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RESEARCH ARTICLE

On representations of the group of order two over local factorial rings in the weakly modular case

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Abstract. We study representations of the group of order 2 over local factorial rings of characteristic not 2 with residue field of characteristic 2. The main results are related to a sufficient condition of wildness of groups.

Introduction

A group G is called wild over an commutative ring K, if the problem of classifying its matrix K-representations contains the problem of classifying the pairs matrices, up to similarity, over a field k. Otherwise, G is called tame over K. When K is a field of characteristic p ($p \geqslant 0$), a finite group G is tame if and only if its every noncyclic abelian p-subgroup has order at most 4 [1]. In particular,

1) in the classical case, when the order of G is not divisible by p, the group G is always tame and even has, up to equivalence, only finite number of indecomposable representations;

2) in the modular case, when the order of G is divisible by p, the group G has only finite number of indecomposable representations if and only if its Sylow p-subgroup G_p is cyclic; when it is not, then G is tame if and only if $p = 2$ and $G_2/[G_2, G_2] \cong (2, 2)$.

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For commutative rings such problem, in general case, is not solved. The first work in this direction is due to the first author [2]. If one talks about integral domains, then in the weakly modular case, i.e. when the order of G is not divisible by the characteristic p of a ring K but is divisible by the characteristic of the residue field [3], a criterion of wildness of G over a local ring K was obtained, in particular, in the following cases:

1) $K = Z'_p$ is the ring of p-adic rational numbers [4];

2) $K = R_p$ is the ring of integers of a finite extension F_p of the field p-adic rational numbers [5];

3) G is a p-group, K is a ring of formal power series in n variables over a complete discrete valuation ring of characteristic 0 with residue field of characteristic p [6].

Wildness of p-groups of order greater than p was studied in [6] for $p > 2$ and in [7, 8] for $p = 2$. Note that the smaller order of the group, the harder to find conditions of its wildness.

In this paper we study the case when the order of G is equal to 2.

1. Formulation of the main results

Let K be a local integral domain with maximal ideal R and residue field k, and G be a group. A matrix representation Γ of G over the free (associative) K-algebra $\Sigma = K\langle x, y\rangle$ is said to be *perfect* if from the equivalence of the representations $\Gamma \otimes T$ and $\Gamma \otimes T'$ of G over K, where T, T' are matrix representations Σ over K, it follows that T and T' are equivalent modulo R. Following Yu. Drozd [9, pp. 70-71] we call the group G *wild over* K if it has a perfect representation over Σ[1].

Recall some definitions on integral domains.

A prime element, or simply a prime, of an integral domain K is, by definition, a non-unit (non-invertible) element c such that whenever $c|ab$ for some $a, b \in K$, then $c|a$ or $c|b$. The element εc with ε to be a unit is called associated to c.

A factorial ring K is an integral domain in which every non-zero non-unit element x can be written as a product of prime elements, uniquely up to order and unit factors. The number $l(x)$ of the prime factors of x is called the length of x.

[1]The problem of allocation of wild objects (relative to different equivalences) has long been one of the main problems of modern representation theory. Besides classical objects (groups, algebras, rings, etc.) there are such well-known objects as directed graphs (quivers) and posets, both with various additional conditions (see, e.g. [10] – [13] for graphs and [14] – [21] for posets).

By *different prime elements* of K we mean non-associated ones.

The aim of this paper is to prove the following theorem.

Theorem 1 (on six twos). *Let G be the group of order 2 and K a local factorial ring of characteristic not 2 with residue field of characteristic 2. If K has 2 different primes and $l(2) > 2$, then G is wild.*

Corollary 1. *Let G be a (finite or infinite) group with a factor group to be a finite 2-group, and K be as in Theorem. Then G is wild.*

2. Auxiliary propositions

In this section K is a local integral domain with maximal ideal R.

Lemma 1. *Let $2 = t_1\, t_2\, t$ (in K), where t_1, t_2 are different primes, $t \in R$, and let*

$$t_1^2 x + t_2^2 y + t_1 t_2 z = 2w \tag{1}$$

for some $x, y, z, w \in K$. Then $x \equiv y \equiv z \equiv 0 \pmod R$.

Proof. From $2 = t_1\, t_2\, t$ and (1),

$$t_2(t_2 y + t_1 z - t_1 t w) = -t_1^2 x \tag{2}$$

whence $t_2 | x$ and therefore $x \equiv 0 \pmod R$. Let $x = t_2 x'$. Then we have from (2) (after reducing by t_2 and elementary transformations) that

$$t_1(z + t_1 x' - tw) = -t_2 y$$

whence $t_1 | y$ and $t_2 | z + t_1 x' - tw$; consequently $y \equiv z \equiv 0 \pmod R$.

Lemma 2. *Let $2 = t_1^2 t$ (in K), where t_1 is a prime, $t \in R$, and let*

$$t_1^2 x + t_2^2 y + t_1 t_2 z = 2w \tag{3}$$

for some $x, y, z, w \in K$ and a prime $t_2 \neq t_1$. Then $x \equiv y \equiv z \equiv 0 \pmod R$.

Proof. From $2 = t_1^2\, t$ and (3),

$$t_1(t_1 x + t_2 z - t_1 t w) = -t_2^2 y \tag{4}$$

whence $t_1 | y$ and therefore $y \equiv 0 \pmod R$. Let $y = t_1 y'$. Then we have from (4) (after reducing by t_1 and elementary transformations) that

$$t_1(x - tw) = -t_2(z + t_2 y')$$

whence $t_2 | x - tw)$ and $t_1 | z + t_2 y'$; consequently $x \equiv z \equiv 0 \pmod R$.

3. Proof of Theorem

Let $G = \langle g|\, g^2 = e\rangle$. It is natural to identify a matrix representations T of $\Sigma = K\langle x, y\rangle$ over K with the ordered pair of matrices $T(x), T(y)$; if these matrices are of size $m \times m$, we say that T is of K-dimension m. Then, for a matrix representation Γ of the group G over K (see above the definition of a wild group) and T of K-dimension m, the matrix $(\Gamma \otimes T)(g)$ is obtained from the matrix $\Gamma(g)$ by change x and y on the matrices $T(x)$ and $T(y)$, and $a \in K$ on the scalar matrix aE_m, where E_m is the identity $m \times m$ matrix.

From the conditions of the theorem it follows immediately that

1) $2 = t_1t_2t$ with t_1, t_2 to be different primes and $t \in R$, or

2) $2 = t_1^2 t$ with t_1 to be a prime and $t \in R$.

Consider first case 1).

We prove that the representation Γ of G over Σ of the form

$$\Gamma : g \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & t_1t_2 & t_1^2 x & 0 \\ 0 & 1 & 0 & t_2^2 & t_1t_2 & t_1^2 y \\ 0 & 0 & 1 & 0 & t_2^2 & t_1t_2 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

is perfect.

Let $T = (A, B)$ and $T' = (A', B')$ be matrix representations of Σ over K of a K-dimension n. Then

$$(\Gamma \otimes T)(g) = \left(\begin{array}{ccc|ccc} E_n & 0 & 0 & t_1t_2E_n & t_1^2 A & 0 \\ 0 & E_n & 0 & t_2^2 E_n & t_1t_2E_n & t_1^2 B \\ 0 & 0 & E_n & 0 & t_2^2 E_n & t_1t_2E_n \\ \hline 0 & 0 & 0 & -E_n & 0 & 0 \\ 0 & 0 & 0 & 0 & -E_n & 0 \\ 0 & 0 & 0 & 0 & 0 & -E_n \end{array}\right) =$$

$$= \begin{pmatrix} E_{3n} & t_1^2 M(A, B) + t_2^2 N + t_1t_2E_{3n} \\ 0 & -E_{3n} \end{pmatrix}$$

and

$$(\Gamma \otimes T')(g) = \left(\begin{array}{ccc|ccc} E_n & 0 & 0 & E_n & t_1^2 A' & 0 \\ 0 & E_n & 0 & t_2^2 E_n & E_n & t_1^2 B' \\ 0 & 0 & E_n & 0 & t_2^2 E_n & E_n \\ \hline 0 & 0 & 0 & -E_n & 0 & 0 \\ 0 & 0 & 0 & 0 & -E_n & 0 \\ 0 & 0 & 0 & 0 & 0 & -E_n \end{array}\right) =$$

$$= \begin{pmatrix} E_{3n} & t_1^2 M(A', B') + t_2^2 N + t_1 t_2 E_{3n} \\ 0 & -E_{3n} \end{pmatrix},$$

where

$$M(A,B) = \begin{pmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix}, \ M(A',B') = \begin{pmatrix} 0 & A' & 0 \\ 0 & 0 & B' \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 0 & 0 & 0 \\ E_n & 0 & 0 \\ 0 & E_n & 0 \end{pmatrix}.$$

Assume that the representations $\Gamma(A,B)$ and $\Gamma(A',B')$ are equivalent, i. e. there exists an invertible matrix C such that $(\Gamma \otimes T)(g)C = C(\Gamma \otimes T')(g)$. So we have the equality

$$\begin{aligned} &\begin{pmatrix} E_{3n} & t_1^2 M(A,B) + t_2^2 N + t_1 t_2 E_{3n} \\ 0 & -E_{3n} \end{pmatrix} \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} = \\ &= \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \begin{pmatrix} E_{3n} & t_1^2 M(A',B') + t_2^2 N + t_1 t_2 E_{3n} \\ 0 & -E_{3n} \end{pmatrix}, \end{aligned} \tag{5}$$

where the partition of

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

on blocks is compatible with those of $(\Gamma \otimes T)(g)$, $(\Gamma \otimes T')(g)$.

The equality (5) is equivalent to the following ones:

$$C_1 + t_1^2 M(A,B) C_3 + t_2^2 N C_3 + t_1 t_2 C_3 = C_1,$$

$$\begin{aligned} &C_2 + t_1^2 M(A,B) C_4 + t_2^2 N C_4 + t_1 t_2 C_4 \\ &\quad = t_1^2 C_1 M(A',B') + t_2^2 C_1 N + t_1 t_2 C_1 - C_2, \end{aligned}$$

$$-C_3 = C_3,$$

$$-C_4 = t_1^2 C_3 M(A',B') + t_2^2 C_3 N + t_1 t_2 C_3 - C_4.$$

In turn, these equations are equivalent to the equations $C_3 = 0$ and

$$t_1^2(M(A,B)C_4 - C_1M(A',B')) + t_2^2(NC_4 - C_1N) + t_1t_2(C_4 - C_1) = -2C_2.$$

By applying Lemma 1 to all scalar equations of the last matrix equation, we easily see that

$$M(A,B)C_4 \equiv C_1M(A',B') \pmod R,$$

$$NC_4 \equiv C_1N \pmod R, \quad C_4 \equiv C_1 \pmod R,$$

or equivalently,

$$M(A,B)C_1 \equiv C_1M(A',B') \pmod R, \tag{6}$$

$$NC_1 \equiv C_1N \pmod R. \tag{7}$$

From $C_3 = 0$ it follows that the block C_1 of the (invertible) matrix C is invertible. Put

$$C_1 = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

and write (7) in the expanded form:

$$\begin{pmatrix} 0 & 0 & 0 \\ E_n & 0 & 0 \\ 0 & E_n & 0 \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \equiv$$

$$\equiv \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ E_n & 0 & 0 \\ 0 & E_n & 0 \end{pmatrix} \pmod R$$

(the partition of C_1 on blocks is compatible with those of N). From this we have

$$\begin{pmatrix} 0 & 0 & 0 \\ C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \end{pmatrix} \equiv \begin{pmatrix} C_{12} & C_{13} & 0 \\ C_{22} & C_{23} & 0 \\ C_{32} & C_{33} & 0 \end{pmatrix} \pmod R,$$

whence $C_{12} \equiv C_{13} \equiv C_{23} \equiv 0 \pmod R$, $C_{11} \equiv C_{22} \equiv C_{33} \pmod R$, $C_{21} \equiv C_{32} \pmod R$, and therefore

$$C_1 \equiv \begin{pmatrix} C_{11} & 0 & 0 \\ C_{21} & C_{11} & 0 \\ C_{31} & C_{21} & C_{11} \end{pmatrix} \pmod R \tag{8}$$

with C_{11} being invertible modulo R.

From (6) and (8),

$$\begin{pmatrix} 0 & A & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} C_{11} & 0 & 0 \\ C_{21} & C_{11} & 0 \\ C_{31} & C_{21} & C_{11} \end{pmatrix} \equiv$$

$$\equiv \begin{pmatrix} C_{11} & 0 & 0 \\ C_{21} & C_{11} & 0 \\ C_{31} & C_{21} & C_{11} \end{pmatrix} \begin{pmatrix} 0 & A' & 0 \\ 0 & 0 & B' \\ 0 & 0 & 0 \end{pmatrix} \pmod{R},$$

or equivalently

$$\begin{pmatrix} AC_{21} & AC_{11} & 0 \\ BC_{31} & BC_{21} & BC_{11} \\ 0 & 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & C_{11}A' & 0 \\ 0 & C_{21}A' & C_{11}B' \\ 0 & C_{31}A' & C_{21}B' \end{pmatrix} \pmod{R}.$$

From this, in particular, we have

$$AC_{11} \equiv C_{11}A' \pmod{R}, \quad BC_{11} \equiv C_{11}B' \pmod{R},$$

as required.

Now consider case 2.

In this case we take as a perfect representation Γ of G over Σ the representation of the same form as in case 1) with t_2 to be any prime element different from t_1 (it exists by the condition of the theorem). Then the proof is analogously to that in case 1), but it is need to use Lemma 2 instead of Lemma 1.

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