^+_{\Gamma(X,\sigma)}(e_1)$ which implies $[u_1, u_2]_X \subset [\sigma(u_1), \sigma(v_1)]_X$. However, the inequality $d_X(V(E_1), V(E_2)) \geqslant 1$ asserts $E([u_1, u_2]_X) - E' \neq \varnothing$. In other words, there exists an edge $e' \notin E'$ such that $e_1 \to e'$ in Γ. Therefore, Γ' is not a weak component in Γ. The obtained contradiction proves that E' is connected.

By Lemma 3, $\Gamma' \simeq \Gamma(X, \sigma)[E'] = \Gamma(X[E'], \mathrm{pr}_{V(E')} \circ \sigma)$. Furthermore, since Γ_1 and Γ_2 are nontrivial digraphs, we have $\Gamma \neq \Gamma'$. This implies $\partial_X V(E') \neq \varnothing$. Fix a vertex $w \in \partial_X V(E')$ and an edge $e \in E_X(w) - E'$. Let E'' be the vertex set of the weak component in $\Gamma(X, \sigma)$ which contains e. Similarly, we can prove that E'' is connected. Also, note that $w \in \partial_X V(E'')$. Finally, the proof of Theorem 4 implies that $\sigma(\partial_X V(E')) \subset$

$\partial_X V(E')$ as well as $\sigma(\partial_X V(E'')) \subset \partial_X V(E'')$. Hence, we can conclude that $\sigma(w) = w$.

Thus, for every weak component Γ' of Γ there exists its realization (X', σ') with $\operatorname{fix} \sigma' \neq \varnothing$. But Γ_1 (as well as Γ_2) is a disjoint union of weak components in Γ. Combining this fact with Remark 1 and Proposition 4, we obtain that $\Gamma_1 \sqcup K_1$ as well as $\Gamma_2 \sqcup K_1$ are M-graphs. □

Example 5. Consider the path $X \simeq P_4$ with $V(X) = \{1, 2, 3, 4\}$, $E(X) = \{12, 23, 34\}$ and its vertex map $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$. Then $\Gamma(X, \sigma)$ has a loop at each vertex, but $\Gamma(X, \sigma) \sqcup K_1$ is not an M-graph.

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