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Flat extension and phantom homology

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Abstract. Phantom homology arises in tight closure theory due to small non-exactness when 'kernel' is not equal to 'image' but 'kernel' is in the tight closure of the 'image'. In this paper we study a typical flat extension, which we call *-flat extension, such that upon tensoring which preserves phantom homology. Along with other properties, we observe that *-flat extension preserves ghost regular sequence, which is a typical 'tight closure' generalization of regular sequence. We also show that in some situations, under *-flat extension, test ideal of the *-flat algebra is the expansion of the test ideal of the base ring.

1. Introduction

Throughout this paper all rings are commutative Noetherian rings of positive prime characteristic $p > 0$ and all modules are finitely generated. Theory of tight closure was developed by M. Hochster and C. Huneke for almost thirty years back and here we explore a typical non-exactness which arises in this theory. For a very good introduction we refer to [5] and [8]. For a general reference we refer to the excellent books [2, 9, 10].

In tight closure theory, we observe a typical non-exactness in a sequence of modules and maps where non-exactness is not very large. For $M' \xrightarrow{g} M \xrightarrow{f} M''$, the non-exactness is of this sort : $\operatorname{im} g \subset \ker f \subset (\operatorname{im} g)^*_M$ where $(\operatorname{im} g)^*_M$ is the tight closure of $\operatorname{im} g$ in M. Here we can say that the sequence is exact at M up to a tight closure. This notion of non-exactness

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was introduced in [5] in context to define phantom homology and we call such a non-exactness a phantom exactness at M. In the case of chain complex of modules, the above situation can be depicted as follows: For a complex $(C., d.)$ over the Noetherian ring R, if $\ker d_i \subset (\operatorname{im} d_{i+1})^*_{C_i}$ then we say that the i-th homology group $H_i(C.)$ is phantom and it is inside $0^*_{C_i/\operatorname{im} d_i}$.

In this paper, we introduce a variant of phantom exactness, call it *-exactness where for $M' \xrightarrow{g} M \xrightarrow{f} M''$ we have $\operatorname{im} g \subset \ker f = (\operatorname{im} g)^*_M$. In section 2, we introduce a flat ring extension which we call *-flat extension, which upon tensoring preserves *-exactness as well as phantom exactness i.e phantom homology. It is a well-known fact that flat extension preserves regular sequence. Observe the definition of a ghost regular sequence [3], which is a typical 'tight closure' generalization of regular sequence. In the similar way of flat extension, here we observe that *-flat extension preserves ghost regular sequence. We discuss few properties of *-flat extension and important results are given in Proposition 2.3 and its corollary, Proposition 2.4 and Theorem 2.6. In section 3, we study the behaviour of test ideal under *-flat extension. It comes out that with some conditions on the rings, test ideal of the *-flat algebra is the expansion of the test ideal of the base ring, see Theorem 3.4 and its corollaries. This result is also comparable with that of Theorem 7.36 of [6], see Remark 3D.

2. Properties of *-flat extension

In this section, we introduce a special kind of flat extension, which we call *-flat extension. In the context of phantom homology, we study some of its properties.

Consider Noetherian ring R and an R-module M. At first, we observe the following fact : Given a submodule $N \subset M$, we can always construct the sequence $N \xrightarrow{i} M \xrightarrow{f} M/N^*_M$ where i is injective and $\ker f = (\operatorname{im} i)^*_M = N^*_M$ and this motivates us to give the following definition which is a special kind of phantom exactness.

Definition. The sequence $M' \xrightarrow{g} M \xrightarrow{f} M''$ is *-exact at M if and only if $\ker f = (\operatorname{im} g)^*_M$.

We introduce the following notion of *-flat extension.

Definition. Let R be a Noetherian ring of characteristic p. We call a ring extension $R \to S$ *-flat R extension or S is *-flat over R, if and only if S is R flat and for an R-module homomorphism $N \xrightarrow{f} M$ with

$\ker f = 0^*_N$, we have an S-module homomorphism $N \otimes_R S \overset{f\otimes 1_S}{\to} M \otimes_R S$ with $\ker(f \otimes 1_S) = \ker f \otimes_R S = 0^*_{N\otimes_R S} = 0^*_N \otimes_R S$.

We observe the following proposition.

Proposition 2.1. *Let $M' \overset{g}{\to} M \overset{f}{\to} M''$ be a sequence of R-modules which is *-exact at M. If S is *-flat over R, then we have a sequence of S-modules $M' \otimes_R S \overset{g\otimes 1_S}{\to} M \otimes_R S \overset{f\otimes 1_S}{\to} M'' \otimes_R S$, such that* $\ker(f \otimes 1_S) = (\mathrm{im}\,(g \otimes 1_S))^*_{M\otimes_R S} = (\mathrm{im}\, g)^*_M \otimes_R S$.

*Conversely, let R be Noetherian ring and S be a flat R algebra such that upon tensoring, S preserves *-exactness then S is *-flat over R.*

*For a flat R-algebra S, S is *-flat over R if and only if it preserves the *-exactness.*

Proof. Assume that S is *-flat. For *-exactness at M we find that $\ker \overline{f} = (\mathrm{im}\, g)^*_M/\mathrm{im}\, g = 0^*_{M/img}$ where $\overline{f} : M/\mathrm{im}\, g \to M''$. For $M' \overset{g}{\to} M \to M/\mathrm{im}\, g \to 0$, we find that $(M/\mathrm{im}\, g) \otimes_R S = (M \otimes_R S)/\mathrm{im}\,(g \otimes 1_S)$. Thus, $\ker(\overline{f} \otimes 1_S) = 0^*_{(M\otimes_R S)/\mathrm{im}\,(g\otimes 1_S)} = (\mathrm{im}\,(g \otimes 1_S))^*_{M\otimes_R S}/\mathrm{im}\,(g \otimes 1_S) = \ker(f \otimes 1_S)/\mathrm{im}\,(g \otimes 1_S)$. So $\ker(f \otimes 1_S) = (\mathrm{im}\,(g \otimes 1_S))^*_{M\otimes_R S}$. Moreover, $\ker(\overline{f} \otimes 1_S) = \ker \overline{f} \otimes_R S = 0^*_{M/\mathrm{im}\, g} \otimes_R S = ((\mathrm{im}\, g)^*_M/\mathrm{im}\, g) \otimes_R S = ((\mathrm{im}\, g)^*_M \otimes_R S)/\mathrm{im}\,(g \otimes 1_S)$. Thus $\ker(f \otimes 1_S) = (\mathrm{im}\,(g \otimes 1_S))^*_{M\otimes_R S} = (\mathrm{im}\, g)^*_M \otimes_R S$

Conversely, take $0 \overset{i}{\to} N \overset{f}{\to} M$ such that $\ker f = 0^*_N = (\mathrm{im}\, i)^*_N$, then after tensoring with S we have $\ker(f \otimes 1_S) = (\mathrm{im}\,(i \otimes 1_S))^*_{N\otimes_R S} = 0^*_{N\otimes_R S} = \ker f \otimes_R S = 0^*_N \otimes_R S$.

Last assertion is immediate from first two assertions. □

Corollary. *For any flat R algebra S, S preserves tight closure upon tensoring if and only if S is a *-flat extension of R.*

Proof. If a flat R algebra S, S preserves tight closure upon tensoring, then S is *-flat over R since it preserves tight closure of zero submodule.

Conversely, let M be an R-module and N be its submodule. Consider *-exact sequence $N \overset{i}{\to} M \overset{f}{\to} M/N^*_M$. From Proposition 2.1 above we find that for a flat R algebra S, S is a *-flat over R if and only if it preserves *-exactness. Thus, we get $N^*_M \otimes_R S = (N \otimes_R S)^*_{M\otimes_R S}$ i.e. S preserves tight closure upon tensoring. □

As an immediate consequence, we have the following result.

Corollary. *Let $R \to S$ be a *-flat extension of R. For a complex $(C., d.)$, if i-th homology group $H_i(C.)$ is $0^*_{C_i/\mathrm{im}\, d_i}$, then we have $H_i(C. \otimes_R S) = 0^*_{(C.\otimes S)_i/\mathrm{im}\,(d_i\otimes S)}$.*

Remark 2A. It is worth noting that if S is flat R algebra then $N^*_M \otimes_R S \subset (N \otimes_R S)^*_{M \otimes_R S}$, which is a generalization of the result of (4.11) Lemma of [5]. Moreover for *-flat R-algebra S, expansions of all tightly closed ideals in R are tightly closed, specifically expansions of all radical ideals are tightly closed.

We present some examples of *-flat extensions.

Example. 1) R is regular if and only if $R \xrightarrow{F^e} R$ is flat if and only if $R \xrightarrow{F^e} R$ is *-flat. Since for R-module M and for its submodule $N \subset M$, $N^{[q]} \subset F^e M$ and $(N^*)^{[q]} = (N^{[q]})^* = N^{[q]}$.

2) For any flat ring homomorphism $R \to S$, if S is weakly F-regular, then $R \to S$ is *-flat. Since for R-modules $N \subset M$, $(N \otimes_R S)^*_{M \otimes_R S} = N \otimes_R S \subset N^*_M \otimes_R S \subset (N \otimes_R S)^*_{M \otimes_R S}$. (Here we mention one way which shows how one can get ring homomorphism $R \to S$ with S as weakly F-regular: Consider a *-flat local ring homomorphism $R \to S$ such that R is weakly F-regular and all ideals of S which are primary to maximal are extended ideals, then S is weakly F-regular. To see this, observe that for any J of S which is primary to maximal in S, $J = IS$ for some I in R and $J^* = (IS)^* = I^*S = IS = J$. (On the other hand faithfully flatness is sufficient for R to be weakly F-regular when S is weakly F-regular, as this implies $I^* = I^*S \cap R \subset (IS)^* \cap R = IS \cap R = I$)).

3) For local $R \to S$, let S be R flat. If R, S satisfy the other conditions of (7.15) of [6], we find S is *-flat (see (a) of (7.15) of [6]). Further in the same way for Artinian ring R and for local $R \to S$, where S is R flat and R, S satisfy the other conditions of (7.12) of [6], we find S is *-flat (see (a) of (7.12) of [6]).

Remark 2B. If S is an R-flat algebra then S is not *-flat in general. For R and a multiplicatively closed set T (or for local R), R_T is *-flat over R ($\hat{R}$ is *-flat over R) if and only if localization (completion) commutes with tight closure.

Proposition 2.2. *Consider a ring homomorphism $R \to S$. For an R-module homomorphism $N \xrightarrow{f} M$ assume* $\ker f \subset 0^*_N$. *If S is *-flat extension of R, then* $\ker(f \otimes 1_S) \subset 0^*_{N \otimes S}$

Proof. Consider the following maps: $g : N \to N/0^*_N$, $f' : N \to N/\ker f$, $p : N/\ker f \to N/0^*_N$ and $i : N/\ker f \to M$. Now $\ker(g \otimes 1_S) = 0^*_{N \otimes_R S}$ and $x \in \ker(f' \otimes 1_S)$ gives $(p \otimes 1_S)(f' \otimes 1_S)(x) = 0 = (g \otimes 1_S)(x)$. Thus $\ker(f \otimes 1_S) \subset 0^*_{N \otimes_R S}$, since flatness of S implies $i \otimes 1_S$ is injective and $\ker(f \otimes 1_S) = \ker(f' \otimes 1_S)$. □

The following proposition gives that *-flat extension not only preserves the *-exactness but also phantom exactness i.e phantom homology.

Proposition 2.3. *For $R \to S$ let S be *-flat over R and let $M' \xrightarrow{g} M \xrightarrow{f} M''$ be sequence of R-modules such that $\ker f \subset (\operatorname{im} g)^*_M$, then $\ker(f \otimes 1_S) \subset (\operatorname{im}(g \otimes 1_S))^*_{M\otimes_R S}$.*

Proof. Consider the following maps: $h : M \to M/(\operatorname{im} g)^*_M$, $f' : M \to M/\ker f$, $p : M/\ker f \to M/(\operatorname{im} g)^*_M$ and $i : M/\ker f \to M''$. Here $i \otimes 1_S$ is injective so $\ker(f \otimes 1_S) = \ker(f' \otimes 1_S) \subset \ker(h \otimes 1_S) = (\operatorname{im}(g \otimes 1_S))^*_{M\otimes_R S}$. □

We have an immediate corollary.

Corollary. *Let $R \to S$ be a *-flat extension of R. For a complex $C.$, if i-th homology group $H_i(C.)$ is phantom then so is $H_i(C. \otimes_R S)$. Thus, *-flat extension preserves the phantom homology.*

We observe that *-flat extension not only preserves phantom exactness but also preserves stably phantom exactness.

Proposition 2.4. *For $R \to S$ let S be *-flat over R, then it preserves stably phantom exactness. More precisely, for sequence of R-modules $M' \xrightarrow{g} M \xrightarrow{f} M''$, if it is stably phantom at M then for the sequence of S modules $M' \otimes_R S \xrightarrow{g\otimes 1_S} M \otimes_R S \xrightarrow{f\otimes 1_S} M'' \otimes_R S$ is also stably phantom at $M \otimes_R S$.*

Proof. Let F^e_R and F^e_S are the Peskine-Szpiro functors for the ring R and S respectively. For all $e \geqslant 0$, we have by hypothesis $F^e_R M' \xrightarrow{F^e_R g} F^e_R M \xrightarrow{F^e_R f} F^e_R M''$ such that $\ker F^e_R f \subset (\operatorname{im} F^e_R g)^*_{F^e_R M}$. Now for $F^e_R M' \otimes_R S = F^e_S(M' \otimes_R S)$ we find $F^e_S(M' \otimes_R S) \xrightarrow{F^e_R g\otimes 1_S} F^e_S(M \otimes_R S) \xrightarrow{F^e_R f\otimes 1_S} F^e_S(M'' \otimes_R S)$ such that $\ker(F^e_R f \otimes 1_S) \subset (\operatorname{im}(F^e_R g \otimes 1_S))^*_{F^e_S(M\otimes_R S)}$ (see Proposition 2.3). Thus $F^e_R f \otimes 1_S = F^e_S(f \otimes 1_S)$ and $F^e_R g \otimes 1_S = F^e_S(g \otimes 1_S)$ and we conclude. □

The above defined *-flat extension has a few properties similar to that of flat algebra. It is transitive like flat extension. Since we are concerned with only finitely generated modules we have to restrict ourselves to the base changes which are only module finite over the original ring.

Proposition 2.5. *Consider a ring homomorphism $R \to S$. If S is *-flat extension of R and T is an R algebra which is finitely generated as an R-module then $S \otimes_R T$ is *-flat extension of T and further for a ring homomorphism $S \to T'$, if T' is *-flat over S, then T' is *-flat over R.*

Proof. Let $N \xrightarrow{f} M$ be T module homomorphism such that $\ker f = 0^*_N$. Thus for $N \otimes_T (T \otimes_R S) \overset{f\otimes 1_{T\otimes_R S}}{\to} M \otimes_T (T \otimes_R S)$ and for $N \otimes_R S \overset{f\otimes 1_S}{\to} M \otimes_R S$ we find $\ker(f \otimes 1_{T\otimes_R S}) = \ker(f \otimes 1_S) = 0^*_{N\otimes_R S} = 0^*_{N\otimes_T(T\otimes_R S)}$ and we conclude. Further if T' is *-flat over S for $(N\otimes_R S)\otimes_S T' \overset{(f\otimes 1_S)\otimes 1'_T}{\to} (M \otimes_R S) \otimes_S T'$ we find $\ker(f \otimes 1_S) \otimes 1'_T) = 0^*_{N\otimes_R S\otimes_S T'} = \ker(f \otimes 1_{T'}) = 0^*_{N\otimes_R T'}$ □

We conclude the section with the application of *-flat extension on ghost M-regular sequence. We recall the definition of a ghost M-regular sequence [3]:

Definition. Let R be a Noetherian ring of prime characteristic $p > 0$ and let M be a R-module. Then we say an element $x \in R$ is *weak ghost M-regular* if $0 :_{F^e(M)} x^{p^e} \subseteq 0^*_{F^e(M)}$ for all $e \geqslant 0$.

A sequence $\mathbf{x} = x_1, \ldots, x_n$ of elements of R is a *weak ghost M-regular sequence* if x_i is weak ghost $(M/(x_1, \ldots, x_{i-1})M)$-regular for $1 \leqslant i \leqslant n$.

Theorem 2.6. *For a ring homomorphism $R \to S$ let S be *-flat over R. For R-module M, if x is a weakly ghost M-regular element, then $x \otimes_R 1_S$ is weakly ghost $M \otimes_R S$-regular. Moreover *-flatness preserves weakly ghost M-regular sequences.*

Proof. For a weakly ghost M-regular element x, we find that the kernel of $f_e : F^e M \to F^e M$ is contained in $0^*_{F^e M}$, where f_e is multiplication by x^{p^e}. Now by Proposition 2.2 we find $\ker(f_e \otimes 1_S) \subset 0^*_{F^e M\otimes S} = 0^*_{F^e_S(M\otimes_R S)}$. To prove the second statement let $M_{i-1} = (x_1, ..x_{i-1})M$ and for $f^i_e : F^e(M/M_{i-1}) \to F^e(M/M_{i-1})$ where f^i_e is multiplication by $x_i^{p^e}$ we find that the kernel of f^i_e is cotained in $0^*_{F^e(M/M_{i-1})}$. Thus we get $f^i_e \otimes_R 1_S : F^e(M/M_{i-1}) \otimes_R S \to F^e(M/M_{i-1}) \otimes_R S$ where $F^e(M/M_{i-1}) \otimes_R S = F^e_S((M \otimes_R S)/(x_1, ..x_{i-1})(M \otimes_R S))$ and by Proposition 2.2 we conclude. □

3. Behaviour of test ideal under *-flat extension

In this section, we study the behaviour of test ideal under *-flat ring extension. At first, we recall the definition of test ideal (see [5]).

Definition. Let R be a Noetherian ring of prime characteristic $p > 0$, the test ideal $\tau(R)$ be the ideal of R which is $\cap_M \mathrm{Ann}_R 0^*_M$ where M runs through all finitely generated R-modules.

Remark 3A. R and $\tau(R) \cap R^0$ is the set of test elements (see (b) of (8.23) of [5]). It turns out that for local ring (R, m, K), $\tau(R)$ is the intersection of the annihilators of 0^*_M where M is an element of family of R-module M of finite length (see (d) of (8.23) Proposition of [5]). Moreover it is sufficient to consider the family of essential extension of K of finite length, whose union is the injective hull of K (see proof of (c) of (8.23) and (d) of (8.23) Proposition of [5]).

In [6] the extension of test ideals has been studied, here we quote the result:

Theorem 3.1 ((7.36) Theorem, [6]). *Let $(R, m, K) \to (S, n, L)$ be a flat local homomorphism of complete local rings such that L/K is separable and the closed fibre S/mS is regular. Then $\tau(S) = \tau(R)S$*

There is a generalization of the above result in [1] when $R \to S$ is a smooth morphism of locally finite type:

Theorem 3.2 ((5.1) Theorem, [1]). *Let $(R, m, K) \to (S, n, L)$ be a (locally finite type) smooth homomorphism of reduced excellent rings of positive characteristic p. Suppose that test ideals commute with localization and that for each maximal ideal $m \subset R$, R_m/mR_m is a perfect field. Then $\tau(S) = \tau(R)S$.*

We need the definition of approximately Gorenstein ring (see [4]).

Definition. A local ring (R, m) is approximately Gorenstein if for every integer $N > 0$ there is an m primary irreducible ideal I_N of R such that $I_N \subset m^N$. The sequence of ideals $\{I_N\}$ is called sequence of small cofinite irreducible.

Remark 3B. It turns out that, for approximately Gorenstein local ring (R, m, K), if E_R is the injective hull of K (which may not be a finitely generated module), then $E_R = \lim_N R/I_N$, where $\{I_N\}$ is the sequence of small cofinite irreducible. Approximately Gorenstein local ring provides examples of a large class of rings, which includes the example of reduced excellent local rings (see (5.2) Theorem of [4]. For a list of approximately Gorenstein local rings, see also (8.6) Discussion of [5].

Proposition 3.3. *Consider a flat local ring homomorphism $(R, m) \to (S, n)$ such that R is approximately Gorenstein and the fibre S/mS is a Gorenstein ring. Then S is approximately Gorenstein ring.*

Moreover, if the ideals $\{I_N\}$ of R is the sequence of small cofinite irreducible and fibre S/mS is a zero dimensional Gorenstein ring, then the ideals $\{I_N S\}$ of S is also the sequence of small cofinite irreducible.

Proof. Let ideals $\{I_N\}$ of R is the sequence of small cofinite irreducible and consider the ring homomorphism $R/I_N \to S/I_N S$. As $R \to S$ is faithfully flat so for $R/I_N \to S/I_N S$ and $I_N S \subset (mS)^N \subset n^N$. Consequently $(S/I_N S)/(m/I_N)(S/I_N S) = S/mS$ is Gorenstein. So by Theorem 23.4 of [10], $S/I_N S$ is Gorenstein. Now by (2.1) Proposition of [4], there exists an irreducible ideal $J_N \subset n^N$ in S such that J_N is n-primary. Thus the ideals $\{J_N\}$ is a sequence of small cofinite irreducible and so S is approximately Gorenstein.

For the second assertion, as $S/I_N S$ becomes zero dimensional Gorenstein ring, the result follows. □

Theorem 3.4. *Consider a local ring homomorphism $(R, m) \to (S, n)$ such that R is approximately Gorenstein and the fibre S/mS is a zero dimensional Gorenstein ring. If S is *-flat extension of R then $\tau(S) \supset \tau(R)S$.*

Proof. From Proposition 2.9 we find that S is approximately Gorenstein and if ideals $\{I_N\}$ of R is a sequence of small cofinite irreducible then so for the ideals $\{I_N S\}$ of S. If E_R is injective hull of R/m and E_S is injective hull of S/n then from Remark 3B, $E_R = \lim_N R/I_N$ and so $E_S = \lim_N S/I_N S = E_R \otimes_R S$. Let for finitely generated R-modules $\{M_\lambda\}$, such that $M_\lambda \subset E_R$ and $E_R = \lim_\lambda M_\lambda$ implies $M_\lambda \otimes_R S \subset E_S$ and $E_S = \lim_\lambda (M_\lambda \otimes_R S)$. As S is *-flat so $0^*_M \otimes_R S = 0^*_{M \otimes_R S}$. Thus $\tau(S) = \cap_\lambda \text{Ann}_S 0^*_{M_\lambda \otimes_R S} = \cap_\lambda (\text{Ann}_R 0^*_{M_\lambda} \otimes_R S) \supset (\cap_\lambda \text{Ann}_R 0^*_{M_\lambda}) \otimes_R S = \tau(R)S$. □

Remark 3C. It has been pointed out in [1] that for the flat local ring homomorphism $R \to S$, if R is complete then $\tau(S) \subset \tau(R)S$, as S is $\cap$-flat by Chevalley's Theorem (see paragraph above (7.18) Theorem of [6]).

Corollary. *Consider complete reduced local ring (R, m). Let (S, n) be another local ring. Consider a local ring homomorphism $R \to S$ such that fibre S/mS is zero dimensional Gorenstein ring. If S is *-flat over R, then $\tau(S) = \tau(R)S$.*

Proof. Remark 3C gives $\tau(S) \subset \tau(R)S$. For other inclusion we can use Remark 3B and then proof follows from the Theorem 3.4. □

Remark 3D. It is to be noted that in above corollary it is sufficient to consider only R to be complete, where in (7.36) Theorem, [6] we need both R and S to be complete. Also instead of S/mS to be regular as it is the required in (7.36) Theorem, [6], here we need S/mS to be a Gorenstein ring of zero dimension. But, we have a relaxation on these conditions of (7.36) Theorem, [6] at the cost of *-flat extension S of R which is a flat extension of R of special type.

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