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RESEARCH ARTICLE

Planarity of a spanning subgraph of the intersection graph of ideals of a commutative ring II, Quasilocal Case

P. Vadhel and S. Visweswaran*

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ABSTRACT. The rings we consider in this article are commutative with identity $1 \neq 0$ and are not fields. Let R be a ring. We denote the collection of all proper ideals of R by $\mathbb{I}(R)$ and the collection $\mathbb{I}(R) \setminus \{(0)\}$ by $\mathbb{I}(R)^*$. Let $H(R)$ be the graph associated with R whose vertex set is $\mathbb{I}(R)^*$ and distinct vertices I, J are adjacent if and only if $IJ \neq (0)$. The aim of this article is to discuss the planarity of $H(R)$ in the case when R is quasilocal.

1. Introduction

The rings considered in this article are commutative with identity which admit at least one nonzero proper ideal. Let R be a ring. As in [4], we denote the collection of all proper ideals of R by $\mathbb{I}(R)$ and the collection $\mathbb{I}(R) \setminus \{(0)\}$ by $\mathbb{I}(R)^*$. Let R be a ring such that $\mathbb{I}(R)^* \neq \varnothing$. Motivated by the work done on the intersection graph of ideals of a ring in the literature (see for example, [1, 6, 10]), in [14], we introduced and investigated the properties of an undirected graph associated with R, denoted by $H(R)$, whose vertex set is $\mathbb{I}(R)^*$ and distinct vertices I, J are adjacent if and only if $IJ \neq (0)$. We denote the set of all maximal ideals of a ring R by $\mathrm{Max}(R)$ and the cardinality of a set A by $|A|$. We denote the set of all units of a ring R by $U(R)$. The intersection graph of ideals of a ring R is denoted by $G(R)$. Observe that $H(R)$ is a spanning

*Corresponding author.

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subgraph of $G(R)$. Inspired by the research work done on the planarity of the intersection graph of ideals of a ring in [10, 11], we characterized rings R with $|\operatorname{Max}(R)| \geqslant 2$ such that $H(R)$ is planar in [12]. We say that a ring R is *quasilocal* (respectively, semiquasilocal) if $|\operatorname{Max}(R)| = 1$ (respectively, $|\operatorname{Max}(R)| < \infty$). A Noetherian quasilocal (respectively, semiquasilocal) ring is refereed to as a *local* (respectively, *semilocal*) ring. The purpose of this article is to characterize quasilocal rings R such that $H(R)$ is planar.

The graphs considered in this article are undirected and simple. Let $G = (V, E)$ be a graph. Recall from [3, Definition 8.1.1] that G is said to be *planar* if G can be drawn in a plane in such a way that no two edges of G intersect in a point other than a vertex of G. For definitions and notations in graph theory that are not mentioned here, the reader can refer either [3] or [9]. In view of Kuratowski's theorem [9, Theorem 5.9] and out of curiosity to know whether the algebraic structure of the ring R plays a role in arriving at the conclusion that $H(R)$ is planar if $H(R)$ satisfies at least one between (C_1) and (C_2), where for each $i \in \{1, 2\}$, the conditions (C_i), (C_i^*) were already introduced in [12]. It is useful to recall them first:

- (C_1) G does not contain K_5 as a subgraph (equivalently, if $\omega(G) \leqslant 4$);
- (C_2) G does not contain $K_{3,3}$ as a subgraph;
- (C_1^*) G satisfies (C_1) and moreover, G does not contain any subgraph homeomorphic to K_5;
- (C_2^*) G satisfies (C_2) and moreover, G does not contain any subgraph homeomorphic to $K_{3,3}$.

Recall that a principal ideal ring is said to be a *special principal ideal ring* (SPIR) if R has a unique prime ideal. If $\mathfrak{m}$ is the unique prime of a SPIR R, then $\mathfrak{m}$ is principal and nilpotent. If R is a SPIR with $\mathfrak{m}$ as its only prime ideal, then we denote it by mentioning that $(R, \mathfrak{m})$ is a SPIR. Let $(R, \mathfrak{m})$ be a quasilocal ring such that $\mathfrak{m}$ is principal and nilpotent. Let $n \geqslant 2$ be least with the property that $\mathfrak{m}^n = (0)$. Then it follows from the proof of (iii) $\Rightarrow$ (i) of [2, Proposition 8.8] that $\{\mathfrak{m}^i | i \in \{1, \ldots, n-1\}\}$ equals $\mathbb{I}(R)^*$ and so, $(R, \mathfrak{m})$ is a SPIR.

Let R be a ring which is not necessarily quasilocal. Recall from [4] that an ideal I of R is said to be an *annihilating ideal* if there exists $r \in R \setminus \{0\}$ such that $Ir = (0)$. Let R be a ring which is not an integral domain. As in [4], we denote the collection of all annihilating ideals of R by $\mathbb{A}(R)$ and the collection $\mathbb{A}(R) \setminus \{(0)\}$ by $\mathbb{A}(R)^*$. Recall from [4] that the *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, is an undirected graph whose vertex set is $\mathbb{A}(R)^*$ and distinct vertices I, J are adjacent if and only if $IJ = (0)$.

Let $G = (V, E)$ be a simple graph. Recall from [3, Definition 1.1.13] that the *complement* of G, denoted by G^c, is a graph whose vertex set is V and distinct vertices x, y are joined by an edge in G^c if and only if there is no edge joining x and y in G. For a graph G, we denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$.

Let R be a ring such that $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. Then $V(H(R)) = V(\mathbb{AG}(R))$. For distinct $I, J \in \mathbb{I}(R)^*$, I, J are adjacent in $H(R)$ if and only if $IJ \neq (0)$ if and only if I, J are adjacent in $(\mathbb{AG}(R))^c$. Hence, $H(R) = (\mathbb{AG}(R))^c$.

Let $G = (V, E)$ be a graph. Recall from [3, Definition 5.1.1] that a nonempty subset S of V is called *independent* if no two vertices of S are adjacent in G. Suppose that there exists $k \in \mathbb{N}$ such that $|S| \leqslant k$ for any independent set S of V. Recall from [3, Definition 5.1.4] that the *independence number* of G, denoted by $\alpha(G)$, is defined as the largest positive integer n such that G contains an independent set S with $|S| = n$. If G contains an independent set containing exactly n vertices for each $n \geqslant 1$, then we define $\alpha(G) = \infty$. For any graph G, it is clear that $\alpha(G) = \omega(G^c)$. Let R be a ring such that $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. Then $H(R) = (\mathbb{AG}(R))^c$ and so, $\omega(H(R)) = \omega((\mathbb{AG}(R))^c) = \alpha(\mathbb{AG}(R))$. Let R be a ring such that $\mathbb{A}(R)^* \neq \varnothing$. In Section 4, we use the results that were proved on $\alpha(\mathbb{AG}(R))$ in [13].

Let $(R, \mathfrak{m})$ be a quasilocal ring which is not a field. The aim of this article is to characterize R such that $H(R)$ is planar. It is clear that if $\mathfrak{m}^2 = (0)$, then $H(R)$ has no edges, and so, $H(R)$ is planar. Hence, in this article, we consider quasilocal rings $(R, \mathfrak{m})$ such that $\mathfrak{m}^2 \neq (0)$. This article consists of four sections.

Section 2 of this article is devoted to state and prove some necessary conditions in order that $H(R)$ satisfies either (C_1) or (C_2). The main result proved in Section 2 is Proposition 2.7 in which it is shown that if $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}$ can be generated by at most two elements and R is necessarily Artinian.

In Section 3, we consider local Artinian rings $(R, \mathfrak{m})$ such that $\mathfrak{m}$ is principal and $\mathfrak{m}^2 \neq (0)$. That is, $(R, \mathfrak{m})$ is a SPIR with $\mathfrak{m}^2 \neq (0)$. It is proved in Theorem 3.3 that $H(R)$ satisfies (C_1), if and only if $H(R)$ satisfies (C_2), if and only if $\mathfrak{m}^9 = (0)$, if and only if $H(R)$ is planar.

In Section 4, we consider Artinian local rings $(R, \mathfrak{m})$ such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$, $\mathfrak{m}^2 \neq (0)$, and try to determine R such that $H(R)$ is planar.

We discuss the planarity of $H(R)$ with the help of several cases.

In case (1), we assume that $a^2 = b^2 = 0$ but $ab \neq (0)$. With this assumption, it is shown in Theorem 4.4 that $H(R)$ satisfies (C_1), if and only if $H(R)$ satisfies (C_2), if and only if $H(R)$ is planar. It is verified that such rings R are such that $|R| \in \{16, 81\}$. With the help of results from [5, 7, 8], in Example 4.5, we provide some examples to illustrate Theorem 4.4.

In case (2), we assume that $a^2 \neq 0$ but $b^2 = ab = 0$. With this assumption, it is proved in Theorem 4.10 that $H(R)$ satisfies both (C_1) and (C_2), if and only if $\mathfrak{m}^3 = (0)$ and $|\frac{R}{\mathfrak{m}}| \leqslant 3$, if and only if $H(R)$ is planar. It is noted in Remark 4.11 if R is such a ring, then $|R| \in \{16, 81\}$ and in Example 4.12, we provide some examples to illustrate Theorem 4.10. In Example 4.14, we provide an example of a local Artinian ring $(R, \mathfrak{m})$ which satisfies the hypotheses of Theorem 4.10 such that $H(R)$ satisfies (C_1) but does not satisfy (C_2) and in Example 4.16, we provide an example of a local Artinian ring $(R, \mathfrak{m})$ which satisfies the hypotheses of Theorem 4.10 such that $H(R)$ satisfies (C_2) but does not satisfy (C_1).

In case (3), we assume that $a^2 \neq 0$, $b^2 \neq 0$, whereas $ab = 0$. With this assumption, it is shown in Theorem 4.22 that $H(R)$ satisfies both (C_1) and (C_2) if and only if $\mathfrak{m}^2 = Ra^2 = Rb^2$ and $|\frac{R}{\mathfrak{m}}| \leqslant 3$ if and only if $H(R)$ is planar. If $(R, \mathfrak{m})$ is a local Artinian ring which satisfies the hypotheses of Theorem 4.22 such that $H(R)$ is planar, then $|R| \in \{16, 81\}$ and in Example 4.23, some examples are provided to illustrate Theorem 4.22. The local Artinian ring $(R, \mathfrak{m})$ provided in Example 4.24 is such that it satisfies the hypotheses of Theorem 4.22 and is such that $H(R)$ satisfies (C_1) but does not satisfy (C_2). In Example 4.26, we provide an example of a local Artinian ring $(R, \mathfrak{m})$ which satisfies the hypotheses of Theorem 4.22 and is such that $H(R)$ satisfies (C_2) but does not satisfy (C_1).

In case (4), we assume that $a^2, ab \in R \setminus \{0\}$, whereas $b^2 = 0$. If $a^2 + ab = 0$, then it is verified that $(R, \mathfrak{m})$ satisfies the hypotheses of Theorem 4.22 and such a R is already determined in Theorem 4.22 such that $H(R)$ is planar. Hence, in case (4), we assume that $a^2 + ab \neq 0$. With this assumption, it is proved in Theorem 4.30 that $H(R)$ satisfies both (C_1) and (C_2) if and only if $H(R)$ satisfies (C_2) if and only if $\mathfrak{m}^3 = (0)$, $\mathfrak{m}^2 = Rab$, and $|\frac{R}{\mathfrak{m}}| = 3$ if and only if $H(R)$ is planar. It is clear that such a ring R satisfies $|R| = 81$ and in Example 4.31, we provide an example to illustrate Theorem 4.30. An example of a local Artinian ring $(R, \mathfrak{m})$ which satisfies the hypotheses of Theorem 4.30 is provided in Example 4.32 and is such that $H(R)$ satisfies (C_1) but $H(R)$ does not satisfy (C_2).

In case (5), we assume that $a^2, b^2, ab \in R \setminus \{0\}$. It is observed that in view of Theorems 4.10 and 4.22, in determining R such that $H(R)$ is planar, we can assume that $a^2 + ab, b^2 + ab \in R \setminus \{0\}$. With these assumptions, some necessary conditions are obtained in order that $H(R)$ to satisfy either (C_1) or (C_2). We are not able to determine R such that $H(R)$ is planar. However with the further assumptions that $\mathfrak{m}^2$ is not principal, $\mathfrak{m}^3 = (0)$, and $a^2 \neq b^2$, it is shown in Theorem 4.42 that $H(R)$

satisfies both (C_1) and (C_2) if and only if $H(R)$ satisfies (C_1) if and only if $\mathfrak{m}^2 = Ra^2 + Rab = Rb^2 + Rab$, $|\frac{R}{\mathfrak{m}}| = 2$, and $\mathfrak{m}^2 \subseteq R(a+b)$ if and only if $H(R)$ is planar. An example of a local Artinian ring $(R, \mathfrak{m})$ is provided in Example 4.43 to illustrate Theorem 4.42.

2. Some necessary conditions for $H(R)$ to satisfy either (C_1) or (C_2)

Let $(R, \mathfrak{m})$ be a quasilocal ring such that $\mathfrak{m} \neq (0)$. We devote this section to determine some necessary conditions for $H(R)$ to satisfy either (C_1) or (C_2).

Lemma 2.1. *Let $n \in \mathbb{N}$. Let $(R, \mathfrak{m})$ be a quasilocal ring. If $\omega(H(R)) \leqslant n$, then $\mathfrak{m}^{2n+1} = (0)$.*

Proof. Assume that $\omega(H(R)) \leqslant n$. As $J(R) = \mathfrak{m}$, it follows from [12, Lemmas 2.5 and 2.10] that $\mathfrak{m}$ is nilpotent. Let $k \in \mathbb{N}$ be least with the property that $\mathfrak{m}^k = (0)$. Suppose that $k > 2n+1$. Observe that $\mathfrak{m}^i \neq \mathfrak{m}^j$ for all distinct $i, j \in \{1, 2, \ldots, k\}$ and $\mathfrak{m}^i \neq (0)$ for each i with $1 \leqslant i < k$. Hence, the subgraph of $H(R)$ induced by $\{\mathfrak{m}^i | i \in \{1, 2, \ldots, n+1\}\}$ is a clique on $n+1$ vertices. This implies that $\omega(H(R)) \geqslant n+1$ and this is a contradiction. Therefore, $k \leqslant 2n+1$ and so, $\mathfrak{m}^{2n+1} = (0)$. □

Lemma 2.2. *Let $(R, \mathfrak{m})$ be a quasilocal ring. If $H(R)$ satisfies either (C_1) or (C_2), then $\mathfrak{m}^9 = (0)$.*

Proof. Assume that $H(R)$ satisfies (C_1). Then $\omega(H(R)) \leqslant 4$. Hence, we obtain from Lemma 2.1 that $\mathfrak{m}^9 = (0)$. Assume that $H(R)$ satisfies (C_2). Then $\omega(H(R)) \leqslant 5$. Therefore, we obtain from Lemma 2.1 that $\mathfrak{m}^{11} = (0)$. Suppose that $\mathfrak{m}^9 \neq (0)$. Then $\mathfrak{m}^i \neq \mathfrak{m}^j$ for all distinct $i, j \in \{1, \ldots, 9\}$. Let $A = \{\mathfrak{m}, \mathfrak{m}^2, \mathfrak{m}^3\}$ and let $B = \{\mathfrak{m}^4, \mathfrak{m}^5, \mathfrak{m}^6\}$. It is clear that $A \cup B \subseteq V(H(R))$, $A \cap B = \emptyset$, and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2). Therefore, we get that $\mathfrak{m}^9 = (0)$. □

Lemma 2.3. *Let $(R, \mathfrak{m})$ be a quasilocal ring such that $\mathfrak{m}$ is nilpotent. If $\{a_\alpha\}_{\alpha \in \Lambda} \subseteq \mathfrak{m}$ is such that $\{a_\alpha + \mathfrak{m}^2 | \alpha \in \Lambda\}$ is a basis of $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ as a vector space over $\frac{R}{\mathfrak{m}}$, then $\mathfrak{m} = \sum_{\alpha \in \Lambda} Ra_\alpha$.*

Proof. By hypothesis, $\mathfrak{m}$ is nilpotent. Let $k \in \mathbb{N}$ be such that $\mathfrak{m}^k = (0)$. As $2^k > k$, it follows that $\mathfrak{m}^{2^k} = (0)$. Let us denote $\sum_{\alpha \in \Lambda} Ra_\alpha$ by I. It follows from the hypothesis on the elements a_α, $\alpha \in \Lambda$ that $\mathfrak{m} = I + \mathfrak{m}^2 = I + (I + \mathfrak{m}^2)^2 = I + \mathfrak{m}^4 = I + \mathfrak{m}^8 = \cdots = I + \mathfrak{m}^{2^k}$. From $\mathfrak{m}^{2^k} = (0)$, it follows that $\mathfrak{m} = I = \sum_{\alpha \in \Lambda} Ra_\alpha$. □

Lemma 2.4. *Let $(R, \mathfrak{m})$ be a quasilocal ring. Let $\{a, b, c\} \subseteq \mathfrak{m}$ be such that $a + \mathfrak{m}^2, b + \mathfrak{m}^2, c + \mathfrak{m}^2$ are linearly independent over $\frac{R}{\mathfrak{m}}$. If at least one among ab, bc, ca is different from 0, then $H(R)$ neither satisfies (C_1) nor satisfies (C_2).*

Proof. We can assume without loss of generality that $ab \neq 0$. If $a^2 \neq 0$, then the subgraph of $H(R)$ induced by $\{Ra, Rb, Ra + Rb, Ra + Rc, \mathfrak{m}\}$ is a clique of five vertices. If $\mathfrak{b}^2 \neq 0$, then the subgraph of $H(R)$ induced by $\{Ra, Rb, Ra + Rb, Rb + Rc, \mathfrak{m}\}$ is a clique on five vertices. If $a^2 = b^2 = (0)$, then the subgraph of $H(R)$ induced by $\{Ra, Rb, R(a+b), Ra+Rb, \mathfrak{m}\}$ is a clique on five vertices. Hence, we arrive at the conclusion that $\omega(H(R)) \geqslant 5$ and so, $H(R)$ does not satisfy (C_1). Let $A = \{Ra, Ra + Rb, Ra + Rc\}$ and let $B = \{Rb, Rb + Rc, \mathfrak{m}\}$. Observe that $A \cup B \subseteq V(H(R))$, $A \cap B = \varnothing$, and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. Therefore, we obtain that $H(R)$ does not satisfy (C_2). □

Lemma 2.5. *Let $(R, \mathfrak{m})$ be a quasilocal ring. Let $\{a, b, c\} \subseteq \mathfrak{m}$ be such that $a+\mathfrak{m}^2, b+\mathfrak{m}^2, c+\mathfrak{m}^2$ are linearly independent over $\frac{R}{\mathfrak{m}}$. If $ab = bc = ca = 0$ and $a^2 \neq 0$, then $H(R)$ neither satisfies (C_1) nor satisfies (C_2).*

Proof. Note that the subgraph of $H(R)$ induced by $\{Ra, R(a + b), Ra + Rb, Ra+Rc, \mathfrak{m}\}$ is a clique on five vertices. This implies that $\omega(H(R)) \geqslant 5$. Hence, we get that $H(R)$ does not satisfy (C_1). (In this part of the proof, we use only the assumptions that $a^2 \neq 0$ and $ab = 0$.) Let $A = \{Ra, R(a + c), Ra + Rc\}$ and let $B = \{R(a + b), Ra + Rb, \mathfrak{m}\}$. It is clear that $A \cup B \subseteq V(H(R))$, $A \cap B = \varnothing$, and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. Therefore, we obtain that $H(R)$ does not satisfy (C_2). □

Lemma 2.6. *Let $(R, \mathfrak{m})$ be a quasilocal ring such that $\mathfrak{m}^2 \neq (0)$. If $H(R)$ satisfies either (C_1) or (C_2), then $\dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) \leqslant 2$.*

Proof. Assume that $H(R)$ satisfies either (C_1) or (C_2). We know from Lemma 2.2 that $\mathfrak{m}^9 = (0)$. Suppose that $\dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) \geqslant 3$. Let $\{a_\alpha | \alpha \in \Lambda\} \subseteq \mathfrak{m}$ be such that $\{a_\alpha + \mathfrak{m}^2 | \alpha \in \Lambda\}$ is a basis of $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ as a vector space over $\frac{R}{\mathfrak{m}}$. By assumption, $|\Lambda| \geqslant 3$ and we know from Lemma 2.3 that $\mathfrak{m} = \sum_{\alpha\in\Lambda} Ra_\alpha$. Hence, $\mathfrak{m}^2 = \sum_{\alpha,\beta\in\Lambda} Ra_\alpha a_\beta$. As $|\Lambda| \geqslant 3$, it follows from Lemma 2.4 that $a_\alpha a_\beta = 0$ for all distinct $\alpha, \beta \in \Lambda$. By hypothesis, $\mathfrak{m}^2 \neq (0)$ and so, $a_\alpha^2 \neq 0$ for some $\alpha \in \Lambda$. In such a case, it follows from Lemma 2.5 that $H(R)$ neither satisfies (C_1) nor satisfies (C_2). This is a contradiction and so, we obtain that $\dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) \leqslant 2$. □

Proposition 2.7. *Let* $(R, \mathfrak{m})$ *be a quasilocal ring such that* $\mathfrak{m}^2 \neq (0)$*. If* $H(R)$ *satisfies either* (C_1) *or* (C_2)*, then* R *is a local Artinian ring,* $\mathfrak{m}^9 = (0)$*, and* $\mathfrak{m}$ *can be generated by at most two elements.*

Proof. Assume that $H(R)$ satisfies either (C_1) or (C_2). We know from Lemma 2.2 that $\mathfrak{m}^9 = (0)$. Hence, it follows that $\mathfrak{m}$ is the only prime ideal of R and so, $\dim R = 0$. We know from Lemmas 2.6 and 2.3 that $\mathfrak{m}$ can be generated by at most two elements. Thus any prime ideal of R is finitely generated and so, we obtain from Cohen's theorem [2, Exercise 1, page 84] that R in Noetherian. Thus R is Noetherian and $\dim R = 0$ and therefore, we obtain from [2, Theorem 8.5] that R is Artinian. This shows that $(R, \mathfrak{m})$ is a local Artinian ring, $\mathfrak{m}^9 = (0)$, and $\mathfrak{m}$ can generated by at most two elements. □

Remark 2.8. Let $(R, \mathfrak{m})$ be a quasilocal ring with $\mathfrak{m}^2 \neq (0)$. If $H(R)$ is planar, then it follows from [9, Theorem 5.9] that $H(R)$ satisfies both (C_1^*) and (C_2^*). Therefore, $H(R)$ satisfies both (C_1) and (C_2) and so, we obtain from Proposition 2.7 that $(R, \mathfrak{m})$ is a local Artinian ring, $\mathfrak{m}^9 = (0)$, and $\mathfrak{m}$ can be generated by at most two elements. Hence, in discussing the planarity of $H(R)$, we assume that $(R, \mathfrak{m})$ is a local Artinian ring and $\mathfrak{m}$ is generated by at most two elements. If $\mathfrak{m}$ is principal, then as is remarked in the introduction, we obtain that $(R, \mathfrak{m})$ is a SPIR.

3. When is $H(R)$ planar if $(R, \mathfrak{m})$ is a SPIR?

Let $(R, \mathfrak{m})$ be a SPIR with $\mathfrak{m}^2 \neq (0)$. The aim of this section is to determine when $H(R)$ is planar.

Lemma 3.1. *Let* $(R, \mathfrak{m})$ *be a* SPIR *with* $\mathfrak{m}^9 = (0)$ *but* $\mathfrak{m}^8 \neq (0)$*. Then* $H(R)$ *is planar.*

Proof. Note that $V(H(R)) = \{v_1 = \mathfrak{m}, v_2 = \mathfrak{m}^6, v_3 = \mathfrak{m}^2, v_4 = \mathfrak{m}^5, v_5 = \mathfrak{m}^3, v_6 = \mathfrak{m}^4, v_7 = \mathfrak{m}^7, v_8 = \mathfrak{m}^8\}$. Observe that $H(R)$ is the union of the cycle $\Gamma : v_1 - v_2 - v_3 - v_4 - v_5 - v_6 - v_1$, the edges $e_1 : v_1 - v_3, e_2 : v_1 - v_4, e_3 : v_1 - v_5, e_4 : v_3 - v_5, e_5 : v_3 - v_6, e_6 : v_1 - v_7$, and the isolated vertex v_8. Observe that Γ can be represented by means of a hexagon. The edges e_1, e_2, e_3 are chords of this hexagon through v_1 and they can be drawn inside the hexagon without any crossing over of the edges. The edges e_4, e_5 are chords of this hexagon through v_3. The edge e_6 joins v_1 with the pendant vertex v_7. The edges e_4, e_5, and e_6 can be drawn outside the hexagon representing Γ in such a way that there are no crossing over of the edges. This proves that $H(R)$ is planar. □

Lemma 3.2. *Let* $(T, \mathfrak{n})$ *be a* SPIR *with* $\mathfrak{n}^2 \neq (0)$ *but* $\mathfrak{n}^9 = (0)$*. Then* $H(T)$ *is planar.*

Proof. If $\mathfrak{n}^8 \neq (0)$, then it follows from Lemma 3.1 that $H(T)$ is planar. Hence, we can assume that $\mathfrak{n}^8 = (0)$. By hypothesis, $\mathfrak{n}^2 \neq (0)$. Let $k \geqslant 2$ be largest with the property that $\mathfrak{n}^k \neq (0)$. Then $k \leqslant 7$. Let us denote the ring $\frac{K[X]}{X^9K[X]}$ by R, where $K[X]$ is the polynomial ring in one variable X over a field K. It is clear that $(R, \mathfrak{m} = \frac{XK[X]}{X^9K[X]})$ is a SPIR with $\mathfrak{m}^9 = (0 + X^9K[X])$ but $\mathfrak{m}^8 \neq (0 + X^9K[X])$. Note that $V(H(T)) = \{\mathfrak{n}^i | i \in \{1, 2, \ldots, k\}\}$ and $V(H(R)) = \{\mathfrak{m}^j | j \in \{1, 2, \ldots, 8\}\}$ and the mapping $f : V(H(T)) \to V(H(R))$ defined by $f(\mathfrak{n}^i) = \mathfrak{m}^i$ is a one-one mapping such that $\mathfrak{n}^i$, $\mathfrak{n}^{i'}$ are adjacent in $H(T)$ implies that $f(\mathfrak{n}^i)$, $f(\mathfrak{n}^{i'})$ are adjacent in $H(R)$. Consider the subgraph g of $H(R)$ induced by $\{f(\mathfrak{n}^i) | i \in \{1, 2, \ldots, k\}\}$. The above arguments imply that $H(T)$ can be identified with a subgraph of g. We know from Lemma 3.1 that $H(R)$ is planar. Since a subgraph of a planar graph is planar, it follows that $H(T)$ is planar. □

Theorem 3.3. *Let* $(R, \mathfrak{m})$ *be a* SPIR *such that* $\mathfrak{m}^2 \neq (0)$*. The following statements are equivalent:*

(i) $H(R)$ *satisfies* (C_1)*.*
(ii) $\mathfrak{m}^9 = (0)$*.*
(iii) $H(R)$ *is planar.*
(iv) $H(R)$ *satisfies* (C_2)*.*
(v) $H(R)$ *satisfies both* (C_1^*) *and* (C_2^*)*.*

Proof. (i) ⇒ (ii) and (iv) ⇒ (ii). We know from Lemma 2.2 that $\mathfrak{m}^9 = (0)$.

(ii) ⇒ (iii). This follows from Lemma 3.2.

(iii) ⇒ (v). This follows from Kuratowski's theorem [9, Theorem 5.9].

The statements (v) ⇒ (i) and (v) ⇒ (iv) are clear. □

4. When is $H(R)$ planar if $(R, \mathfrak{m})$ is a local Artinian ring such that $\mathfrak{m}^2 \neq (0)$ and $\mathfrak{m}$ is not principal?

In this section, we focus on Artinian local rings $(R, \mathfrak{m})$ with $\mathfrak{m}^2 \neq (0)$, $\mathfrak{m}$ is not principal, and try to characterize them such that $H(R)$ is planar. If $H(R)$ is planar, then we know from Remark 2.8 that there exist $a, b \in \mathfrak{m}$ such that $\mathfrak{m} = Ra + Rb$. First, it is useful to have the following Remark.

Remark 4.1. Let $(R, \mathfrak{m})$ be a local Artinian ring. We know from [2, Proposition 8.4] that $\mathfrak{m}$ is nilpotent and so, $\mathbb{I}(R)^* = \mathbb{A}(R)^*$. Hence, as is noted in the introduction, we obtain that $H(R) = (\mathbb{AG}(R))^c$ and so,

$\omega(H(R)) = \omega((\mathbb{AG}(R))^c) = \alpha(\mathbb{AG}(R))$. Observe that $H(R)$ satisfies (C_1) if and only if $\omega(H(R)) \leqslant 4$ if and only if $\alpha(\mathbb{AG}(R)) \leqslant 4$. Hence, we use the results from [13] in determining R such that $H(R)$ satisfies (C_1).

For the sake of convenience, we discuss the planarity of $H(R)$ with the help of several cases.

Case (1): $a^2 = b^2 = 0$ but $ab \neq 0$

Lemma 4.2. *Let $(R, \mathfrak{m})$ be a local Artinian ring such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ with $a^2 = b^2 = 0$ but $ab \neq 0$. If $H(R)$ satisfies either (C_1) or (C_2), then $|\frac{R}{\mathfrak{m}}| \leqslant 3$.*

Proof. By hypothesis, $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$. Therefore, it follows that $\{a + \mathfrak{m}^2, b + \mathfrak{m}^2\}$ is a basis of $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ as a vector space over $\frac{R}{\mathfrak{m}}$.

Suppose that $|\frac{R}{\mathfrak{m}}| \geqslant 4$. Then either $2 \in \mathfrak{m}$ or $2 \notin \mathfrak{m}$. If $2 \in \mathfrak{m}$, then $1 + \mathfrak{m} = -1 + \mathfrak{m}$. If $2 \notin \mathfrak{m}$, then $|\frac{R}{\mathfrak{m}}| \geqslant 5$. Thus in any case, it is possible to find $r, s \in R\backslash\mathfrak{m}$ such that $r \pm 1, s \pm 1, r - s \in R\backslash\mathfrak{m}$. As $ab \neq 0$, it follows that $(a+b)(a-rb) = (1-r)ab \neq 0$ and $(a+b)(a-sb) = (1-s)ab \neq 0$. Observe that the subgraph of $H(R)$ induced by $\{Ra, Rb, R(a+b), R(a-rb), \mathfrak{m}\}$ is a clique on five vertices. Hence, $H(R)$ does not satisfy (C_1). Let $A = \{Ra, Rb, R(a+b)\}$ and let $B = \{R(a-rb), R(a-sb), \mathfrak{m}\}$. It is clear that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This implies that $H(R)$ does not satisfy (C_2). Thus if $H(R)$ satisfies either (C_1) or (C_2), then $|\frac{R}{\mathfrak{m}}| \leqslant 3$. □

Lemma 4.3. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.2. If $H(R)$ satisfies either (C_1) or (C_2), then the following hold.*

(i) $|\frac{R}{\mathfrak{m}}| \in \{2, 3\}$ *and* $|R| \in \{16, 81\}$.

(ii) $V(H(R)) = \{Ra, Rb, R(a+b), Rab, \mathfrak{m}\}$ *in the case* $|\frac{R}{\mathfrak{m}}| = 2$ *and* $H(R)$ *is planar.*

(iii) $V(H(R)) = \{Ra, Rb, R(a+b).R(a+2b).Rab, \mathfrak{m}\}$ *in the case* $|\frac{R}{\mathfrak{m}}| = 3$ *and* $H(R)$ *is planar.*

Proof. Note that $V(H(R)) = \mathbb{I}(R)^*$. Assume that $H(R)$ satisfies either (C_1) or (C_2). Then we know from Lemma 4.2 that $|\frac{R}{\mathfrak{m}}| \leqslant 3$.

(i) As $|\frac{R}{\mathfrak{m}}| \leqslant 3$, it follows that $|\frac{R}{\mathfrak{m}}| \in \{2, 3\}$. It was shown in the proof of (ii) $\Rightarrow$ (i) of [13, Lemma 4.4] that $|R| \in \{16, 81\}$.

(ii) Suppose that $|\frac{R}{\mathfrak{m}}| = 2$. It was verified in the proof of (ii) $\Rightarrow$ (i) of [13, Lemma 4.4] that $V(H(R)) = \{Ra, Rb, R(a+b), Rab, \mathfrak{m}\}$. Observe that $\mathfrak{m}^2 = Rab$ and $\mathfrak{m}^3 = (0)$. Hence, Rab is an isolated vertex of $H(R)$. It is clear that the subgraph of $H(R)$ induced by $\{Ra, Rb, R(a+b), \mathfrak{m}\}$

is a clique on four vertices. Since K_4 is planar, it follows that $H(R)$ is planar.

(iii) Suppose that $|\frac{R}{\mathfrak{m}}| = 3$. We know from the proof of (ii) $\Rightarrow$ (i) of [13, Lemma 4.4] that $V(H(R)) = \{v_1 = Ra, v_2 = Rb, v_3 = R(a+b), v_4 = \mathfrak{m}, v_5 = R(a+2b), v_6 = Rab\}$. As $\mathfrak{m}^2 = Rab$ and $\mathfrak{m}^3 = (0)$, it follows that Rab is an isolated vertex of $H(R)$. It is not hard to verify that $H(R)$ is the union of the cycle $\Gamma : v_1 - v_2 - v_3 - v_4 - v_5 - v_1$, the edges $e_1 : v_1 - v_3, e_2 : v_1 - v_4, e_3 : v_2 - v_4, e_4 : v_2 - v_5$, and the isolated vertex v_6. Observe that Γ can be represented by means of a pentagon. The edges e_1, e_2 are chords of this pentagon through v_1 and they can be drawn inside this pentagon. The edges e_3, e_4 are chords of this pentagon through v_2 and they can be drawn outside this pentagon in such a way that there are no crossing over of the edges. This proves that $H(R)$ is planar. □

Theorem 4.4. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.2. The following statements are equivalent:*

(i) *$H(R)$ satisfies (C_1).*
(ii) *$|\frac{R}{\mathfrak{m}}| \in \{2, 3\}$ and $|R| \in \{16, 81\}$.*
(iii) *$H(R)$ is planar.*
(iv) *$H(R)$ satisfies (C_2).*
(v) *$H(R)$ satisfies both (C_1^*) and (C_2^*).*

Proof. The statements (i) $\Rightarrow$ (ii) and $(iv) \Rightarrow (ii)$ follow from Lemma 4.3(i).

(ii) $\Rightarrow$ (iii). If $|\frac{R}{\mathfrak{m}}| = 2$, then we know from Lemma 4.3(ii) that $H(R)$ is planar. If $|\frac{R}{\mathfrak{m}}| = 3$, then we know from Lemma 4.3(iii) that $H(R)$ is planar.

(iii) $\Rightarrow$ (v). This follows from Kuratowski's theorem [9, Theorem 5.9].

The statements (v) $\Rightarrow$ (i) and (v) $\Rightarrow$ (iv) are clear. □

With the help of results from [5, 7, 8], we mention in Example 4.5, finite local rings $(R, \mathfrak{m})$ such that each one of them satisfies the hypotheses of Theorem 4.4 and the statement (ii) of Theorem 4.4. For any ring S, we denote the polynomial ring in one variable X (respectively, in two variables X, Y) over S by $S[X]$ (respectively, by $S[X, Y]$). For any prime number p and $n \geqslant 1$, we denote the finite field containing exactly p^n elements by $\mathbb{F}_{p^n}$. For any $n \geqslant 2$, we denote the ring of integers modulo n by $\mathbb{Z}_n$.

Example 4.5.

(i) $T = \mathbb{F}_2[X, Y]$, $I = TX^2 + TY^2$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;
(ii) $T = \mathbb{Z}_4[X, Y], I = TX^2 + T(XY - 2) + TY^2$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;
(iii) $T = \mathbb{Z}_4[X], I = TX^2$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{T2+TX}{I})$;
(iv) $T = \mathbb{F}_3[X, Y], I = TX^2 + TY^2$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;

(v) $T = \mathbb{Z}_9[X, Y], I = TX^2 + T(XY - 3) + TY^2$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;
(vi) $T = \mathbb{Z}_9[X], I = TX^2$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{T3+TX}{I})$;
(vii) $T = \mathbb{Z}_9[X], I = T(X^2 - 3X)$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{T3+T(X+3)}{I})$;
(viii) $T = \mathbb{Z}_9[X], I = T(X^2 + 3X)$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{T3+T(X-3)}{I})$.

Case (2): $a^2 \neq 0$ but $b^2 = ab = 0$

Lemma 4.6. *Let $(R, \mathfrak{m})$ be a local Artinian ring such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ with $a^2 \neq 0$ but $b^2 = ab = 0$. If $H(R)$ satisfies (C_1), then $|\frac{R}{\mathfrak{m}}| \leqslant 3$.*

Proof. Assume that $H(R)$ satisfies (C_1). That is, $\omega(H(R)) \leqslant 4$. It is already noted in Remark 4.1 that $\omega(H(R)) = \alpha(\mathbb{AG}(R))$. Thus $\alpha(\mathbb{AG}(R)) \leqslant 4$ and so, we obtain from [13, Lemma 4.8] that $|\frac{R}{\mathfrak{m}}| \leqslant 3$. □

Lemma 4.7. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.6. If $H(R)$ satisfies (C_2), then $\mathfrak{m}^3 = (0)$.*

Proof. Assume that $H(R)$ satisfies (C_2). Suppose that $\mathfrak{m}^3 \neq (0)$. It is clear from the hypotheses on a, b that $\mathfrak{m}^2 = Ra^2$ and $\mathfrak{m}^3 = Ra^3$. Hence, $a^3 \neq 0$. Let $A = \{Ra, R(a + b), \mathfrak{m}\}$ and $B = \{Ra^2, R(a^2 + b), Ra^2 + Rb\}$. Observe that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2). Therefore, we obtain that $\mathfrak{m}^3 = (0)$. □

Lemma 4.8. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.6. Suppose that $\mathfrak{m}^3 = (0)$. If $|\frac{R}{\mathfrak{m}}| = 2$, then $H(R)$ is planar.*

Proof. We know from the proof of [13, Lemma 3.11] that $V(H(R)) = \{Ra, Rb, R(a+b), R(a^2+b), Ra^2, Ra^2+Rb, \mathfrak{m}\}$. Since $b\mathfrak{m} = (0)$ and $\mathfrak{m}^3 = (0)$, it follows that each member from $W = \{Rb, R(a^2+b), Ra^2, Ra^2+Rb\}$ is an isolated vertex of $H(R)$. It is clear that $H(R)$ is the union of the cycle $\Gamma : Ra - R(a+b) - \mathfrak{m} - Ra$ of length 3 and W. Therefore, we obtain that $H(R)$ is planar. □

Lemma 4.9. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.6. Suppose that $\mathfrak{m}^3 = (0)$. If $|\frac{R}{\mathfrak{m}}| = 3$, then $H(R)$ is planar.*

Proof. Note that $\frac{R}{\mathfrak{m}} = \{0 + \mathfrak{m}, 1 + \mathfrak{m}, 2 + \mathfrak{m}\}$ and $\mathfrak{m}^2 = \{0, a^2, 2a^2\}$. Since $\dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = 2$, we get that $|\frac{\mathfrak{m}}{\mathfrak{m}^2}| = 9$. Therefore, $|\mathfrak{m}| = 27$. Let $A = \{0, 1, 2\}$. Observe that $\mathfrak{m} = \{xa + yb + za^2 | x, y, z \in A\}$. Let $I \in \mathbb{I}(R)^*$. If $I \subseteq \mathfrak{m}^2$, then it is clear that $I = \mathfrak{m}^2$. Suppose that $I \not\subseteq \mathfrak{m}^2$. Then there exists $m \in I \setminus \mathfrak{m}^2$. It is clear that $m = xa + yb + za^2$ for some $x, y, z \in A$ with at least one between x, y is nonzero. If $x \in \{1, 2\}$, then from $b\mathfrak{m} = (0)$, it follows that $am = xa^2 \in I$ and so, $a^2 \in I$. In such a case, $Ra^2 = \mathfrak{m}^2 \subset I$. Hence, $\dim_{\frac{R}{\mathfrak{m}}}(\frac{I}{\mathfrak{m}^2}) = 1$ or 2. If $\dim_{\frac{R}{\mathfrak{m}}}(\frac{I}{\mathfrak{m}^2}) = 2$, then $I = \mathfrak{m}$. If $\dim_{\frac{R}{\mathfrak{m}}}(\frac{I}{\mathfrak{m}^2}) = 1$, then $I = Rm = R(a+x^{-1}yb+x^{-1}za^2) = R(a+x^{-1}yb+x^{-1}za(a+x^{-1}yb))$. Since $1 + x^{-1}za \in U(R)$, it follows that $I = R(a + x^{-1}yb)$. Hence, in this case, we obtain that $I \in \{Ra, R(a+b), R(a+2b)\}$. If $x = 0$, then $y \in \{1, 2\}$. Therefore, $m = yb + za^2 = y(b + y^{-1}za^2)$ and so, $Rm = R(b + y^{-1}za^2)$. Let us denote Rm by C. Since $\mathfrak{m} = Ra+Rm$, it follows that $\frac{\mathfrak{m}}{C} = \frac{R}{C}(a+C)$ is principal and it is clear that $(\frac{\mathfrak{m}}{C})^3 = (0 + C)$. Therefore, it follows from the proof of (iii) $\Rightarrow$ (i) of [2, Proposition 8.8] that $\mathbb{I}(\frac{R}{C})^* = \{\frac{\mathfrak{m}}{C}, (\frac{\mathfrak{m}}{C})^2\}$. Since $\mathfrak{m} \supseteq I \supseteq C$, it follows that $I \in \{C, \mathfrak{m}^2 + C, \mathfrak{m}\}$. Therefore, we get that $I \in \{Rb, R(b+a^2), R(b+2a^2), Rb+Ra^2, \mathfrak{m}\}$. It is now clear from the above given arguments that $V(H(R)) = \{v_1 = Ra, v_2 = R(a+b), v_3 = R(a+2b), v_4 = \mathfrak{m}, v_5 = Rb, v_6 = R(a^2+b), v_7 = R(2a^2+b), v_8 = Ra^2, v_9 = Ra^2 + Rb\}$. Since $b\mathfrak{m} = (0)$ and $\mathfrak{m}^3 = (0)$, it is clear that each vertex from $\{v_5, v_6, v_7, v_8, v_9\}$ is an isolated vertex of $H(R)$. Observe that the subgraph g of $H(R)$ induced by $\{v_1, v_2, v_3, v_4\}$ is a clique on four vertices. Therefore, we get that $H(R)$ is the union of g and the set of all isolated vertices of $H(R)$. As K_4 is planar, we obtain that $H(R)$ is planar. □

Theorem 4.10. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.6. The following statements are equivalent:*

(i) *$H(R)$ satisfies both (C_1) and (C_2).*
(ii) *$|\frac{R}{\mathfrak{m}}| \leqslant 3$ and $\mathfrak{m}^3 = (0)$.*
(iii) *$H(R)$ is planar.*
(iv) *$H(R)$ satisfies both (C_1^*) and (C_2^*).*

Proof. (i) $\Rightarrow$ (ii) Assume that $H(R)$ satisfies both (C_1) and (C_2). Then we obtain from Lemmas 4.6 and 4.7 that $|\frac{R}{\mathfrak{m}}| \leqslant 3$ and $\mathfrak{m}^3 = (0)$.

(ii) $\Rightarrow$ (iii) Assume that $|\frac{R}{\mathfrak{m}}| \leqslant 3$ and $\mathfrak{m}^3 = (0)$. If $|\frac{R}{\mathfrak{m}}| = 2$, then we obtain from Lemma 4.8 that $H(R)$ is planar. If $|\frac{R}{\mathfrak{m}}| = 3$, then we obtain from Lemma 4.9 that $H(R)$ is planar.

(iii) $\Rightarrow$ (iv) This follows from Kuratowski's theorem [9, Theorem 5.9].

(iv) $\Rightarrow$ (i) This is clear. □

Remark 4.11. Let $(R, \mathfrak{m})$ be a local Artinian ring satisfying the hypotheses of Lemma 4.6 and the statement (ii) of Theorem 4.10. Note that $|\mathfrak{m}^2| = |\frac{R}{\mathfrak{m}}|$, $|\frac{\mathfrak{m}}{\mathfrak{m}^2}| = (|\frac{R}{\mathfrak{m}}|)^2$, $|\mathfrak{m}| = (|\frac{R}{\mathfrak{m}}|)^3$, and $|R| = (|\frac{R}{\mathfrak{m}}|)^4$. Hence, $|R| = 16$ if $|\frac{R}{\mathfrak{m}}| = 2$ and $|R| = 81$ if $|\frac{R}{\mathfrak{m}}| = 3$. With the help of the work presented in [5, 7, 8], in Example 4.12, we mention examples of local Artinian rings

$(R, \mathfrak{m})$ such that $(R, \mathfrak{m})$ satisfies the hypotheses of Lemma 4.6 and the statement(ii) of Theorem 4.10.

Example 4.12.
(i) $T = \mathbb{F}_2[X, Y]$, $I = TX^3 + TXY + TY^2$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;
(ii) $T = \mathbb{Z}_4[X, Y], I = T(X^2 - 2) + TXY + TY^2 + T(2X)$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;
(iii) $T = \mathbb{Z}_4[X], I = T(2X) + TX^3$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+T2}{I})$;
(iv) $T = \mathbb{Z}_8[X], I = T(2X) + TX^2$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{T2+TX}{I})$;
(v) $T = \mathbb{F}_3[X, Y], I = TX^3 + TXY + TY^2$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;
(vi) $T = \mathbb{Z}_9[X, Y], I = T(X^2 - 3) + TXY + TY^2 + T(3X)$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;
(vii) $T = \mathbb{Z}_9[X], I = T(3X) + TX^3$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+T3}{I})$;
(viii) $T = \mathbb{Z}_{27}[X], I = T(3X) + TX^2$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{T3+TX}{I})$.

Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Theorem 4.4. Then it is shown in Theorem 4.4 that $H(R)$ satisfies (C_1) if and only if $H(R)$ satisfies (C_2). We provide some examples to illustrate that for a local Artinian ring $(R, \mathfrak{m})$ which satisfies the hypotheses of Lemma 4.6, the statement $H(R)$ satisfies (C_1) and the statement $H(R)$ satisfies (C_2) can happen to be not equivalent.

Lemma 4.13. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.6. Suppose that $\mathfrak{m}^4 = (0)$, $\mathfrak{m}^3 \neq (0)$, and $|\frac{R}{\mathfrak{m}}| = 2$. Then the following hold.*
(i) $|R| = 32$.
(ii) $H(R)$ *satisfies* (C_1).
(iii) $H(R)$ *does not satisfy* (C_2).

Proof. Note that $\mathfrak{m} = Ra + Rb$, $a^2 \neq 0$ but $b^2 = ab = 0$, and so, $\mathfrak{m}^i = Ra^i$ for each $i \geqslant 2$. By hypothesis, $a^3 \neq 0$, $a^4 = 0$, and $|\frac{R}{\mathfrak{m}}| = 2$.

(i) It follows from $|\mathfrak{m}^3| = 2, |\frac{\mathfrak{m}^2}{\mathfrak{m}^3}| = 2$, $|\frac{\mathfrak{m}}{\mathfrak{m}^2}| = 4$ that $|\mathfrak{m}| = 16$ and so, $|R| = 32$.

(ii) Let $A = \{0, 1\}$. Note that $\mathfrak{m} = \{xa+yb+za^2+wa^3 | x, y, z, w \in A\}$. It is not hard to verify that $V(H(R)) = \{Ra, Rb, R(a + b), Ra^2, R(a^2 + b), Ra^2 + Rb, Ra^3, R(a^3 + b), Ra^3 + Rb, \mathfrak{m}\}$. It follows from $a^3 \neq 0$ and $b\mathfrak{m} = (0)$ that the subgraph of $H(R)$ induced by $\{Ra, R(a+b), R(a^2+b), \mathfrak{m}\}$ is a clique. Therefore, $\omega(H(R)) \geqslant 4$. Observe that each member from $W = \{Rb, Ra^3, R(a^3+b), Ra^3+Rb\}$ is an isolated vertex of $H(R)$. Let $U \subseteq V(H(R))$ be such that the subgraph of $H(R)$ induced by U is a clique. It is clear that $U \subseteq V(H(R)) \backslash W = \{Ra, R(a+b), Ra^2, R(a^2+b), Ra^2+Rb, \mathfrak{m}\}$. It follows from $a^4 = 0$ and $b\mathfrak{m} = (0)$ that at most one vertex from

$\{Ra^2, R(a^2+b), Ra^2+Rb\}$ can belong to U. Therefore, we get that $|U| \leqslant 4$. This shows that $\omega(H(R)) \leqslant 4$ and so, $H(R)$ satisfies (C_1). Indeed, $\omega(H(R)) = 4$.

(iii) As $\mathfrak{m}^3 \neq (0)$, we obtain from Lemma 4.7 that $H(R)$ does not satisfy (C_2). □

In Example 4.14, we provide from [5, page 476], an example of a local Artinian ring $(R, \mathfrak{m})$ which satisfies the hypotheses of Lemma 4.6 and is such that $H(R)$ satisfies (C_1) but it does not satisfy (C_2).

Example 4.14. Let $T = \mathbb{F}_2[X, Y]$ and $I = TX^4 + TXY + TY^2$. Observe that $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$ is a local Artinian ring and it satisfies the hypotheses of Lemma 4.6 with $a = X + I$ and $b = Y + I$. Moreover, note that $\mathfrak{m}^3 \neq (0)$, $\mathfrak{m}^4 = (0)$, and $|\frac{R}{\mathfrak{m}}| = 2$. Hence, we obtain from Lemma 4.13 that $H(R)$ satisfies (C_1) but it does not satisfy (C_2).

Lemma 4.15. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.6. Suppose that $\mathfrak{m}^3 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 4$. Then $H(R)$ satisfies (C_2) but it does not satisfy (C_1).*

Proof. Note that there exist $r, s \in R \setminus \mathfrak{m}$ such that $\frac{R}{\mathfrak{m}} = \{0 + \mathfrak{m}, 1 + \mathfrak{m}, r + \mathfrak{m}, s + \mathfrak{m}\}$. Observe that $\mathfrak{m}^2 = Ra^2$, $|\mathfrak{m}^2| = 4$, $|\frac{\mathfrak{m}}{\mathfrak{m}^2}| = 16$, and $|\mathfrak{m}| = 64$. Let $A = \{0, 1, r, s\}$. Note that $\mathfrak{m} = \{xa + yb + za^2 | x, y, z \in A\}$. It can be shown that $V(H(R)) = \{Ra, Rb, R(a+b), R(a+rb), R(a+sb), Ra^2, R(a^2+b), R(ra^2+b), R(sa^2+b), Ra^2+Rb, \mathfrak{m}\}$. Since $b\mathfrak{m} = (0)$ and $\mathfrak{m}^3 = (0)$, it follows that each vertex from $W = \{Rb, Ra^2, R(a^2+b), R(ra^2+b), R(sa^2+b), Ra^2+Rb\}$ is an isolated vertex of $H(R)$. It follows from $a^2 \neq 0$ that the subgraph of $H(R)$ induced by $\{Ra, R(a+b), R(a+rb), R(a+sb), \mathfrak{m}\}$ is a clique on five vertices. Observe that $H(R)$ is the union of a clique on five vertices and W. Therefore, we get that $H(R)$ satisfies (C_2) but it does not satisfy (C_1). □

Example 4.16. Let $T = \mathbb{F}_4[X, Y]$ and $I = TX^3 + TXY + TY^2$. Observe that $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$ is a local Artinian ring which satisfies the hypotheses of Lemma 4.6 with $a = X + I$ and $b = Y + I$. Moreover, $\mathfrak{m}^3 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 4$. Therefore, we obtain from Lemma 4.15 that $H(R)$ satisfies (C_2) but it does not satisfy (C_1).

Case (3): $a^2 \neq 0$, $b^2 \neq 0$, whereas $ab = 0$

Lemma 4.17. *Let $(R, \mathfrak{m})$ be a local Artinian ring such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ with $a^2 \neq 0, b^2 \neq (0)$, whereas $ab = 0$. If $H(R)$ satisfies (C_1), then Ra^2 and Rb^2 are comparable under the inclusion relation.*

Proof. Assume that $H(R)$ satisfies (C_1). That is, $\omega(H(R)) \leqslant 4$. It is noted in Remark 4.1 that $\omega(H(R)) = \alpha(\mathbb{AG}(R))$. Hence, $\alpha(\mathbb{AG}(R)) \leqslant 4$ and therefore, we obtain from [13, Lemma 4.12] that Ra^2 and Rb^2 are comparable under the inclusion relation. □

Lemma 4.18. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.17. If $H(R)$ satisfies (C_2), then $\mathfrak{m}^3 = (0)$ and $|\frac{R}{\mathfrak{m}}| \leqslant 3$.*

Proof. Assume that $H(R)$ satisfies (C_2). Now, $\mathfrak{m} = Ra + Rb$ and from $ab = 0$, it follows that $\mathfrak{m}^2 = Ra^2 + Rb^2$, and $\mathfrak{m}^3 = Ra^3 + Rb^3$. First, we show that $a^3 = 0$. Suppose that $a^3 \neq 0$. Then $Ra^2 \not\subseteq Rb$. Let $A = \{Ra, Rb, \mathfrak{m}\}$ and let $B = \{R(a + b), R(a^2 + b), Ra^2 + Rb\}$. Note that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2). Therefore, $a^3 = 0$ and similarly, it can be shown that $b^3 = 0$. Hence, we obtain that $\mathfrak{m}^3 = (0)$.

We next verify that $|\frac{R}{\mathfrak{m}}| \leqslant 3$. Suppose that $|\frac{R}{\mathfrak{m}}| > 3$. Then it is possible to find $r, s \in R \backslash \mathfrak{m}$ such that $r - 1, s - 1, r - s \in R \backslash \mathfrak{m}$. Let $A = \{Ra, Rb, \mathfrak{m}\}$ and let $B = \{R(a + b), R(a + rb), R(a + sb)\}$. Note that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2). Therefore, we obtain that $|\frac{R}{\mathfrak{m}}| \leqslant 3$. □

Lemma 4.19. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypothesis of Lemma 4.17. If $H(R)$ satisfies both (C_1) and (C_2), then $Ra^2 = Rb^2$.*

Proof. Assume that $H(R)$ satisfies both (C_1) and (C_2). We know from Lemma 4.17 that either $Ra^2 \subseteq Rb^2$ or $Rb^2 \subseteq Ra^2$ and from Lemma 4.18, we know that $\mathfrak{m}^3 = (0)$. Without loss of generality, we can assume that $Rb^2 \subseteq Ra^2$. Then $b^2 = ra^2$ for some $r \in R$. As $b^2 \neq 0$ and $\mathfrak{m}^3 = (0)$, it follows that $r \in U(R)$ and so, $a^2 = r^{-1}b^2$. This implies that $Ra^2 \subseteq Rb^2$ and hence, we obtain that $Ra^2 = Rb^2$. □

Lemma 4.20. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.17. If $\mathfrak{m}^2 = Ra^2 = Rb^2$ and $|\frac{R}{\mathfrak{m}}| = 2$, then $|R| = 16$ and $H(R)$ is planar.*

Proof. From $ab = 0$ and $\mathfrak{m}^2 = Ra^2 = Rb^2$, it follows that $\mathfrak{m}^3 = (0)$. As $|\frac{R}{\mathfrak{m}}| = 2$, we obtain that $|\mathfrak{m}^2| = 2$, $|\frac{\mathfrak{m}}{\mathfrak{m}^2}| = 4$, and hence, $|\mathfrak{m}| = 8$. It is now clear that $|R| = 16$. Let $A = \{0, 1\}$. Observe that $\mathfrak{m} = \{xa + yb + za^2 | x, y, z \in A\}$. It is not hard to verify that $V(H(R)) = \{v_1 = Ra, v_2 =$

$R(a+b), v_3 = Rb, v_4 = \mathfrak{m}, v_5 = Ra^2\}$. From $\mathfrak{m}^3 = (0)$, it follows that v_5 is an isolated vertex of $H(R)$. Note that $H(R)$ is the union of the cycle $\Gamma : v_1 - v_2 - v_3 - v_4 - v_1$, the edge $e_1 : v_2 - v_4$, and the isolated vertex v_5. The cycle Γ can be represented by means of a rectangle, the edge e_1 is a diagonal of the rectangle representing Γ. It is now clear that $H(R)$ is planar. □

Lemma 4.21. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.17. If $\mathfrak{m}^2 = Ra^2 = Rb^2$ and $|\frac{R}{\mathfrak{m}}| = 3$, then $|R| = 81$ and $H(R)$ is planar.*

Proof. It follows as in the proof of Lemma 4.20 that $\mathfrak{m}^3 = (0)$. From the assumption that $|\frac{R}{\mathfrak{m}}| = 3$, we get that $|\mathfrak{m}^2| = 3$, $|\frac{\mathfrak{m}}{\mathfrak{m}^2}| = 9$, $|\mathfrak{m}| = 27$, and so, $|R| = 81$. Let $A = \{0, 1, 2\}$. Observe that $\mathfrak{m} = \{xa + yb + za^2 | x, y, z \in A\}$. It is not hard to verify that $V(H(R)) = \{v_1 = Ra, v_2 = R(a+b), v_3 = Rb, v_4 = R(a+2b), v_5 = \mathfrak{m}, v_6 = Ra^2\}$. It is clear that $a^2 \in \{b^2, 2b^2\}$, and v_6 is an isolated vertex of $H(R)$. Note that $H(R)$ is the union of the cycle $\Gamma : v_1 - v_2 - v_3 - v_4 - v_1$, the edges $e_i : v_i - v_5$ for each $i \in \{1, 2, 3, 4\}$, the edge $e_5 : v_2 - v_4$ in the case $a^2 = 2b^2$, and the isolated vertex v_6. The cycle Γ can be represented by means of a rectangle and the vertex v_5 can be plotted inside this rectangle and the edges e_i for $i \in \{1, 2, 3, 4\}$ can be drawn inside the rectangle representing Γ in such a way that there are no crossing over of the edges and the edge e_5 if it exists can be drawn outside the rectangle representing Γ. This shows that $H(R)$ is planar. □

Theorem 4.22. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.17. The following statements are equivalent:*

(i) *$H(R)$ satisfies both (C_1) and (C_2).*
(ii) *$\mathfrak{m}^2 = Ra^2 = Rb^2$ and $|\frac{R}{\mathfrak{m}}| \leqslant 3$.*
(iii) *$H(R)$ is planar.*
(iv) *$H(R)$ satisfies both (C_1^*) and (C_2^*).*

Proof. (i) ⇒ (ii) We know from Lemma 4.18 that $|\frac{R}{\mathfrak{m}}| \leqslant 3$ and from Lemma 4.19, we know that $Ra^2 = Rb^2$. It follows from $ab = 0$ that $\mathfrak{m}^2 = Ra^2$.

(ii) ⇒ *(iii)* Note that $|\frac{R}{\mathfrak{m}}| \in \{2, 3\}$. Therefore, we obtain from Lemmas 4.20 and 4.21 that $H(R)$ is planar.

(iii) ⇒ (iv) This follows from Kuratowski's theorem [9, Theorem 5.9].

(iv) ⇒ (i) This is clear. □

With the help of results from [5, 7, 8], in Example 4.23, we provide examples of local Artinian rings $(R, \mathfrak{m})$ such that $(R, \mathfrak{m})$ satisfies the hypotheses of Lemma 4.17 and the statement (ii) of Theorem 4.22.

Example 4.23.

(i) $K \in \{\mathbb{F}_2, \mathbb{F}_3\}$. Let $T = K[X,Y], I = T(X^2 - Y^2) + TXY$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;

(ii) $T = \mathbb{Z}_4[X,Y], I = T(X^2 - 2) + TXY + T(Y^2 - 2) + T(2X)$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;

(iii) $T = \mathbb{Z}_9[X,Y], I = T(X^2 - 3) + TXY + T(Y^2 - 3) + T(3X)$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+TY}{I})$;

(iv) $T = \mathbb{Z}_4[X], I = T(X^2 - 2X)$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+T(X-2)}{I})$;

(v) $T = \mathbb{Z}_9[X,Y], I = T(X^2 - 3X)$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+T(X-3)}{I})$;

(vi) $T = \mathbb{Z}_8[X], I = T(2X) + T(X^2 - 4)$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+T2}{I})$;

(vii) $T = \mathbb{Z}_{27}[X], I = T(3X) + T(X^2 - 9)$, and $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+T3}{I})$.

In Example 4.24, we provide an example from [5, page 478] of a local Artinian ring $(R, \mathfrak{m})$ which satisfies the hypotheses of Lemma 4.17 and is such that $H(R)$ satisfies (C_1) but it does not satisfy (C_2).

Example 4.24. Let $T = \mathbb{Z}_8[X]$ and $I = T(2X) + T(X^3 - 4)$. Let $R = \frac{T}{I}$ and $\mathfrak{m} = \frac{TX+T2}{I}$. Then $(R, \mathfrak{m})$ is a local Artinian ring which satisfies the hypotheses of Lemma 4.17 and is such that $H(R)$ satisfies (C_1) but $H(R)$ does not satisfy (C_2).

Proof. Observe that $\mathfrak{m} = Ra + Rb$, where $a = X + I$ and $b = 2 + I$ and $\mathfrak{m}$ is not principal. Note that $a^2 \neq 0 + I, b^2 \neq 0 + I, ab = 0 + I$, and $a^3 = 4 + I \neq 0 + I$ and $\mathfrak{m}^4 = (0 + I)$. This shows that $(R, \mathfrak{m})$ is a local Artinian ring and it satisfies the hypotheses of Lemma 4.17. Observe that $Rb^2 \subset Ra^2, \mathfrak{m}^3 \neq (0 + I)$, and $|\frac{R}{\mathfrak{m}}| = 2$. Hence, it follows from (ii) $\Rightarrow$ (i) of [13, Proposition 4.13] that $\alpha(\mathbb{AG}(R)) = 4$ and so, $\omega(H(R)) = 4$. Therefore, we obtain that $H(R)$ satisfies (C_1). As $\mathfrak{m}^3 \neq (0 + I)$, it follows from Lemma 4.18 that $H(R)$ does not satisfy (C_2). □

We next proceed to give an example from [5, page 479] in Example 4.26 of a local Artinian ring $(R, \mathfrak{m})$ which satisfies the hypotheses of Lemma 4.17 and is such that $H(R)$ satisfies (C_2) but $H(R)$ does not satisfy (C_1). We use Lemma 4.25 in the verification of Example 4.26.

Lemma 4.25. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.17. Suppose that $\mathfrak{m}^2$ is not principal. The following statements are equivalent:*

(i) *$H(R)$ satisfies (C_2).*

(ii) *$\mathfrak{m}^3 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 2$.*

Proof. Observe that $\mathfrak{m}^2 = Ra^2 + Rb^2$. By hypothesis, $\mathfrak{m}^2$ is not principal. Hence, $Ra^2 \not\subseteq Rb^2$. We claim that $Ra^2 \not\subseteq Rb$. For if $Ra^2 \subseteq Rb$, then

$Ra^2 \subseteq \mathfrak{m}b$ and this implies that $Ra^2 \subseteq (Ra + Rb)b = Rb^2$. This is a contradiction and so, $Ra^2 \not\subseteq Rb$.

(i) $\Rightarrow$ (ii) Assume that $H(R)$ satisfies (C_2). We know from Lemma 4.18 that $\mathfrak{m}^3 = (0)$. Suppose that $|\frac{R}{\mathfrak{m}}| > 2$. Then it is possible to find $r \in R$ such that $r, r-1 \in R \setminus \mathfrak{m}$. Let $A = \{Rb, R(a^2+b), \mathfrak{m}\}$ and let $B = \{R(a+b), R(a+rb), Ra^2 + Rb\}$. Note that $A \cap B = \emptyset$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2). Therefore, we obtain that $|\frac{R}{\mathfrak{m}}| = 2$.

(ii) $\Rightarrow$ (i) Assume that $\mathfrak{m}^3 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 2$. Note that $|\mathfrak{m}^2| = 4, |\frac{\mathfrak{m}}{\mathfrak{m}^2}| = 4$, and so, $|\mathfrak{m}| = 16$. Let $A = \{0, 1\}$. It is clear that $\mathfrak{m} = \{xa + yb + za^2 + wb^2 | x, y, z, w \in A\}$. It can be shown as in the proof of Lemma 4.9 that $V(H(R)) = \{v_1 = Ra, v_2 = \mathfrak{m}, v_3 = R(a+b), v_4 = Ra + Rb^2, v_5 = R(a+b^2), v_6 = Rb, v_7 = Ra^2 + Rb, v_8 = R(a^2+b), v_9 = Ra^2, v_{10} = Rb^2, v_{11} = R(a^2+b^2), v_{12} = Ra^2 + Rb^2\}$. We next verify that $H(R)$ satisfies (C_2). Note that the subgraph of $H(R)$ induced by $\{v_1, v_2, v_3, v_4, v_5\}$ is a clique on five vertices and the subgraph of $H(R)$ induced by $\{v_2, v_3, v_6, v_7, v_8\}$ is a clique on five vertices. Hence, $H(R)$ does not satisfy (C_1). Suppose that $H(R)$ does not satisfy (C_2). Then it is possible to find subsets A_1, B_1 of $V(H(R))$ such that $|A_1| = |B_1| = 3$, $A_1 \cap B_1 = \emptyset$ and each vertex of A_1 is adjacent to each vertex of B_1 in $H(R)$. As $\mathfrak{m}^3 = (0)$, it follows that each vertex from $W = \{v_9, v_{10}, v_{11}, v_{12}\}$ is an isolated vertex of $H(R)$. Let $S = \{v_1, v_4, v_5\}$ and let $T = \{v_6, v_7, v_8\}$. Note that $v_i \in S$ is not adjacent to any vertex of T in $H(R)$ for each $i \in \{1, 4, 5\}$. Now, $A_1 \cup B_1 \subseteq S \cup T \cup \{v_2, v_3\}$. It is clear that at least one member of S must be in $A_1 \cup B_1$. Without loss of generality, we can assume that $v_1 \in A_1$. Then $B_1 \subseteq \{v_2, v_3, v_4, v_5\}$. Hence, at least one between v_4 and v_5 must be in B_1. Observe that $T \cap B_1 = \emptyset$. As $|T| = 3$, it follows at least one member of T must be in A_1. This is a contradiction since both v_4 and v_5 are not adjacent to any member of T in $H(R)$. This proves that $H(R)$ satisfies (C_2). □

Example 4.26. Let $T = \mathbb{Z}_8[X]$, $I = T(2X) + TX^3$, and $R = \frac{T}{I}$. Let $\mathfrak{m} = \frac{T2+TX}{I}$. Then $(R, \mathfrak{m})$ is a local Artinian ring which satisfies the hypotheses of Lemma 4.17 and is such that $H(R)$ satisfies (C_2) but $H(R)$ does not satisfy (C_1).

Proof. Observe that $\mathfrak{m} = Ra + Rb$, where $a = 2 + I$ and $b = X + I$ and $\mathfrak{m}$ is not principal. It is clear that $a^2 \neq 0 + I, b^2 \neq 0 + I, ab = 0 + I$, and $\mathfrak{m}^3 = (0 + I)$. Hence, $(R, \mathfrak{m})$ is a local Artinian ring which satisfies the hypotheses of Lemma 4.17. Observe that $\mathfrak{m}^2$ is not principal, $\mathfrak{m}^3 = (0+I)$, and $|\frac{R}{\mathfrak{m}}| = 2$. Therefore, we obtain from (ii) $\Rightarrow$ (i) of Lemma 4.25 that

$H(R)$ satisfies (C_2). It is noted in the proof of (ii) $\Rightarrow$ (i) of Lemma 4.25 that $H(R)$ does not satisfy (C_1). One can apply the following another argument to arrive at the fact that $H(R)$ does not satisfy (C_1). As Ra^2 and Rb^2 are not comparable under the inclusion relation, we obtain from Lemma 4.17 that $H(R)$ does not satisfy (C_1). $\square$

Case (4): $a^2 \neq 0$, $ab \neq 0$, whereas $b^2 = 0$

Let $(R, \mathfrak{m})$ be a local Artinian ring such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ with $a^2 \neq 0, ab \neq 0$, whereas $b^2 = 0$. We next try to determine R such that $H(R)$ is planar. Suppose that $a^2 + ab = 0$. Let $x = a$ and let $y = a + b$. Observe that $\mathfrak{m} = Rx + Ry$ with $x^2 \neq 0, y^2 = ab \neq 0$, and $xy = 0$. In Theorem 4.22, it is shown that $H(R)$ is planar if and only if $\mathfrak{m}^2 = Rx^2 = Ry^2$ and $|\frac{R}{\mathfrak{m}}| \leqslant 3$. Hence, in this case, in characterizing R such that $H(R)$ is planar, we can assume that $a^2 + ab \neq 0$.

Lemma 4.27. *Let $(R, \mathfrak{m})$ be a local Artinian ring such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ with $a^2 \neq 0, ab \neq 0, a^2 + ab \neq 0$, whereas $b^2 = 0$. If $H(R)$ satisfies (C_2), then the following hold.*
(i) $\mathfrak{m}^3 = (0)$.
(ii) $\mathfrak{m}^2 = Rab$.

Proof. Assume that $H(R)$ satisfies (C_2).

(i) As $b^2 = 0$, it follows that $\mathfrak{m}^3 = Ra^3 + Ra^2b$. First, we show that $a^2b = 0$. Suppose that $a^2b \neq 0$. Then it is clear that $Ra^2 \not\subseteq Rb$. Let $A = \{Ra, Ra^2, R(a+b)\}$ and let $B = \{Rb, Ra^2 + Rb, \mathfrak{m}\}$. Note that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This contradicts the assumption that $H(R)$ satisfies (C_2). Therefore, $a^2b = 0$. We next verify that $a^3 = 0$. Suppose that $a^3 \neq 0$. We claim that $Ra^2 \not\subseteq Rb$. For, if $Ra^2 \subseteq Rb$, then $a^2 \in \mathfrak{m}b = Rab$. This implies that $a^3 \in Ra^2b = (0)$. This is in contradiction to the assumption that $a^3 \neq 0$. Therefore, $Ra^2 \not\subseteq Rb$. Let $A_1 = \{Rb, Ra^2, Ra^2 + Rb\}$ and let $B_1 = \{Ra, R(a+b), \mathfrak{m}\}$. Observe that $A_1 \cap B_1 = \varnothing$ and the subgraph of $H(R)$ induced by $A_1 \cup B_1$ contains $K_{3,3}$ as a subgraph. This is a contradiction and so, $a^3 = 0$. Therefore, $\mathfrak{m}^3 = Ra^3 + Ra^2b = (0)$.

(ii) We know from (i) that $\mathfrak{m}^3 = (0)$. We assert that $Ra^2 \subseteq Rb$. Suppose that $Ra^2 \not\subseteq Rb$. Let $A_2 = \{Ra, R(a+b), \mathfrak{m}\}$ and let $B_2 = \{Rb, R(a^2 + b), Ra^2 + Rb\}$. It is clear that $A_2 \cap B_2 = \varnothing$ and the subgraph of $H(R)$ induced by $A_2 \cup B_2$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2). Therefore, we obtain that $Ra^2 \subseteq Rb$ and so, $a^2 \in \mathfrak{m}b = Rab$. Hence, $\mathfrak{m}^2 = Ra^2 + Rab = Rab$. $\square$

Lemma 4.28. *Let* $(R, \mathfrak{m})$ *be a local Artinian ring which satisfies the hypotheses of Lemma 4.27. If* $H(R)$ *satisfies* (C_2)*, then* $|\frac{R}{\mathfrak{m}}| = 3$.

Proof. Assume that $H(R)$ satisfies (C_2). We first verify that $|\frac{R}{\mathfrak{m}}| \leqslant 4$. Suppose that $|\frac{R}{\mathfrak{m}}| > 4$. Then it is possible to find $r, s, t \in R \setminus \mathfrak{m}$ such that $r-1, s-1, t-1, r-s, s-t, r-t \in R \setminus \mathfrak{m}$. From $ab \neq 0$, it follows that at least two among $a^2 + (r+1)ab, a^2 + (s+1)ab, a^2 + (t+1)ab$ must be different from 0. Without loss of generality, we can assume that $a^2 + (r+1)ab \neq 0$ and $a^2 + (s+1)ab \neq 0$. Let $A = \{Rb, R(a+b), \mathfrak{m}\}$ and let $B = \{R(a+rb), R(a+sb), Ra\}$. Observe that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2). Therefore, $|\frac{R}{\mathfrak{m}}| \leqslant 4$.

We next verify that $|\frac{R}{\mathfrak{m}}| \notin \{2, 4\}$. We know from Lemma 4.27 that $\mathfrak{m}^3 = (0)$ and $\mathfrak{m}^2 = Rab$. Hence, $a^2 = uab$ for some $u \in U(R)$. Suppose that $|\frac{R}{\mathfrak{m}}| = 2$. Then $u = 1 + m$ for some $m \in \mathfrak{m}$. This implies that $a^2 = (1+m)ab = ab$ and as $2 \in \mathfrak{m}$, we obtain that $a^2 + ab = a^2 - ab = 0$. This contradicts the hypothesis that $a^2 + ab \neq 0$. Hence, $|\frac{R}{\mathfrak{m}}| \neq 2$ and so, $|\frac{R}{\mathfrak{m}}| \geqslant 3$. We next verify that $|\frac{R}{\mathfrak{m}}| \neq 4$. Suppose that $|\frac{R}{\mathfrak{m}}| = 4$. Then we can find $r \in R \setminus \mathfrak{m}$ such that $r^2 + r + 1 \in \mathfrak{m}$ and $\frac{R}{\mathfrak{m}} = \{0 + \mathfrak{m}, 1 + \mathfrak{m}, r + \mathfrak{m}, (r+1) + \mathfrak{m}\}$. From $a^2 = uab$ for some $u \in U(R)$ and $a^2 + ab \neq 0$, it follows that either $a^2 = rab$ or $a^2 = (r+1)ab$. Without loss of generality, we can assume that $a^2 = rab$. Let $A_1 = \{Ra, R(a+rb), \mathfrak{m}\}$ and let $B_1 = \{Rb, R(a+b), R(a+(r+1)b)\}$. Note that $A_1 \cap B_1 = \varnothing$ and the subgraph of $H(R)$ induced by $A_1 \cup B_1$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2). Therefore, $|\frac{R}{\mathfrak{m}}| \neq 4$ and so, $|\frac{R}{\mathfrak{m}}| = 3$. □

Lemma 4.29. *Let* $(R, \mathfrak{m})$ *be a local Artinian ring which satisfies the hypotheses of Lemma 4.27. If* $\mathfrak{m}^3 = (0), \mathfrak{m}^2 = Rab$*, and* $|\frac{R}{\mathfrak{m}}| = 3$*, then* $|R| = 81$ *and* $H(R)$ *is planar.*

Proof. Observe that $|\mathfrak{m}^2| = 3$, $|\frac{\mathfrak{m}}{\mathfrak{m}^2}| = 9$, and so, $|\mathfrak{m}| = 27$. Hence, we obtain that $|R| = 81$. From $\mathfrak{m}^3 = (0)$ and $\mathfrak{m}^2 = Rab$, it follows that $a^2 = uab$ for some $u \in U(R)$. It follows from the hypothesis $a^2 + ab \neq 0$ that $a^2 = ab$. Let $A = \{0, 1, 2\}$. Note that $\mathfrak{m} = \{xa + yb + zab | x, y, z \in A\}$. It is not hard to verify that $V(H(R)) = \{v_1 = Ra, v_2 = Rb, v_3 = R(a+2b), v_4 = R(a+b), v_5 = \mathfrak{m}, v_6 = Rab\}$. Note that $H(R)$ is the union of the cycle $\Gamma : v_1 - v_2 - v_3 - v_4 - v_5 - v_1$, the edges $e_1 : v_4 - v_1, e_2 : v_4 - v_2, e_3 : v_5 - v_2, e_4 : v_5 - v_3$, and the isolated vertex v_6. The cycle Γ can be represented by means of a pentagon, the edges e_1, e_2 are chords of this pentagon through v_4 and they can be drawn inside this pentagon, the edges e_3, e_4 are chords of this pentagon through v_5 and they can be drawn outside this pentagon in such a way that there are no crossing over of the edges. This proves that $H(R)$ is planar. □

Theorem 4.30. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Lemma 4.27. The following statements are equivalent:*
(i) *$H(R)$ satisfies both (C_1) and (C_2).*
(ii) *$H(R)$ satisfies (C_2).*
(iii) *$\mathfrak{m}^3 = (0)$, $\mathfrak{m}^2 = Rab$, and $|\frac{R}{\mathfrak{m}}| = 3$.*
(iv) *$H(R)$ is planar.*
(v) *$H(R)$ satisfies both (C_1^*) and (C_2^*).*

Proof. (i) ⇒ (ii) This is clear.

(ii) ⇒ (iii) Assume that $H(R)$ satisfies (C_2). We know from Lemma 4.27 that $\mathfrak{m}^3 = (0)$ and $\mathfrak{m}^2 = Rab$. From Lemma 4.28, we know that $|\frac{R}{\mathfrak{m}}| = 3$.

(iii) ⇒ (iv) This follows from Lemma 4.29.

(iv) ⇒ (v) This follows from Kuratowski's theorem [9, Theorem 5.9].

(v) ⇒ (i) This is clear. □

We provide an example in Example 4.31 to illustrate Theorem 4.30.

Example 4.31. Let $T = \mathbb{Z}_9[X]$ and $I = T(X^2 - 3X)$. Then $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+T3}{I})$ is a local Artinian ring which satisfies the hypotheses of Lemma 4.27 with $a = X + I$ and $b = 3 + I$ and moreover, $|\frac{R}{\mathfrak{m}}| = 3$. It is clear that $a^2 = ab$ and so, $\mathfrak{m}^2 = Rab$ and from $a^2b = 0 + I$, it follows that $\mathfrak{m}^3 = Ra^2b + Rab^2 = (0 + I)$. Hence, $(R, \mathfrak{m})$ satisfies the hypotheses of Lemma 4.27 and also the statement (iii) of Theorem 4.30.

In Example 4.32, we provide an example from [5, page 477] of a local Artinian ring $(R, \mathfrak{m})$ which satisfies the hypotheses of Lemma 4.27 and is such that $H(R)$ satisfies (C_1) but $H(R)$ does not satisfy (C_2).

Example 4.32. Let $T = \mathbb{Z}_4[X]$ and $I = T(2X^2) + T(X^3 - 2X)$. Then $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX+T2}{I})$ is a local Artinian ring which satisfies the hypotheses of Lemma 4.27 and is such that $H(R)$ satisfies (C_1) but $H(R)$ does not satisfy (C_2).

Proof. It is clear that $\mathfrak{m} = Ra + Rb$, where $a = X + I$ and $b = 2 + I$, $\mathfrak{m}^4 = (0)$, and $\mathfrak{m}$ is not principal. Thus $(R, \mathfrak{m})$ is a local Artinian ring and it satisfies the hypotheses of Lemma 4.27. Observe that $\mathfrak{m}^2 = Ra^2 + Rab = Ra^2 + Ra^3 = Ra^2$, and $\mathfrak{m}^3 = Ra^3 \neq (0 + I)$, and $|\frac{R}{\mathfrak{m}}| = 2$. Note that $|\mathfrak{m}^3| = 2, |\frac{\mathfrak{m}^2}{\mathfrak{m}^3}| = 2$, and $|\frac{\mathfrak{m}}{\mathfrak{m}^2}| = 4$. Therefore, $|\mathfrak{m}| = 16$ and so, $|R| = 32$. It now follows from (ii) ⇒ (i) of [13, Proposition 4.24] that $\alpha(\mathbb{AG}(R)) = 4$.

Therefore, we obtain that $\omega(H(R)) = 4$. This shows that $H(R)$ satisfies (C_1). As $\mathfrak{m}^3 \neq (0 + I)$, we obtain from Lemma 4.27(i) that $H(R)$ does not satisfy (C_2). □

Case (5): $a^2, b^2, ab \in R \setminus \{0\}$

Let $(R, \mathfrak{m})$ be a local Artinian ring such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ with $a^2, b^2, ab \in R \setminus \{0\}$ and try to determine R such that $H(R)$ is planar. If $a^2 + ab = b^2 + ab = 0$, then with $x = a, y = a + b$, we get that $\mathfrak{m} = Rx + Ry$ and note that $x^2 \neq 0$, whereas $y^2 = xy = 0$. Such Artinian local rings are already characterized in Theorem 4.10 such that $H(R)$ is planar. Hence, in determining rings R such that $H(R)$ is planar, we can assume without loss of generality that $a^2 + ab \neq 0$. Suppose that $b^2 + ab = 0$. With $x_1 = a + b, y_1 = b$, we obtain that $\mathfrak{m} = Rx_1 + Ry_1$, $x_1^2 \neq 0, y_1^2 \neq 0$, and $x_1y_1 = 0$. In Theorem 4.22, such rings R are characterized such that $H(R)$ is planar. Therefore, in determining rings R such that $H(R)$ is planar, we can assume that $a^2 + ab \neq 0$ and $b^2 + ab \neq 0$.

Remark 4.33. Let $(R, \mathfrak{m})$ be a local Artinian ring such that $\mathfrak{m}$ is not principal but $\mathfrak{m} = Ra + Rb$ for some $a, b \in \mathfrak{m}$ such that $a^2, b^2, ab, a^2 + ab, b^2 + ab \in R \setminus \{0\}$. Then $H(R)$ satisfies (C_1) if and only if $\omega(H(R)) = 4$.

Proof. Note that the subgraph of $H(R)$ induced by $\{Ra, Rb, R(a+b), \mathfrak{m}\}$ is a clique on four vertices. Therefore, we get that $\omega(H(R)) \geqslant 4$. Thus $H(R)$ satisfies (C_1) if and only if $\omega(H(R)) \leqslant 4$ if and only if $\omega(H(R)) = 4$. □

Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. We first obtain some necessary conditions in order that $H(R)$ to satisfy either (C_1) or (C_2).

Lemma 4.34. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. Then the following hold.*

(i) *If $H(R)$ satisfies (C_1), then $\mathfrak{m}^5 = (0)$ and moreover, $\mathfrak{m}^3$ and $\mathfrak{m}^4$ are principal.*

(ii) *If $H(R)$ satisfies (C_2), then $\mathfrak{m}^4 = (0)$.*

Proof. Assume that $H(R)$ satisfies (C_1). We know from Remark 4.33 that $\omega(H(R)) = 4$. Thus $\alpha(\mathbb{AG}(R)) = \omega(H(R)) = 4$. In such a case, we know from [13, Lemma 4.32] that $\mathfrak{m}^5 = (0)$. Moreover, $\mathfrak{m}^3$ and $\mathfrak{m}^4$ are principal.

(ii) Assume that $H(R)$ satisfies (C_2). As $\mathfrak{m}^2 \neq (0)$, it follows from Nakayama's lemma [2, Proposition 2.6] that $\mathfrak{m}^2 \neq \mathfrak{m}^3$. Suppose that $\mathfrak{m}^4 \neq (0)$. Then either $\mathfrak{m}^3 a \neq (0)$ or $\mathfrak{m}^3 b \neq (0)$. Without loss of generality, we can assume that $\mathfrak{m}^3 a \neq (0)$. We assert that $\mathfrak{m}^3 b = (0)$. Suppose that $\mathfrak{m}^3 b \neq (0)$. Let $A = \{Ra, Rb, \mathfrak{m}\}$ and let $B = \{R(a+b), \mathfrak{m}^2, \mathfrak{m}^3\}$. Note that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This contradicts the assumption that $H(R)$ satisfies (C_2). Therefore, $\mathfrak{m}^3 b = (0)$. Note that $\mathfrak{m}^3(a+b) = \mathfrak{m}^3 a \neq (0)$. Let $A_1 = \{Ra, R(a+b), \mathfrak{m}\}$ and let $B_1 = \{Rb, \mathfrak{m}^2, \mathfrak{m}^3\}$. Observe that $A_1 \cap B_1 = \varnothing$ and the subgraph of $H(R)$ induced by $A_1 \cup B_1$ contains $K_{3,3}$ as a subgraph. This is a contradiction and so, we obtain that $\mathfrak{m}^4 = (0)$. □

Lemma 4.35. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. If $H(R)$ satisfies (C_2), then $Ra^2 \subseteq Rb$ and $Rb^2 \subseteq Ra$.*

Proof. Assume that $H(R)$ satisfies (C_2). We first verify that $Ra^2 \subseteq Rb$. Suppose that $Ra^2 \nsubseteq Rb$. We claim that either $a^3 \neq 0$ or $a^2 b \neq 0$. Suppose that $a^3 = a^2 b = 0$. Let $A = \{Ra, Rb, \mathfrak{m}\}$ and let $B = \{R(a+b), R(a^2 + b), Ra^2 + Rb\}$. Note that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This contradicts the assumption that $H(R)$ satisfies (C_2). Therefore, either $a^3 \neq (0)$ or $a^2 b \neq 0$. We consider the following cases.

Case 1: $a^3 \neq 0$ and $a^2 b \neq 0$. Let $A_1 = \{Ra, Rb, \mathfrak{m}\}$ and let $B_1 = \{R(a+b), Ra^2, Ra^2 + Rb\}$. Note that $A_1 \cap B_1 = \varnothing$ and the subgraph of $H(R)$ induced by $A_1 \cup B_1$ contains $K_{3,3}$ as a subgraph. This is impossible.

Case 2: $a^3 \neq 0$ whereas $a^2 b = 0$. Let $A_2 = \{Ra, R(a+b), \mathfrak{m}\}$ and let $B_2 = \{Rb, Ra^2, Ra^2 + Rb\}$. Observe that $A_2 \cap B_2 = \varnothing$ and the subgraph of $H(R)$ induced by $A_2 \cup B_2$ contains $K_{3,3}$ as a subgraph. This is impossible.

Case 3: $a^3 = 0$ whereas $a^2 b \neq 0$. Let $A_3 = \{Rb, R(a+b), \mathfrak{m}\}$ and let $B_3 = \{Ra, Ra^2, Ra^2 + Rb\}$. Note that $A_3 \cap B_3 = \varnothing$ and the subgraph of $H(R)$ induced by $A_3 \cup B_3$ contains $K_{3,3}$ as a subgraph. This is a contradiction.

Thus if $H(R)$ satisfies (C_2), then $Ra^2 \subseteq Rb$. Similarly, it can be shown that $Rb^2 \subseteq Ra$. □

Lemma 4.36. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. Suppose that $\mathfrak{m}^2$ is not principal. If $H(R)$ satisfies (C_1), then $\mathfrak{m}^4 = (0)$ and $|\frac{R}{\mathfrak{m}}| \leqslant 3$.*

Proof. Assume that $H(R)$ satisfies (C_1). Then we know from Remark 4.33 that $\alpha(\mathbb{AG}(R)) = 4$. Hence, we obtain from [13, Lemma 4.33] that $\mathfrak{m}^4 = (0)$ and $|\frac{R}{\mathfrak{m}}| \leqslant 3$. □

Lemma 4.37. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. Suppose that $\mathfrak{m}^2$ is not principal. If $H(R)$ satisfies (C_2), then $|\frac{R}{\mathfrak{m}}| \leqslant 3$.*

Proof. Assume that $H(R)$ satisfies (C_2). We know from Lemma 4.35 that $Ra^2 \subseteq Rb$ and $Rb^2 \subseteq Ra$. From $Ra^2 \subseteq Rb$, it follows that $Ra^2 \subseteq \mathfrak{m}b = (Ra + Rb)b = Rab + Rb^2$. Hence, $\mathfrak{m}^2 = Ra^2 + Rab + Rb^2 = Rab + Rb^2$. Similarly, it follows from $Rb^2 \subseteq Ra$ that $\mathfrak{m}^2 = Ra^2 + Rab$. By hypothesis, $\mathfrak{m}^2$ is not principal. Therefore, for any $r \in R \setminus \mathfrak{m}$, $a^2 + rab, ab + rb^2 \neq 0$. We now verify that $|\frac{R}{\mathfrak{m}}| \leqslant 3$. Suppose that $|\frac{R}{\mathfrak{m}}| > 3$. Then it is possible to find $r, s \in R \setminus \mathfrak{m}$ such that $r-1, s-1, r-s \in R \setminus \mathfrak{m}$. Let $A = \{Ra, Rb, \mathfrak{m}\}$ and let $B = \{R(a+b), R(a+rb), R(a+sb)\}$. Note that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2). Therefore, we obtain that $|\frac{R}{\mathfrak{m}}| \leqslant 3$. □

Lemma 4.38. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. Suppose that $\mathfrak{m}^2$ is not principal and $|\frac{R}{\mathfrak{m}}| = 3$. If $H(R)$ satisfies (C_2), then $\mathfrak{m}^3 = (0)$.*

Proof. Assume that $\mathfrak{m}^2$ is not principal, $|\frac{R}{\mathfrak{m}}| = 3$, and $H(R)$ satisfies (C_2). We know from the proof of Lemma 4.37 that $\mathfrak{m}^2 = Ra^2 + Rab = Rb^2 + Rab$. Since $\mathfrak{m}^2$ is not principal, it follows that $a^2 - ab, b^2 - ab \neq 0$. We verify that $\mathfrak{m}^3 = (0)$. Suppose that $\mathfrak{m}^3 \neq (0)$. As $\mathfrak{m}^3 = \mathfrak{m}^2 a + \mathfrak{m}^2 b$, it follows that either $\mathfrak{m}^2 a \neq (0)$ or $\mathfrak{m}^2 b \neq (0)$. Without loss of generality, we can assume that $\mathfrak{m}^2 a \neq (0)$. We consider the following cases.

Case 1: $\mathfrak{m}^2 b \neq (0)$. Let $A = \{Ra, Rb, \mathfrak{m}\}$ and let $B = \{R(a+b), R(a-b), \mathfrak{m}^2\}$. Observe that $A \cap B = \varnothing$ and the subgraph of $H(R)$ induced by $A \cup B$ contains $K_{3,3}$ as a subgraph. This contradicts the assumption that $H(R)$ satisfies (C_2).

Case 2: $\mathfrak{m}^2 b = (0)$. In this case, $\mathfrak{m}^2(a+b) = \mathfrak{m}^2(a-b) = \mathfrak{m}^2 a \neq (0)$. Let $A_1 = \{Ra, R(a+b), R(a-b)\}$ and let $B_1 = \{Rb, \mathfrak{m}, \mathfrak{m}^2\}$. Note that $A_1 \cap B_1 = \varnothing$ and the subgraph of $H(R)$ induced by $A_1 \cup B_1$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $H(R)$ satisfies (C_2).

Thus if $H(R)$ satisfies (C_2), then $\mathfrak{m}^3 = (0)$. □

Lemma 4.39. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. Suppose that $\mathfrak{m}^3 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 2$. If $H(R)$ satisfies (C_1), then either $a^2 = b^2$ or $\mathfrak{m}^2 \subseteq R(a+b)$.*

Proof. Assume that $H(R)$ satisfies (C_1). Then we know from Remark 4.33 that $\omega(H(R)) = 4$. Hence, $\alpha(\mathbb{AG}(R)) = 4$. Therefore, we obtain from the proof of (i) $\Rightarrow$ (ii) of [13, Proposition 4.34] that either $a^2 = b^2$ or $\mathfrak{m}^2 \subseteq R(a+b)$. (This part of the proof in the proof of [13, Proposition 4.34] holds even if $\mathfrak{m}^2$ is principal.) □

Lemma 4.40. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. Suppose that $\mathfrak{m}^3 = (0)$, $\mathfrak{m}^2$ is not principal, and $|\frac{R}{\mathfrak{m}}| = 3$. Then $H(R)$ does not satisfy (C_1).*

Proof. Assume that $H(R)$ satisfies (C_1). We know from Remark 4.33 that $\omega(H(R)) = 4$. Indeed, it is noted in the proof of Remark 4.33 that the subgraph of $H(R)$ induced by $W = \{Ra, Rb, R(a+b), \mathfrak{m}\}$ is a clique on four vertices. Therefore, $\alpha(\mathbb{AG}(R)) = 4$. In such a case, it is verified in the proof of [13, Lemma 4.32] that $\mathfrak{m}^2 = Ra^2 + Rab = Rb^2 + Rab$. By hypothesis, $\mathfrak{m}^3 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 3$. Observe that $\{a^2, ab\}$ (respectively, $\{b^2, ab\}$) is linearly independent over $\frac{R}{\mathfrak{m}}$. Therefore, $a^2 - ab, b^2 - ab \in R \setminus \{0\}$. Note that $R(a-b) \notin W$. If $a^2 - b^2 \neq 0$, then the subgraph of $H(R)$ induced by $W \cup \{R(a-b)\}$ is a clique on five vertices. Suppose that $a^2 = b^2$. Observe that $(a+b)^2 = 2(a^2 + ab) \neq 0$. We assert that $a^2 \notin R(a+b)$. For if $a^2 \in R(a+b)$, then $a^2 = m(a+b)$ for some $m \in \mathfrak{m}$. This implies that $a^2 = (xa+yb)(a+b) = (x+y)(a^2+ab)$ for some $x, y \in R$. It follows from $\mathfrak{m}^3 = (0)$ and $a^2 \neq 0$ that $x + y \in U(R)$. Hence, we obtain that $ab \in Ra^2$. This is impossible since by hypothesis, $\mathfrak{m}^2$ is not principal. Therefore, we get that $a^2 \notin R(a+b)$. Note that $R(a+b) + Ra^2 \notin W$ and the subgraph of $H(R)$ induced by $W \cup \{R(a+b) + Ra^2\}$ is a clique on five vertices. This proves that $\omega(H(R)) \geqslant 5$ and so, $H(R)$ does not satisfy (C_1). □

Remark 4.41. Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. Suppose that $\mathfrak{m}^3 = (0)$ and $|\frac{R}{\mathfrak{m}}| = 2$. If $a^2 = b^2$, then with $x = a, y = a + b$, we get that $\mathfrak{m} = Rx + Ry$ and moreover, $x^2 \neq 0, y^2 = 0, xy \neq 0$ and furthermore, $x^2 + xy = ab \neq 0$. In such a case, we know from Theorem 4.30 that $H(R)$ is not planar. Hence, in determining rings R such that $H(R)$ is planar, we assume that $a^2 \neq b^2$.

Theorem 4.42. *Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. Suppose that $\mathfrak{m}^3 = (0), \mathfrak{m}^2$ is not principal, and $a^2 \neq b^2$. Then the following statements are equivalent:*

(i) *$H(R)$ satisfies both (C_1) and (C_2).*
(ii) *$H(R)$ satisfies (C_1).*
(iii) *$\mathfrak{m}^2 = Ra^2 + Rab = Rb^2 + Rab$, $|\frac{R}{\mathfrak{m}}| = 2$, and $\mathfrak{m}^2 \subseteq R(a+b)$.*
(iv) *$H(R)$ is planar.*
(v) *$H(R)$ satisfies both (C_1^*) and (C_2^*).*

Proof. (i) $\Rightarrow$ (ii). This is clear.

(ii) $\Rightarrow$ (iii) Assume that $H(R)$ satisfies (C_1). We know from Lemma 4.36 that $|\frac{R}{\mathfrak{m}}| \leqslant 3$. It is already noted in the proof of Lemma 4.40 that $\mathfrak{m}^2 = Ra^2 + Rab = Rb^2 + Rab$ (the proof of this assertion is independent of the number of elements in $\frac{R}{\mathfrak{m}}$). By hypothesis, $a^2 \neq b^2$. If $|\frac{R}{\mathfrak{m}}| = 3$, then it is already observed in the proof of Lemma 4.40 that $\omega(H(R)) \geqslant 5$. This is in contradiction to the assumption that $H(R)$ satisfies (C_1). Therefore, $|\frac{R}{\mathfrak{m}}| = 2$. In such a case, we know from Lemma 4.39 that $\mathfrak{m}^2 \subseteq R(a+b)$.

(iii) $\Rightarrow$ (iv) By hypothesis, $\mathfrak{m}^3 = (0)$ and $\mathfrak{m}^2$ is not principal. We are assuming that $\mathfrak{m}^2 = Ra^2 + Rab = Rb^2 + Rab$, $|\frac{R}{\mathfrak{m}}| = 2$, and $\mathfrak{m}^2 \subseteq R(a+b)$. Observe that $\{a^2, ab\}$ is linearly independent over $\frac{R}{\mathfrak{m}}$. Hence, $|\mathfrak{m}^2| = 4$ and it is clear that $|\frac{\mathfrak{m}}{\mathfrak{m}^2}| = 4$. Therefore, $|\mathfrak{m}| = 16$. Let $A = \{0, 1\}$. Note that $\mathfrak{m} = \{xa + yb + za^2 + wab | x, y, z, w \in A\}$. It can be easily verified that $V(H(R)) = \{v_1 = Ra, v_2 = Rb, v_3 = R(a+b), v_4 = \mathfrak{m}, v_5 = Ra^2, v_6 = Rb^2, v_7 = Rab, v_8 = \mathfrak{m}^2\}$. Observe that the subgraph of $H(R)$ induced by $\{v_1, v_2, v_3, v_4\}$ is a clique on four vertices and it follows from $\mathfrak{m}^3 = (0)$ that $\{v_5, v_6, v_7, v_8\}$ is the set of all isolated vertices of $H(R)$. This shows that $H(R)$ is the union of a clique on $\{v_1, v_2, v_3, v_4\}$ and the set of all its isolated vertices. As K_4 is planar, it follows that $H(R)$ is planar.

(iv) $\Rightarrow$ (v). This follows from Kuratowski's theorem [9, Theorem 5.9].

(v) $\Rightarrow$ (i). This is clear. □

We now provide an example from [5, page 479] in Example 4.43 to illustrate Theorem 4.42.

Example 4.43. Let $T = \mathbb{Z}_8[X]$ and $I = T(4X) + T(X^2 - 2X - 4)$. Then $(R = \frac{T}{I}, \mathfrak{m} = \frac{TX + T2}{I})$ is a local Artinian ring which satisfies the hypotheses of Theorem 4.42 and the statement (iii) of Theorem 4.42.

Proof. Note that $\mathfrak{m} = Ra + Rb$ with $a = X + I$ and $b = 2 + I$ and $\mathfrak{m}$ is not principal. Note that $\mathfrak{m}^3 = (0)$. Thus $(R, \mathfrak{m})$ is a local Artinian ring which satisfies the hypotheses of Remark 4.33. Observe that $\mathfrak{m}^2$ is not principal and $a^2 \neq b^2$. Moreover, $\mathfrak{m}^2 = Ra^2 + Rab = Rb^2 + Rab$, $|\frac{R}{\mathfrak{m}}| = 2$ and $\mathfrak{m}^2 \subseteq R(a+b)$. Therefore, $(R, \mathfrak{m})$ is a local Artinian ring which satisfies the hypotheses of Theorem 4.42 and the statement (iii) of Theorem 4.42. □

Let $(R, \mathfrak{m})$ be a local Artinian ring which satisfies the hypotheses of Remark 4.33. Suppose that $\mathfrak{m}^2$ is principal. We are not able to determine R such that $H(R)$ is planar.

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Contact information

P. Vadhel, **S. Visweswaran**

Department of Mathematics, Saurashtra University, Rajkot, 360 005 India
E-Mail(s): pravin_2727@yahoo.com, s_visweswaran2006@yahoo.co.in

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