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Variants of the lattice of partitions of a countable set

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Abstract. In this paper we consider the ordered by inclusion lattice $\mathrm{Part}(M)$ of all partitions of a countable set M. The lattice $\mathrm{Part}(M)$ is a semigroup with respect to the operation $\wedge$ which maps two partitions to their greatest lower bound. We obtain necessary and sufficiency conditions for isomorphism of two variants of the semigroup $\mathrm{Part}(M)$.

Introduction

Let S be a semigroup and $a \in S$. A binary operation $*_a$ defined on the set S by $x *_a y = xay$ $(x, y \in S)$ is associative. This operation $*_a$ is called a *sandwich operation* and the semigroup $(S, *_a)$ is called a *variant* of S or a *sandwich semigroup*.

Lyapin initiated the study of variants in his monograph [1]. In that work he studied transformation semigroups. Variants of other types of semigroups were studied by various authors, see, for example, papers [2–7], and Chapter 13 in [8] and the references therein.

Let M be a countable set. Let $\mathrm{Part}(M)$ be the set of all partitions of the set M. On the set $\mathrm{Part}(M)$ we can define a natural partial order. For any $\pi_1, \pi_2 \in \mathrm{Part}(M)$ we say that $\pi_1 \leqslant \pi_2$ if and only if from the fact that $x, y \in M$ belong to the same block of the partition π_1 it follows that x and y belong to the same block of the partition π_2.

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The set $\mathrm{Part}(M)$ is a complete lattice with respect to this partial order, the zero element of this lattice is the partition such that each block of it is a one-element block and the identity is the one-block, in which case this block is M.

Let $\wedge$ be the operation of taking the greatest lower bound of two elements. The set $\mathrm{Part}(M)$ is a commutative idempotent semigroup with respect to the operation $\wedge$. Note that, the inequality $\rho \leqslant \tau$ holds if and only if $\rho \wedge \tau = \rho$.

Let S be a commutative idempotent semigroup with zero 0 and let $a \in S$ be a fixed element. For any element x from the interval $S_{[0,a]} = \{x \in S \mid a \cdot x = x\}$, by $\Omega_a(x)$ we denote the set $\{y \in S \mid a \cdot y = x\}$. Let the *weight* $\omega_a(x)$ of an element x be defined by $\omega_a(x) = |\Omega_a(x)|$.

Theorem 1 ([10]). *Let S be a commutative idempotent semigroup with zero. Two variants $(S, *_a)$ and $(S, *_b)$ are isomorphic if and only if there exists a weight-preserving isomorphism of intervals $S_{[0,a]}$ and $S_{[0,b]}$.*

In this paper we study variants of the semigroup $(\mathrm{Part}(M), \wedge)$. The main result of the paper is Theorem 16, it gives an isomorphism criterion for variants of the semigroup $(\mathrm{Part}(M), \wedge)$.

1. Preliminaries

Let L be a partially ordered set. Recall that the *height* of L is the least upper bound on the chain length in L. In a partially ordered set with zero let the *rank* of an element a be the height of the interval $[0, a]$ and denote it by $\mathrm{rank}\,(a)$. An element of rank 1 is called an *atom*. For a partition $\rho \in \mathrm{Part}(M)$ we will denote by $a(\rho)$ the number of atoms in the interval $[0, \rho]$. Let Part_k denote the lattice of all partitions of a k-element set.

We call a singleton block of a partition a *trivial* block. A partition $\rho \in \mathrm{Part}(M)$ has *type* $(l_1, l_2, \ldots, l_k, \ldots, l_\infty)$ if it contains l_i blocks of cardinality i, for each positive integer i, and l_∞ blocks of infinite cardinality. We call a partition a *k–uniblock* if it has a single non-trivial block of size k and we denote the type of k-uniblock by $(l_1, 0, \ldots, 0, 1_k, 0, \ldots, 0)$. A block is *finite* if $k \in \mathbb{N}$. Note that any atom from the lattice $\mathrm{Part}(M)$ has type $(\infty, 1_2, 0, \ldots, 0)$, and it is 2–uniblock. We will denote by $(l_2, \ldots, l_k, \ldots, l_\infty)$ the *1-reduced* type of a partition.

A partition σ is a *covering* for a partition ρ if $\sigma > \rho$ and there does not exist any partition χ, such that $\sigma > \chi > \rho$. In other words, if we divide one block from σ into two smaller blocks, then we obtain the partition ρ.

Let $(M_i)_{i\in I}$ be the collection of all pairwise different blocks of the partition ρ then ρ is represented by $\bigcup_{i\in I} M_i$.

The next lemma is obvious.

Lemma 2. *Let $\rho = \bigcup_{i\in I} M_i$ be a partition of M. Let $[0,\rho]$ be an interval from the lattice* $\mathrm{Part}(M)$. *Then $[0,\rho]$ is isomorphic to the Cartesian product $\prod_{i\in I} \mathrm{Part}(M_i)$, such that $\mathrm{Part}(M_i)$ are lattices of partitions of blocks M_i, $i \in I$.*

Since for any $k \in \mathbb{N}$ the height of the lattice Part_k is equal to $k-1$ and taking into account Lemma 2, we get the next lemma.

Lemma 3. *Let ρ be a partition of a n-element set A. Then the height of the interval $[0,\rho]$ is equal to $n-m$, where m is the number of blocks in the partition ρ.*

Lemma 4. *Let ρ be a partition of rank $k-1$. Which contains more than one non-trivial block. Then the interval $[0,\rho]$ and the lattice Part_k have different numbers of atoms.*

Proof. Let $M_1, M_2, \ldots, M_n$ be all non-trivial blocks of the partition ρ such that they have cardinalities $m_1, m_2, \ldots, m_n$, respectively. Hence by Lemma 3, we have:

$$m_1 + m_2 + \cdots + m_n = n + k - 1. \tag{1}$$

Suppose that the interval $[0,\rho]$ and the lattice Part_k have the same number of atoms. Then

$$\binom{m_1}{2} + \binom{m_2}{2} + \cdots + \binom{m_n}{2} = \binom{k}{2}. \tag{2}$$

Using (1), we can rewrite equation (2) as follows

$$\sum_{i=1}^{n} m_i^2 = k^2 - k + \sum_{i=1}^{n} m_i = k^2 + n - 1. \tag{3}$$

On the other hand, squaring both sides in (1), we get

$$\sum_{i=1}^{n} m_i^2 + \sum_{i\neq j} m_i m_j = n^2 + k^2 + 1 + 2nk - 2n - 2k. \tag{4}$$

Combining (3) and (4), we obtain

$$\sum_{i\neq j} m_i m_j = n^2 + 2nk - 3n - 2k + 2. \tag{5}$$

Assume that $m_i = 1 + a_i$. Then

$$\sum_{i\neq j} m_i m_j = \sum_{i\neq j}(1+a_i)(1+a_j) = n^2 - n + 2(n-1)\sum_{i=1}^{n} a_i + \sum_{i\neq j} a_i a_j$$
$$= n^2 - n + 2(n-1)(k-1) + \sum_{i\neq j} a_i a_j.$$

Finally, taking into account (5), we get

$$\sum_{i\neq j} a_i a_j = (n^2 + 2nk - 3n - 2k + 2) - (n^2 - n + 2(n-1)(k-1)) = 0.$$

However, the sum $\sum_{i\neq j} a_i a_j$ is positive by assumption. This contradiction concludes the proof. $\square$

Lemma 5. *Let partitions μ and σ be covering partitions for a $(k+1)$–uniblock ρ. If μ and σ have different types, then the intervals $[0,\mu]$ and $[0,\sigma]$ contain different numbers of k–rank partitions.*

Proof. A covering partition for ρ has type $(l_1,\ldots,0,1_{k+2},0,\ldots,0)$ or $(t_1,1_2,0,\ldots,0,1_{k+1},0,\ldots,0)$. Without restriction of generality we can assume that μ has the first type and σ has the second type.

The partition μ is a $(k+2)$–uniblock. Any k–rank partition from the interval $[0,\mu]$ can be obtained by dividing one block into two smaller blocks. Obviously, this can be done in $2^{k+1}-1$ ways.

The partition σ has two non–trivial blocks B and C such that B is a 2–element block and C is a $(k+1)$–element block. A k–rank partition from the interval $[0,\sigma]$ can be obtained by dividing one of these blocks into two smaller blocks. The block B can be divided uniquely and there are 2^k-1 ways to divide the block C into two smaller blocks. Hence in this case we have 2^k partitions of rank k.

For each positive integer k the inequality $2^{k+1}-1 > 2^k$ holds. This completes the proof. $\square$

Recall from [9] that an isomorphism of a lattice L_1 into a lattice L_2 is a bijection $\varphi : L_1 \to L_2$ such that

$$a \leqslant b \ \Leftrightarrow\ \varphi(a) \leqslant \varphi(b) \tag{6}$$

for all $a,b\in L_1$. By the bijectivity of φ and (6) we have that φ maps the greatest lower bound to the greatest lower bound.

Hence for any $a, b \in L_1$

$$\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b). \tag{7}$$

Conversely, since in any lattice the inequality $a \leqslant b$ holds if $a \wedge b = a$, by the bijectivity of φ and (7) it follows that

$$a \leqslant b \;\Leftrightarrow\; a \wedge b = a \;\Leftrightarrow\; \varphi(a) \wedge \varphi(b) = \varphi(a) \;\Leftrightarrow\; \varphi(a) \leqslant \varphi(b).$$

Thus a bijection $\varphi : L_1 \to L_2$ is an isomorphism from the lattice $(L_1, \leqslant)$ to the lattice $(L_2, \leqslant)$ if and only if φ is an isomorphism from the semigroup $(L_1, \wedge)$ to the semigroup $(L_2, \wedge)$. In particular, further we will consider the isomorphism of lattices of partitions in both meanings which are introduced above.

The next lemma follows from the definition of the isomorphism of lattices as partially ordered sets and the fact that the inverse mapping of an isomorphism is an isomorphism.

Lemma 6. *Let a mapping $\varphi : \mathrm{Part}(M) \to \mathrm{Part}(N)$ be an isomorphism of lattices. Then the following holds*

(a) $\mathrm{rank}(\varphi(\rho)) = \mathrm{rank}(\rho)$;

(b) *if ρ is an atom of the lattice* $\mathrm{Part}(M)$*, then $\varphi(\rho)$ is an atom of the lattice* $\mathrm{Part}(N)$;

(c) $a(\varphi(\rho)) = a(\rho)$;

(d) *if a partition $\sigma \in \mathrm{Part}(M)$ is a covering partition for ρ, then the partition $\varphi(\sigma)$ is a covering partition for $\varphi(\rho)$.*

Corollary 7. *Let a mapping $\varphi : \mathrm{Part}(M) \to \mathrm{Part}(N)$ ba an isomorphism of lattices. If a partition $\rho \in \mathrm{Part}(M)$ is a finite k–uniblock, then $\varphi(\rho)$ is a finite k–uniblock.*

Proof. Let a partition ρ be a k–uniblock, then the interval $[0, \rho]$ is isomorphic to the lattice Part_k. Moreover by Lemma 6 (a), $\mathrm{rank}(\varphi(\rho)) = \mathrm{rank}(\rho)$. If a partition $\varphi(\rho)$ was not a k–uniblock, then by Lemma 4 the interval $[0, \varphi(\rho)]$ and the lattice Part_k would have a different number of atoms. But, by Lemma 6 (c), we have that $a(\varphi(\rho)) = a(\rho)$. Hence $\varphi(\rho)$ is a finite k–uniblock. □

Lemma 8. *Let μ and σ be partitions of types $(l_1, l_2, l_3, \dots, l_\infty)$ and $(t_1, t_2, t_3, \dots, t_\infty)$, respectively. If the interval $[0, \mu]$ is isomorphic to the interval $[0, \sigma]$, then, for each $k \in \mathbb{N}$ such that $k \geqslant 2$, we have $l_k = t_k$.*

Proof. Let μ be a partition of the form $\bigcup_{i\in I} A_i$ and σ be a partition of the form $\bigcup_{j\in J} B_j$. Let $\varphi : [0,\mu] \to [0,\sigma]$ be an isomorphism of lattices of partitions. Note that an isomorphism maps atoms to atoms and a covering partition for $\tau \in [0,\mu]$ to a covering partition for $\varphi(\tau)$. Moreover, an isomorphism preserves ranks of elements.

For any $k \geqslant 2$ and any k–element block A_i from the partition μ, the interval $[0,\mu]$ contains k–uniblock μ_{A_i} such that it has only one non-trivial block A_i. Similarly, we define uniblocks σ_{B_j} for blocks of partition σ.

Since different blocks of a partition do not intersect, it follows that any covering partition for k–uniblock μ_{A_i} is not a uniblock. Hence all covering partitions for μ_{A_i} have a type $(l_1, 1_2, 0, \ldots, 0, 1_k, 0, \ldots, 0)$. By Corollary 7 the partition $\varphi(\mu_{A_i})$ is a k–uniblock. Then all covering partitions for $\varphi(\mu_{A_i})$ have type $(p_1, 1_2, 0, \ldots, 0, 1_k, 0, \ldots, 0)$ or type $(q_1, 0, \ldots, 0, 1_{k+1}, 0, \ldots, 0)$.

A partition of the last type is a $(k+1)$–uniblock. By Lemma 6 and Corollary 7 the preimage (inverse image) of this partition is a $(k+1)$–uniblock such that it is a covering partition for μ_{A_i}. Since the last statement contradicts to the previously proved, it follows that all covering partitions for $\varphi(\mu_{A_i})$ have type $(p_1, 1_2, 0, \ldots, 0, 1_k, 0, \ldots, 0)$. Hence $\varphi(\mu_{A_i})$ has the form σ_{B_j} for some k–element block B_j of the partition σ.

Thus the isomorphism φ induce an injective mapping from the set of k–element blocks of the partition μ to the set of k–element blocks of the partition σ. Since inverse mapping φ^{-1} is an isomorphism, we have that for any $k \geqslant 2$ we have a bijection from k–element blocks of the partition μ to k–element blocks of the partition σ. □

Lemma 9. *A partition μ contains an infinite block if and only if the interval $[0,\mu]$ has an infinite increasing chain*

$$0 < \tau_1 < \tau_2 < \tau_3 < \cdots , \tag{8}$$

such that, for any k, the partition τ_k has rank k, and the interval $[0,\tau_k]$ contains $\binom{k+1}{2}$ atoms.

Proof. Let A be an infinite block of the partition μ. Consider an infinite increasing chain

$$\{a_0, a_1\} \subset \{a_0, a_1, a_2\} \subset \{a_0, a_1, a_2, a_3\} \subset \cdots$$

of subsets $A_k = \{a_0, a_1, a_2, \ldots, a_k\}$ of the set A. For any subset A_k let $\mu_{A_k} \in [0,\mu]$ be a $(k+1)$–uniblock such that A_k is the non-trivial block from this $(k+1)$–uniblock. Since the number of atoms for the interval

μ_{A_k} is equal to the number of atoms for the lattice Part_{k+1}, we have that $a(\mu_{A_k}) = \binom{k+1}{2}$. Obviously, the partition μ_{A_k} has rank k. Hence the chain

$$0 < \mu_{A_1} < \mu_{A_2} < \mu_{A_3} < \cdots$$

of partitions satisfies the conditions of the lemma.

Suppose the interval $[0, \mu]$ contains an infinite increasing chain (8) such that it satisfies the conditions of the lemma. Since the rank of the interval τ_k and the height of the lattice Part_{k+1} are equal, and, moreover, $[0, \tau_k]$ and Part_{k+1} have the same number of atoms, then according to Lemma 4, we see that the partition τ_k is a uniblock. By A_k we denote the non-trivial block of the partition τ_k. Since partitions τ_k form a chain, we have that blocks A_k also form a chain with respect to inclusion. Hence all sets A_k must be contained in the same block A of the partition μ. Then, since A contains an infinite set $\bigcup_{k\geqslant 1} A_k$, it follows that the block A is infinite. □

Lemma 10. *Let μ be a partition of type $(l_1, l_2, \ldots, l_\infty)$, where $k \in \mathbb{N}$. Then $l_\infty = k$ if and only if there exist k different chains $L_1, \ldots, L_k$ from the interval $[0, \mu]$ such that these chains satisfy the following conditions:*

(a) *any chain L_i is an infinite increasing chain and it has the form $0 < \tau_1^i < \tau_2^i < \tau_3^i < \cdots$;*

(b) *for any partition τ_t^i the equalities* $\mathrm{rank}(\tau_t^i) = t$ *and* $a(\tau_t^i) = \binom{t+1}{2}$ *hold;*

(c) *if $i \neq j$ and for partitions τ_i^k, τ_j^m and for some partition ν from $[0, \mu]$, the inequalities $\nu > \tau_i^p$ and $\nu > \tau_j^q$ hold, then ν is a finite uniblock, and the number k of such chains is maximal possible.*

Proof. Necessity. Let $l_\infty = k$ and $A^1, A^2, \cdots, A^k$ be all infinite blocks of the partition μ. Similarly to the proof of Lemma 9, for any block A^i, we can construct a chain $0 < \tau_1^i < \tau_2^i < \tau_3^i < \cdots$ such that it satisfies conditions (a) and (b).

Suppose that the inequalities $\nu > \tau_i^p$ and $\nu > \tau_j^q$ holds for partitions τ_i^p, τ_j^q and ν from $[0, \mu]$, moreover, suppose that ν is a finite uniblock. Partitions τ_i^p and τ_j^q are finite uniblocks by the construction. Let $A_p^i \subset A^i$, $A_q^j \subset A^j$ and B be the non-trivial blocks of τ_i^p, τ_j^q and ν. Since $\nu > \tau_i^p$ and $\nu > \tau_j^q$, it follows that $A_p^i \subset B$ and $A_q^j \subset B$. Hence B has a non-empty intersection with each of the blocks A^i and A^j such that $A^i \cap A^j = \varnothing$. On the other hand, by the inequality $\nu \leqslant \mu$ and since ν is a uniblock, it follows that B is contained in one of the blocks of the partition μ. This contradiction shows that the constructed chains satisfy the condition (c).

Suppose that there are $k+1$ chains $L_1, \ldots, L_{k+1}$ such that each of this chains satisfies conditions (a) and (b). By the proof of Lemma 9 all partitions τ_i^m are uniblocks. We denote the single non-trivial block of the partition τ_i^m by A_m^i. Then $\bigcup_{n\geqslant 1} A_n^i$ is contained in some infinite block of the partition μ. Hence there exist p, q and r such that $\bigcup_{n\geqslant 1} A_n^p$ and $\bigcup_{n\geqslant 1} A_n^q$ are contained in A^r. Let ν be a finite uniblock such that $A_1^p \cup A_1^q$ is the single non-trivial block of this uniblock. Then $\nu \in [0,\mu]$ and we have $\nu > \tau_1^p$ and $\nu > \tau_1^q$. Hence condition (c) is not satisfied for the chains $L_1, \ldots, L_{k+1}$.

Sufficiency. Suppose that the k chains $L_1, \ldots, L_k$ satisfy conditions (a) – (c). Similarly to the proof of Lemma 9, for any chain L_i we can construct an infinite block A^i of the partition μ. By condition (c), it follows that different blocks correspond to different chains. Hence μ has at least k infinite blocks. On the other hand, if there are more then k infinite blocks, then, similarly to above proof, we will be able to construct more than k chains such that this chains satisfying conditions (a) – (c). Hence $l_\infty = k$. □

Corollary 11. *Let μ and σ be partitions of types $(l_1, l_2, \ldots, l_\infty)$ and $(t_1, t_2, \ldots, t_\infty)$, respectively. If the interval $[0,\mu]$ is isomorphic to the interval $[0,\sigma]$, then $l_\infty = t_\infty$.*

Proof. Let $\varphi : [0,\mu] \to [0,\sigma]$ be an isomorphism. If $l_\infty = k$ and $k \in \mathbb{N}$, then the interval $[0,\mu]$ contains k different chains $L_1, \ldots, L_k$ such that these chains satisfy the conditions of Lemma 10. Let $\varphi(L_i)$ be an infinite increasing chain of the form $0 < \varphi(\tau_1^i) < \varphi(\tau_2^i) < \varphi(\tau_3^i) < \cdots$. Then, by Corollary 7, the chains $\varphi(L_1), \ldots, \varphi(L_k)$ in the interval $[0,\sigma]$ also satisfy the conditions of Lemma 10. Hence $t_\infty = k$.

If $l_\infty = \infty$, then the interval $[0,\mu]$ contains infinitely many chains L_1, L_2, $L_3, \ldots$ such that these chains satisfy conditions (a) – (c) of Lemma 10. It follows that the chains $\varphi(L_1)$, $\varphi(L_2)$, $\varphi(L_3), \ldots$ also satisfy the same conditions, and thus $t_\infty = \infty$. □

2. The main result

Proposition 12. *Let* Part(M) *be the lattice of partitions of a countable set M. Two intervals $[0,\mu]$ and $[0,\sigma]$ of the lattice* Part(M) *are isomorphic if and only if the partitions μ and σ have the same 1-reduced type.*

Proof. Necessity. Let $[0,\mu]$ and $[0,\sigma]$ be two isomorphic intervals of the lattice *Part*(*M*). Assume that μ and σ are partitions of types $(l_1, l_2, \ldots, l_\infty)$

and $(t_1, t_2, \ldots, t_\infty)$, respectively. Then, by Lemma 8, for any positive integer k such that $k \geqslant 2$, we have $l_k = t_k$ and, by Corollary 11, we have $l_\infty = t_\infty$. Hence μ and σ have the same type up to the number of trivial blocks.

Sufficiency. Let μ and σ be partitions of types $(l_1, m_2, m_3, \ldots, m_\infty)$ and $(t_1, m_2, m_3, \ldots, m_\infty)$ respectively. We denote by A and B the union of all non-trivial blocks of partitions μ and σ, respectively.

Since 1-reduced types of the partitions μ and σ are equal, it follows that there exists a bijection $\pi : A \to B$ which maps non-trivial blocks of the partition μ to the non-trivial blocks of the partition σ. In other words, for any block $C = \{c_1, c_2, \ldots\}$ of the partition μ the set $\pi(C) = \{\pi(c_1), \pi(c_2), \ldots\}$ is a block of the partition σ. Any partition $\tau \in [0, \mu]$ has the form $\tau = \left(\bigcup_{a \in M\setminus A}\{a\}\right) \bigcup \left(\bigcup_{i \in I} M_i\right)$. It is easily seen that the mapping

$$\varphi : [0, \mu] \to [0, \sigma], \qquad \tau \mapsto \left(\bigcup_{b \in M\setminus B} \{b\}\right) \bigcup \left(\bigcup_{i \in I} \pi(M_i)\right)$$

is an isomorphism from the interval $[0, \mu]$ onto the interval $[0, \sigma]$. □

Corollary 13. *If partitions μ and σ have the same type, then there exists an isomorphism from the interval $[0, \mu]$ to the interval $[0, \sigma]$ such that this isomorphism is induced by a permutation on the set M.*

Proof. Let $\pi : A \to B$ be a bijection from the proof of Proposition 12. If partitions μ and σ have the same type, then π can be extended to a permutation π' on the set M. □

Let us consider the commutative idempotent semigroup $(\mathrm{Part}(M), \wedge)$. As defined in the introduction, for a fixed partition $\rho \in \mathrm{Part}(M)$ the weight $\omega_\rho(\chi)$ of a partition $\chi \in [0, \rho]$ is the number

$$\omega_\rho(\chi) = |\{\xi \in \mathrm{Part}(M) \mid \rho \ \wedge \xi = \chi\}|.$$

Proposition 14 (On weights of partitions)**.** (*a*) *If a partition ρ contains infinitely many blocks, then the weight $\omega_\rho(\chi)$ of each element $\chi \in [0, \rho]$ is the continuum $\mathfrak{c}$.*

(*b*) *If a partition ρ contains a finite number n of blocks, then $\omega_\rho(\rho)$ equals B_n, the n-th Bell number.*

Proof. a) Let ρ be a partition of M of the form $M = \bigcup_{i\in I} M_i$, where I is infinite. Let $\chi \in [0,\rho]$ be of the form $M = \bigcup_{i\in I}\bigcup_{j\in J_i} N_{ij}$, where $M_i = \bigcup_{j\in J_i} N_{ij}$ is a partition of a block M_i. Let us consider a partition ξ such that unions of blocks N_{ij} are blocks of ξ, moreover, no block of the partition ξ contains two blocks N_{ij} with the same first index. Since the set I is countable, we see that the number of this partitions is $\mathfrak{c}$. On the other hand, by the construction of ξ, we have that the intersection of any block of the partition ξ with any block of the partition ρ is either empty or it is a block of the form N_{ij} that is a block of the partition χ. Hence $\rho \wedge \xi = \chi$. Thus the cardinality of the set $\{\xi \in \mathrm{Part}(M) \mid \rho \wedge \xi = \chi\}$ is $\mathfrak{c}$, and hence $\omega(\chi) = \mathfrak{c}$.

b) By the definition $\omega_\rho(\rho) = |\{\xi \in \mathrm{Part}(M) \mid \rho \wedge \xi = \rho\}|$. In other words, $\omega_\rho(\rho)$ is equals to the cardinality of the set of partitions ξ from the interval $[\rho, 1]$. Since the partition ρ contains n blocks it follows that $[\rho, 1]$ is isomorphic to the lattice Part_n, then it has the cardinality B_n. □

By Propositions 12 and 14, we have the next corollary.

Corollary 15. *If intervals $[0,\mu]$ and $[0,\sigma]$ are isomorphic and if weights $\omega_\mu(\mu)$, $\omega_\sigma(\sigma)$ are the same, then one of the following holds:*

(a) *both μ and σ have infinite number of blocks;*

(b) *the partitions μ and σ have the same finite number of blocks and the same type.*

Theorem 16 (Isomorphism criterion for variants of the lattice of partitions)**.** *Let Part_M be the lattice of partitions of a countable set M. The variants $(\mathrm{Part}_M, *_\mu)$ and $(\mathrm{Part}_M, *_\sigma)$ are isomorphic if and only if one of the following conditions holds:*

(a) *the partitions μ and σ both have infinitely many blocks and the same 1-reduced type*;

(b) *the partitions μ and σ both have finitely many blocks and the same type.*

Proof. From Theorem 1 it follows that the variants $(\mathrm{Part}(M), *_\mu)$ and $(\mathrm{Part}(M), *_\sigma)$ are isomorphic if and only if there exists a weight-preserving isomorphism $\varphi : [0,\mu] \to [0,\sigma]$. Since μ and σ are the maximum elements in the interval $[0,\mu]$ and $[0,\sigma]$, respectively, Lemma 6 shows that φ maps μ to σ. Moreover, by Proposition 14 and Corollary 15 we have that $\omega_\mu(\mu) = \omega_\sigma(\sigma)$.

By Proposition 14, we have that the variants $(\mathrm{Part}(M), *_\mu)$ and $(\mathrm{Part}(M), *_\sigma)$ are isomorphic if partitions μ and σ have a countable number of blocks or the same finite number of blocks. Let us consider both cases.

(1) The partitions μ and σ both have countable many blocks. By Proposition 12 the intervals $[0,\mu]$ and $[0,\sigma]$ are isomorphic if and only if the partitions μ and σ have the same 1-reduced type. Moreover, by Proposition 14 weights of all elements in these intervals are the continuum. Thus any isomorphism from $[0,\mu]$ to $[0,\sigma]$ preserves weights. In this case the variants $(\mathrm{Part}(M), *_\mu)$ and $(\mathrm{Part}(M), *_\sigma)$ are isomorphic if and only if partitions μ and σ have the same 1-reduced type.

(2) The partitions μ and σ both have finitely many blocks. Since these partitions have the same type, it follows that μ and σ have the same number of trivial blocks. Then, by Corollary 13, there exists an isomorphism from $[0,\mu]$ to $[0,\sigma]$ such that this isomorphism is induced by a permutation of the set M. Obviously, this isomorphism preserves weights of elements. Hence, in this case, the variants $(\mathrm{Part}(M), *_\mu)$ and $(\mathrm{Part}(M), *_\sigma)$ are isomorphic if and only if the partitions μ and σ have the same type. □

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