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Equivalence of Carter diagrams

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Abstract. We introduce the equivalence relation ρ on the set of Carter diagrams and construct an explicit transformation of any Carter diagram containing l-cycles with $l > 4$ to an equivalent Carter diagram containing only 4-cycles. Transforming one Carter diagram Γ_1 to another Carter diagram Γ_2 we can get a certain intermediate diagram Γ' which is not necessarily a Carter diagram. Such an intermediate diagram is called a connection diagram. The relation ρ is the equivalence relation on the set of Carter diagrams and connection diagrams. The properties of connection and Carter diagrams are studied in this paper. The paper contains an alternative proof of Carter's classification of admissible diagrams.

1. Introduction

1.1. Cycles

1.1.1. Surprising cycles and dotted edges. Let W be a Weyl group and $\varPhi$ the root system associated with W. Let us connect the non-orthogonal simple roots in $\varPhi$ with each other. We get a graph called a *Dynkin diagram*. One may want to connect all (not only simple) non-orthogonal roots with each other. How does the graph thus obtained look like?

The graphs thus obtained are the beautiful color computer-generated pictures given on John Stembridge's home page (based on Peter McMullen's drawings). These pictures are projections of the root system of $\varPhi$

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into the Coxeter plane[1], see [Stm07]. Though beautiful, these graphs are not easy to grasp: for example, in the picture of the root system E_8, there are 6720 edges, see [Ma10].

To see some details in Stembridge's pictures, one can confine oneself to only connected subsets of linearly independent roots. Then the graphs greatly simplified. Essentially, such diagrams were presented by Carter in 1972, in [Ca70], [Ca72]. These graphs are said to be *admissible diagrams* and are designed to characterize elements of the Weyl group, see definition in § 1.2.1.

Each element $w \in W$ can be expressed in the form

$$w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}, \text{ where } \alpha_i \in \varPhi, \tag{1.1}$$

and $s_{\alpha_i} \in W$ are reflections corresponding to not necessarily simple roots $\alpha_i \in \varPhi$.

Carter proved that k in the decomposition (1.1) is the smallest if and only if the subset of roots $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ is linearly independent; such a decomposition is said to be *reduced*. The admissible diagram corresponding to the given element w is not unique, since the reduced decomposition of the element w is not unique.

When I first got acquainted with admissible diagrams I was surprised by the fact that these diagrams contain cycles, though the extended Dynkin diagram $\widetilde{A}_l$, cannot be a part of any admissible diagram (Lemma A.1). It turned out that the cycles in admissible diagrams essentially differ from the cycle $\widetilde{A}_l$. Namely, in such a cycle, there are necessarily two pairs of roots: A pair with a positive inner product together with a pair with a negative inner product. This does not happen for $\widetilde{A}_l$.

This observation motivated me to distinguish such pairs of roots: Let us draw the *dotted* (resp. *solid*) edge $\{\alpha, \beta\}$ if $(\alpha, \beta) > 0$ (resp. $(\alpha, \beta) < 0$), see Figure 1. Let the diagrams with properties of admissible diagrams and containing dotted edges be called *Carter diagrams*. Up to dotted edges, the classification of Carter diagrams coincides with the classification of admissible diagrams. Recall that $(\alpha, \beta) > 0$ (resp. $(\alpha, \beta) < 0$) means[2] that the angle between roots α and β is acute (resp. obtuse). For the

[1] The *Coxeter plane* P is the span of the real and imaginary parts of an eigenvector for the Coxeter element $\mathbf{C}$ with eigenvalue $\cos(\frac{2\pi}{h}) + i \cdot \sin(\frac{2\pi}{h})$, where h is the Coxeter number associated with the root system $\varPhi$.

[2] Here and below, (.) is the *quadratic Tits form*, i.e. the symmetric bilinear form associated with given Weyl group W and the corresponding Cartan matrix, see [St08, §2.1.1].

Dynkin diagrams, all angles between simple roots are obtuse, thus all edges are solid.

1.1.2. Theorem on eliminating long cycles. There are different decompositions (1.1) of w: They can be obtained from each other by some transformations. Transforming one Carter diagram Γ_1 to another Carter diagram Γ_2 we can get a certain intermediate diagram Γ' which is not necessarily a Carter diagram. Such an intermediate diagram will be called a *connection diagram*. This term reflects the fact that such a diagram describes only the connectivity between roots, nothing more. In this paper, we study properties of connection diagrams and Carter diagrams.

Consider an example of basic properties of connection and Carter diagrams. Let $\{\alpha_1, \alpha_2, \alpha_3\}$ be 3 linearly independent and mutually orthogonal roots. There do not exist two non-connected roots β and γ connected to every α_i in such a way that $\{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\}$ is a linearly independent quintuple. First of all, any cycle of linearly independent roots contains an odd number of dotted edges. Let n_1, n_2, n_3 be the odd numbers of dotted edges in every cycle $\{\alpha_i, \beta, \alpha_j, \gamma\}$, where $1 \leqslant i < j \leqslant 3$. Therefore, $n_1 + n_2 + n_3$ is odd, contradicting the fact that every dotted edge appears twice (Corollary 2.4, Figure 12(a)).

Such properties allow us to simplify the classification of Carter diagrams. The main result obtained in this direction is the following one: *Any Carter diagram containing l-cycles, where $l > 4$, is equivalent to another Carter diagram containing only* 4*-cycles* (Theorem 3.1). To realize this equivalence, we construct an explicit transformation mapping each Carter diagram with long cycles into a certain Carter diagram containing only 4-cycles, see § 3. By Theorem 3.1, we eliminate Carter diagrams containing l-cycles with $l > 4$. For the pairs of equivalent diagrams, see Table 2.

1.1.3. Classification of simply-laced Carter diagrams with cycles. The paper contains the alternative proof of the Carter's classification of admissible diagrams. The classification of simply-laced Carter diagrams with cycles is based on the following facts:

(i) the diagram containing any non-Dynkin diagram which is a tree (in particular, any extended Dynkin diagram) is not a Carter diagram (Proposition 2.1).

(ii) the diagram containing two cycles with a bridge of length > 1 is not a Carter diagram (Proposition 2.3(i)).

(iii) the diagram containing two intersecting cycles, both of which with length > 4, is not a Carter diagram; one of which is 4-cycle and the second one is a cycle of length > 6, is not a Carter diagram (Proposition 2.3(iii)).

(iv) the diagram which can be equivalently transformed into a diagram of type (i), (ii) or (iii) is not a Carter diagram. We use this fact in Lemma 2.5.

(v) the Carter diagrams containing cycles of length > 4 can be excluded from Carter's list (Theorem 3.1).

1.2. Diagrams

1.2.1. Admissible and Carter diagrams. Let Φ be the root system associated with a Weyl group W; let s_{α_i} be the reflection in W corresponding to *not necessarily simple root* $\alpha_i \in \Phi$. Each element $w \in W$ can be expressed in the form

$$w = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}, \text{ where } \alpha_i \in \Phi, \tag{1.2}$$

We denote by $l_C(w)$ the smallest value k in any expression like (1.2), see [Ca72, p. 3]. We always have $l_C(w) \leqslant l(w)$. Recall that $l(w)$ is the smallest value k in any expression like (1.2) such that all roots α_i are *simple*. The decomposition (1.2) is called *reduced* if $l_C(s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}) = k$.

Lemma 1.1 ([Ca72, Lemma 3]). *Let* $\alpha_1, \alpha_2, \dots, \alpha_k \in \Phi$. *Then* $s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}$ *is reduced if and only if* $\alpha_1, \alpha_2, \dots, \alpha_k$ *are linearly independent.* □

A diagram Γ is said to be *admissible*, see [Ca72, p. 7], if

$$\begin{aligned} &(a) \text{ The nodes of } \Gamma \text{ correspond to a set of linearly independent roots in } \Phi. \\ &(b) \text{ If a subdiagram of } \Gamma \text{ is a cycle, then it contains an even number of nodes.} \end{aligned} \tag{1.3}$$

Any admissible diagram Γ is said to be a *Carter diagram* if any edge connecting a pair of roots $\{\alpha, \beta\}$ with inner product $(\alpha, \beta) > 0$ (resp. $(\alpha, \beta) < 0$) is drawn as dotted (resp. solid) edge. Let

$$S = \{\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_h\} \tag{1.4}$$

be any set of linearly independent, not necessarily simple, roots associated with Γ, where roots of the set $S_\alpha := \{\alpha_i \mid i = 1, \dots, k\}$ are mutually orthogonal, roots of the set $S_\beta := \{\beta_j \mid j = 1, \dots, h\}$ are also mutually orthogonal. According to (1.3(a)), there exists the set (1.4) of linearly independent roots. Thanks to (1.3(b)), such a partitioning into the union of two mutually orthogonal sets S_α and S_β is possible. The set S is said to be a Γ-*associated* set of roots. Let

$$w = w_1 w_2, \quad \text{where} \quad w_1 = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_k}, \quad w_2 = s_{\beta_1} s_{\beta_2} \dots s_{\beta_h}. \quad (1.5)$$

Since S is linearly independent, the decomposition (1.5) is reduced, see Lemma 1.1, and $k + h = l_C(w)$. The element w is said to be Γ-*associated*, and also S-*associated*. The decomposition (1.5) is said to be a *bicolored decomposition*. The set of roots S_α (resp. S_β) is said to be the α-*set* (resp. β-*set*) of roots corresponding to the bicolored decomposition (1.5).

1.2.2. Connection diagrams. Let Γ be the diagram characterizing connections between roots of a certain set S of linearly independent and not necessarily simple roots, o be the order of reflections in the decomposition (1.2). The pair (Γ, o) is said to be a *connection diagram*. We omit indicating order o in the description of the connection diagram if the order of reflections in the decomposition (1.2) is clear. The connection diagram determines the element w (and its inverse w^{-1}) obtained as the product of all reflections associated with the diagram, while the order o (resp. o^{-1}) describes the order of reflections in the decomposition of w (resp. w^{-1}). The element w is called the *semi-Coxeter element associated with the connection diagram* (Γ, o), or (Γ, o)-*semi-Coxeter element*, see [CE72].

Connection diagrams describe connected sets that may contain cycles, not necessarily even. Converting a Carter diagram Γ_1 into another Carter diagram Γ_2 we sometimes get connection diagrams (but not Carter diagrams), and the "evenness"of cycles is violated during this conversion, see § 3.

The Dynkin diagrams in this paper appear in two ways: (1) associated with Weyl groups (customary use); (2) representing conjugacy classes (CCl), i.e, a Carter diagram which looks like (and actually is) a Dynkin diagram. In a few cases Dynkin diagrams represent two (and even three!) conjugacy classes.

For the Carter diagrams and connection diagrams, we distinguish acute and obtuse angles between roots. Recall that a *solid edge* indicates an obtuse angle between the roots exactly as for Dynkin diagrams. A *dotted edge* indicates an acute angle between the roots considered, see § 1.1.1 and Figure 1.

1.2.3. The 4-cycles in Carter diagrams and connection diagrams. The Carter diagram for a 4-cycle in Figure 1 determines a bicolored decomposition:

$$w = s_{\alpha_1} s_{\alpha_2} s_{\beta_1} s_{\beta_2}.$$

Here, w is the $D_4(a_1)$-associated element, where $D_4(a_1)$ denotes a 4-cycle, see [Ca72]. The diagrams in Figure 1 differ in the order. In the case of Carter diagrams, the order is *trivial* (related with a given bicolored decomposition) and we do not indicate it. The connection diagram in Figure 1 has order $o = \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$:

$$w_o = s_{\alpha_1} s_{\beta_1} s_{\alpha_2} s_{\beta_2}. \tag{1.6}$$

In (1.6), w_o is the $(\mathcal{G}_4, o)$-associated element, where $\mathcal{G}_4$ is a 4-cycle. We will omit the index o of the element w_o if the order o is clear from the context.

Remark 1.2. Hereafter, we suppose that every cycle contains only one dotted edge. Otherwise, we apply reflections $\alpha \longmapsto -\alpha$. These operations do not change the element w since $s_\alpha = s_{-\alpha}$. In this case, every dotted edge with an endpoint vertex α is changed to the solid one, the cycle with all edges solid cannot occur, see Lemma A.1. Note also that the dotted edge can be moved to any other edge of the cycle by means of reflections.

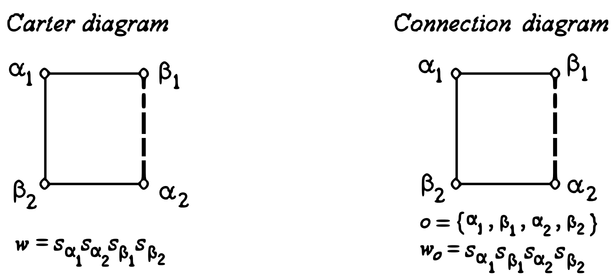


Figure 1. The Carter diagram $D_4(a_1)$ and connection diagram $(\mathcal{G}_4, o)$

The semi-Coxeter elements generated by reflections $s_{\alpha_1}, s_{\alpha_2}, s_{\beta_1}, s_{\beta_2}$ constitute exactly two conjugacy classes, w and w_o being their representatives. In the basis $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$, we have:

$$w = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 \end{pmatrix}, \qquad w_o = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 \end{pmatrix} \tag{1.7}$$

and their characteristic polynomials are:

$$\chi(w) = x^4 + 2x^2 + 1, \qquad \chi(w_o) = x^4 + x^3 + x + 1. \tag{1.8}$$

1.2.4. Transformation of 4-cycles. Denote by $\overset{u}{\simeq}$ the conjugation $w \longrightarrow u^{-1}wu$. Let us transform the element w_o from (1.6):

$$\begin{aligned} w_o = s_{\alpha_1}s_{\beta_1}s_{\alpha_2}s_{\beta_2} = s_{\alpha_1+\beta_1}s_{\alpha_1}s_{\alpha_2}s_{\beta_2} &\overset{s_{\alpha_1+\beta_1}}{\simeq} s_{\alpha_1}s_{\alpha_2}s_{\beta_2}s_{\alpha_1+\beta_1} \\ = s_{\alpha_1}s_{\alpha_2}s_{\alpha_1+\beta_1+\beta_2}s_{\beta_2} &= s_{\alpha_1}s_{\alpha_2}s_{-(\alpha_1+\beta_1+\beta_2)}s_{\beta_2}. \end{aligned} \tag{1.9}$$

We have:

$$\begin{gathered} (\alpha_1+\beta_1+\beta_2, \alpha_1) = (\alpha_1,\alpha_1) + (\beta_1,\alpha_1) + (\beta_2,\alpha_1) = 1 - \frac{1}{2} - \frac{1}{2} = 0, \\ (\alpha_1+\beta_1+\beta_2, \alpha_2) = (\beta_1,\alpha_2) + (\beta_2,\alpha_2) = \frac{1}{2} - \frac{1}{2} = 0, \\ (\alpha_1+\beta_1+\beta_2, \beta_2) = 1 - \frac{1}{2} = \frac{1}{2}. \end{gathered} \tag{1.10}$$

Hence, the roots $\{\alpha_1, \alpha_2, -(\alpha_1+\beta_1+\beta_2)\}$ are mutually orthogonal, so in (1.9), we obtained a bicolored decomposition. Thus, the connection diagram ($\mathcal{G}_4$, o_1) is reduced to the Carter diagram (D_4, o_4), which is also the Dynkin diagram D_4, see Figure 2. That is why, in (1.8) the characteristic polynomial $\chi(w_{o_1}) = x^4 + x^3 + x + 1 = (x^3+1)(x+1)$ is equal to the characteristic polynomial of the D_4-associated element, see [Ca72, Table 3], or [St08, Table 1].

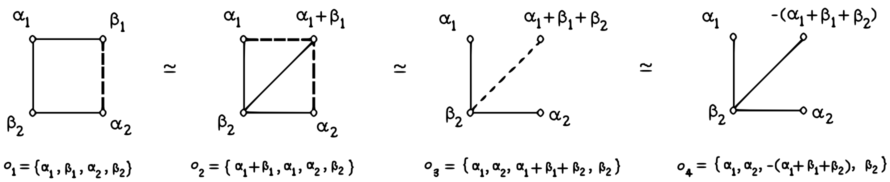


FIGURE 2. Eliminating of the cycle. The equivalence of ($\mathcal{G}_4$, o_1) and (D_4, o_4)

1.3. Equivalence

1.3.1. Three transformations. Talking about a certain diagram Γ we actually have in mind a set of roots with orthogonality relations as it is prescribed by the diagram Γ. We try to find some common properties of sets of roots (from the root systems associated with the simple Lie algebras) and diagrams associated with these sets. These diagrams are not necessarily Dynkin diagrams since sets of roots we study are not necessarily sets of simple roots and are not root subsystems. We use the term *Dynkin diagram* to describe connected sets of linearly independent simple roots in the root system. Similarly, *Carter diagrams* describe connected sets of linearly independent roots, not necessarily simple, and such that any cycle is even.

Same as Dynkin diagrams describe simple Lie algebras, Carter diagrams describe conjugacy classes in Weyl groups.

First of all, in this paper we will see that any Carter diagram with cycles of any length can be transformed into an *equivalent Carter diagram* with cycles of length 4. The equivalence of connection diagrams (and, in particular, of Carter diagrams) is discussed in § 1.3.2. Below we consider a rather natural set of three transformations operating on connection diagrams: Similarities, conjugations and s-permutations.

Similarity. This is replacing a root with the opposite one:

$$\alpha \longmapsto -\alpha. \tag{1.11}$$

Two connection diagrams obtained from each other by a sequence of reflections (1.11), are said to be *similar* connection diagrams, see Figure 3. An equivalence transformation of connection diagrams obtained by a sequence of reflections (1.11) is said to be a *similarity transformation* or *similarity.*

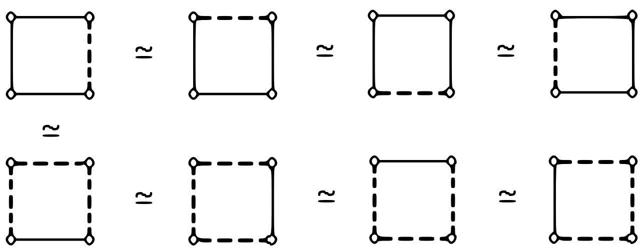


Figure 3. Eight similar 4-cycles equivalent to $D_4(a_1)$

By applying similarity (1.11) any solid edge with an endpoint vertex being α can be changed to a dotted one and vice versa; this does not change, however, the corresponding reflection:

$$s_\alpha = s_{-\alpha}.$$

Remark 1.3 (On trees)**.** For the set $\{\alpha_1, \dots, \alpha_i, \alpha_{i+1}, \dots \alpha_n\}$ forming a tree, we may assume that, up to the similarity, all non-zero inner products (α_i, α_j) are negative. Indeed, if $(\alpha_i, \alpha_j) > 0$, we apply similarity transformation $\alpha_j \longmapsto -\alpha_j$, consider all inner products $(\alpha_k, \alpha_j) > 0$ and repeat similarity transformations $\alpha_k \longmapsto -\alpha_k$ if necessary. This process converges since the diagram is a tree.

Conjugation. Let (Γ, o) be a connection diagram, $S = \{\alpha_1 \dots, \alpha_n\}$ a Γ-associated set. A conjugation sends all roots of a given set S to another set by means of the same element T from the Weyl group:

$$\alpha_1 \longmapsto T\alpha_1, \quad \dots, \quad \alpha_n \longmapsto T\alpha_n. \tag{1.12}$$

Then

$$s_{\alpha_i} \longmapsto s_{T\alpha_i} = T s_{\alpha_i} T^{-1} \text{ for } i = 1, \dots, n, \quad \text{and} \quad \prod_i s_{\alpha_i} \longmapsto \prod_i s_{T\alpha_i}.$$

If $o = \{\alpha_{i_1} \dots, \alpha_{i_n}\}$ is an order of roots, then the conjugation T sends o into $To = \{T\alpha_{i_1} \dots, T\alpha_{i_n}\}$. Let Γ be a Carter diagram. Since T preserves relations between roots, T preserves Γ and the Γ-associated conjugacy class.

s-Permutation. The "evenness"of cycles is not violated by similarities (1.11) and conjugations (1.12). It can be violated by the transformations of the third type, we call them *s-permutations*:

$$s_\alpha s_\beta = \begin{cases} s_\beta s_{\alpha+\beta} = s_{\alpha+\beta} s_\alpha & \text{for } (\alpha, \beta) < 0, \\ s_\beta s_{\alpha-\beta} = s_{\alpha-\beta} s_\alpha & \text{for } (\alpha, \beta) > 0, \\ s_\beta s_\alpha & \text{for } (\alpha, \beta) = 0. \end{cases} \tag{1.13}$$

Relations (1.13) take place only for a simply-laced connection between vertices α and β. In the general case, the s-permutations satisfy the following relation:

$$s_\alpha s_\beta = s_\beta s_{s_\beta(\alpha)} = s_{s_\alpha(\beta)} s_\alpha.$$

Clearly, the s-permutation (1.13) is non-trivial only if α and β are connected. A non-trivial s-permutation (1.13) yields a new set of roots in which α (or β) is changed to $\alpha + \beta$ or $\alpha - \beta$ according to whether the edge $\{\alpha, \beta\}$ is solid or dotted. For the new set, we also draw the diagram which is not necessarily a Carter diagram anymore but is a certain connection diagram.

The set of transformations (1.11), (1.12) and (1.13) operates on a connection diagram Γ and the root subset S associated with the diagram Γ. Similarities (1.11) change a given connection diagram to a similar one; conjugations (1.12) preserve connection diagrams; s-permutations (1.13) essentially change connection diagrams. However, both similarities and s-permutations preserve the element w associated with the given diagram. Transformations (1.11), (1.12) and (1.13) *preserve the conjugacy class* containing w and also *preserve the linear independence* of the roots constituting the subset S.

1.3.2. The equivalence of connection diagrams. Similarities, conjugations and s-permutations are said to be *equivalence transformations*. The equivalence transformations preserve associated conjugacy classes. Connection diagrams (Γ_1, o_1) and (Γ_2, o_2) are said to be *equivalent* if for any (Γ_1, o_1)-associated element w_1, there exists a (Γ_2, o_2)-associated element w_2' such that w_2' can be obtained from w_1 by means of equivalence transformations, and for any (Γ_2, o_2)-associated element w_2, there exists a (Γ_2, o_1)-associated element w_1' such that w_2 can be obtained from w_1' by means of equivalence transformations. In this case, we will write

$$\begin{aligned}&(\Gamma_1, o_1) \simeq (\Gamma_2, o_2),\\ &w_1 \simeq w_2', \quad w_2 \simeq w_1'.\end{aligned}$$

Such a definition of the equivalence of connection diagrams does not require the uniqueness of the conjugacy class associated with Γ_1 (resp. Γ_2). However, if one of diagrams Γ_1 and Γ_2 determines a single conjugacy class, the same holds for another diagram. Indeed, let $\{w_1\}$ be a single Γ_1-associated conjugacy class and w_2, w_2' be arbitrary Γ_2-associated elements, i.e., $w_2 \simeq w_1$, and $w_2' \simeq w_1$. Then by transitivity, we have $w_2 \simeq w_2'$. For example, it will be shown in § 3.1 that

$$\begin{aligned}&E_8(b_3) \simeq E_8(a_3), \qquad E_7(b_2) \simeq E_7(a_2),\\ &D_6(b_2) \simeq D_6(a_2), \qquad E_8(b_5) \simeq E_8(a_5).\end{aligned} \tag{1.14}$$

Some of admissible and Carter diagram may be equivalent to a connection diagram and vice versa. In § 3, we use this fact in the process of excluding diagrams with cycles of length > 4 from Carter's list [Ca72, p. 10], see Theorem 3.1. We exclude a number of diagrams from possible candidates for the role of admissible or Carter diagram, since they have a subdiagram equivalent to an extended Dynkin diagram, a case which cannot be (Proposition 2.1, Lemma 2.5).

1.3.3. Two Γ-associated conjugacy classes. There exist Γ-associated elements w_1 and w_2 such that $w_1 \not\simeq w_2$. For example, the Carter diagram A_3 determines two different conjugacy classes in D_l, see Figure 4; for details, see [St10, B.2].

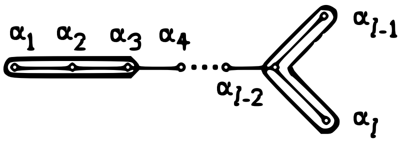


FIGURE 4. Elements $s_{\alpha_1}s_{\alpha_3}s_{\alpha_2}$ and $s_{\alpha_{l-1}}s_{\alpha_l}s_{\alpha_{l-2}}$ are not conjugate

1.3.4. Two non-conjugate Γ-associated sets. Let $S_1 = \{\varphi_1, \dots, \varphi_n\}$ and $S_2 = \{\delta_1, \dots, \delta_n\}$ be two Γ-associated sets of roots. The sets S_1 and S_2 are said to be *conjugate* if there exists an element $T \in W$ such that $T : \varphi_i \longmapsto \delta_i$ for $i = 1, \dots, n$. In this case, we write

$$S_1 \simeq S_2 \quad \text{and} \quad TS_1 = S_2.$$

Let w_1 (resp. w_2) be any S_1-associated (resp. S_2-associated) element. If $S_1 \simeq S_2$, then $w_1 \simeq w_2$.

There exist, however, conjugate elements w_1 and w_2 such that $S_1 \not\simeq S_2$. Consider two 4-cycles in D_6:

$$\begin{aligned}\mathcal{C}_1 &= \{e_1 + e_2, e_4 - e_1, e_1 - e_2, e_2 - e_3\},\\ \mathcal{C}_2 &= \{e_1 + e_2, e_4 - e_1, e_3 - e_4, e_2 - e_3\}.\end{aligned}$$

These sets are non-conjugate: $\mathcal{C}_1 \not\simeq \mathcal{C}_2$, see Figure 5 and [St10, B.1.2], but the $\mathcal{C}_1$-associated element $w_1 = s_{e_1+e_2}s_{e_1-e_2}s_{e_4-e_1}s_{e_2-e_3}$ and the $\mathcal{C}_2$-associated element $w_2 = s_{e_1+e_2}s_{e_3-e_4}s_{e_4-e_1}s_{e_2-e_3}$ are conjugate.

1.3.5. Bridges. Consider Carter diagrams containing intersecting cycles, i.e., cycles having a common path, see Figure 6(a). There are three cycles in this figure, see (1.15). To speak about intersecting cycles we choose the *two shortest ones*. In the case of Figure 6(a), we throw away from consideration the cycle $\mathcal{C}_3$, where

$$\begin{aligned}\mathcal{C}_1 &= \{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_n\},\\ \mathcal{C}_2 &= \{\beta_1, \alpha_4, \beta_m, \alpha_5, \beta_2, \alpha_2\},\\ \mathcal{C}_3 &= \{\alpha_1, \beta_1, \alpha_4, \beta_m, \alpha_5, \beta_2, \alpha_3, \beta_n\}.\end{aligned} \tag{1.15}$$

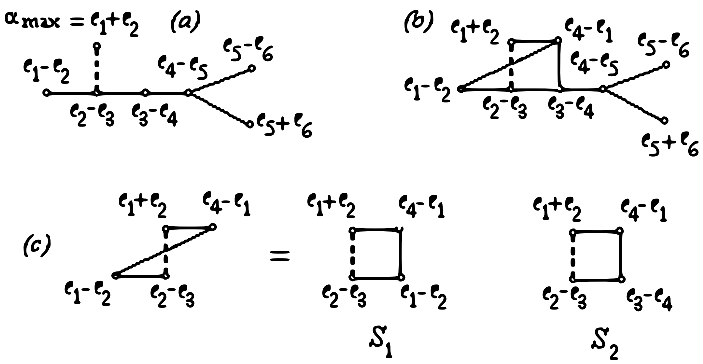


FIGURE 5. Equivalence of the $\mathcal{C}_1$-associated element w_1 and the $\mathcal{C}_2$-associated element w_2

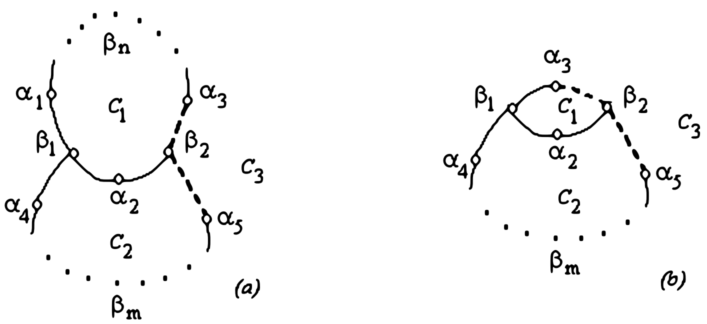


FIGURE 6. Intersecting cycles

Then $\mathcal{C}_1$ and $\mathcal{C}_2$ have the common path $\{\beta_1, \alpha_2, \beta_2\}$. We denote this path by $\mathcal{C}_1 \cap \mathcal{C}_2$. It remains to consider the case, where 2 cycles have the same length, see Figure 6(b) and (1.16).

$$\begin{aligned} \mathcal{C}_1 &= \{\alpha_3, \beta_1, \alpha_2, \beta_2\}, \\ \mathcal{C}_2 &= \{\beta_1, \alpha_4, \beta_m, \alpha_5, \beta_2, \alpha_2\}, \\ \mathcal{C}_3 &= \{\alpha_3, \beta_1, \alpha_4, \beta_m, \alpha_5, \beta_2\}. \end{aligned} \tag{1.16}$$

In Figure 6(b), lengths of $\mathcal{C}_2$ and $\mathcal{C}_3$ coincide. Then the choice of C_2 or C_3 does not matter. The common path will be called a *bridge*. For the pair $\{\mathcal{C}_1, \mathcal{C}_2\}$ (resp. $\{\mathcal{C}_1, \mathcal{C}_3\}$), the bridge is as follows:

$$\mathcal{C}_1 \cap \mathcal{C}_2 = \{\beta_1, \alpha_2, \beta_2\}, \qquad (\text{resp. } \mathcal{C}_1 \cap \mathcal{C}_3 = \{\beta_1, \alpha_3, \beta_2\}).$$

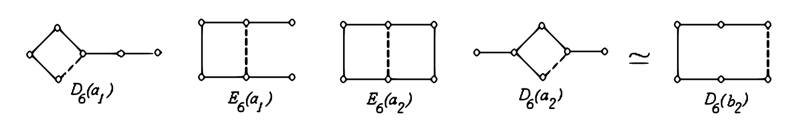
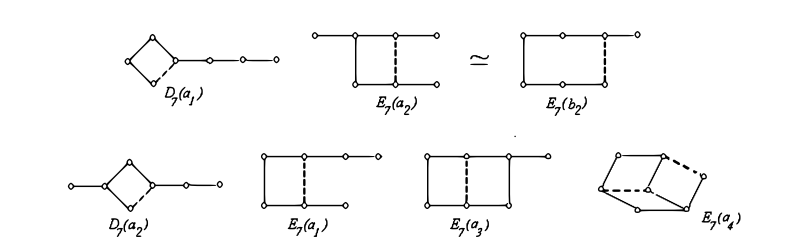
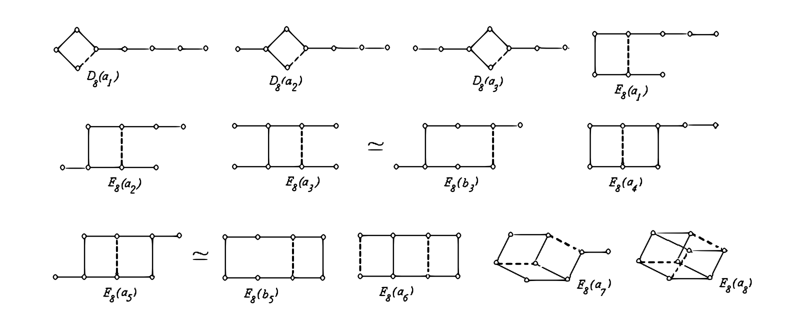
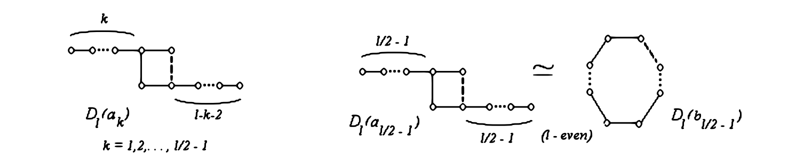
4-5	
6	
7	
8	
$l > 8$	

TABLE 1. The simply-laced Carter diagrams with cycles

2. Classification of Carter diagrams

In this section, we add new arguments to obtain the list of Carter diagrams: We use the statement on intersecting cycles, Proposition 2.3; we exclude diagrams with cycles of length > 4, see Theorem 3.1. The following proposition states that any Carter diagram, or connection diagram, without cycles is a Dynkin diagram.

Proposition 2.1 (Lemma 8, [Ca72])**.** *Let* Γ *be a Carter diagram or connection diagram. If* Γ *is a tree, then* Γ *is the Dynkin diagram.*

For the proof and examples, see § A.2.1.

Due to this proposition, to classify the Carter diagrams, it suffices to consider only diagrams with cycles.

Remark 2.2. For G_2 and A_l, there are no Carter diagrams with cycles. Indeed, for G_2, this fact is trivial, since there at most two linearly independent roots; for A_l, see A.2.2.

2.1. For the multiply-laced case, only a 4-cycle is possible

Consider a multiply-laced diagram containing cycles. If the root system $\varPhi$ contains a cycle, then $\varPhi$ constitutes the 4-cycle with one dotted edge, [Ca72, p. 13]. This case occurs in F_4, see Figure 7.

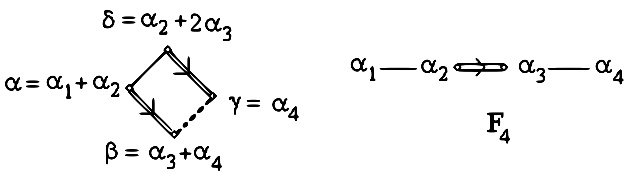


Figure 7. The 4-cycle root subset in F_4. The angle $(\widehat{\beta, \gamma})$ is acute

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the simple roots in F_4, then the quadruple

$$\alpha = \alpha_1 + \alpha_2, \quad \beta = \alpha_3 + \alpha_4, \quad \gamma = \alpha_4, \quad \delta = \alpha_2 + 2\alpha_3$$

constitutes such a 4-cycle. The values of the Tits form on the corresponding pairs of roots are as follows:

$$\begin{aligned}
&(\alpha, \beta) = (\alpha_1 + \alpha_2, \alpha_3 + \alpha_4) = (\alpha_2, \alpha_3) = -1, \\
&(\beta, \gamma) = (\alpha_3 + \alpha_4, \alpha_4) = (\alpha_4, \alpha_4) - (\alpha_3, \alpha_4) = 1 - \frac{1}{2} = \frac{1}{2} \quad \text{(dotted edge)}, \\
&(\gamma, \delta) = (\alpha_4, \alpha_2 + 2\alpha_3) = 2(\alpha_4, \alpha_3) = -1, \\
&(\delta, \alpha) = (\alpha_2 + 2\alpha_3, \alpha_1 + \alpha_2) = (\alpha_2, \alpha_2) + (\alpha_2, \alpha_1) + 2(\alpha_2, \alpha_3) \\
&\qquad = 2 - 1 - 2 = -1.
\end{aligned}$$

In § A.3, we prove that for multiply-laced cases, there are no other Carter diagrams with cycles.

2.1.1. Two intersecting cycles in the simply-laced case. From the foregoing in this section, it suffices to consider only simply-laced diagrams. First of all, we discuss Carter diagrams containing intersecting cycles and bridges, see § 1.3.5.

Proposition 2.3 (On intersecting cycles and bridges). (i) *Let* Γ *be any Carter diagram, or connection diagram, containing two cycles with bridge* $\mathcal{P}$*. Then* $\mathcal{P}$ *consists of exactly one edge.*

(ii) *Let* Γ *be any Carter diagram. Let* P_1, $P_2 \subset \Gamma$ *be two paths stemming from the opposite vertices of a* 4*-cycle in* Γ*; let* α_1 *(resp.* α_2*) be the vertex lying in* P_1 *(resp.* P_2*). The diagram obtained from* Γ *by adding the edge* $\{\alpha_1, \alpha_2\}$ *is not a Carter diagram, see Figure* 8.

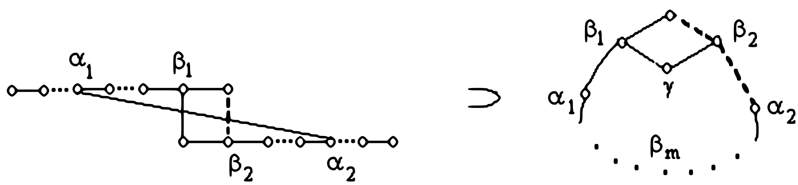


FIGURE 8. Two paths stemming from the opposite vertices of a 4-cycle

(iii) *Let* Γ *be any Carter diagram containing two intersecting cycles. Then one of the cycles consists of* 4 *vertices, and the other one can contain only* 4 *or* 6 *edges.*

Proof. (i) Every cycle contains an odd number of dotted edges, otherwise by several reflections we get a cycle containing only solid edges, a case which cannot happen, see Lemma A.1. Let n_1 be the number of dotted edges in the top cycle: $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_n\}$, and n_2 the number of dotted edges in the bottom cycle: $\{\alpha_4, \beta_1, \alpha_2, \beta_2, \alpha_5, \beta_m\}$. Both n_1 and n_2 are odd. Suppose the bridge $\mathcal{P}$ with endpoints β_1 and β_2 contains an additional vertex α_2 (i.e., $\mathcal{P} = \{\beta_1, \alpha_2, \beta_2\}$, see Figure 6($a$) or Figure 6($b$)). After discarding the vertex α_2 we get a bigger cycle $\mathcal{C}_3 = \{\alpha_1, \beta_1, \alpha_4, \beta_m, \alpha_5, \beta_2, \alpha_3, \beta_n\}$; in the generic case of the bridge $\mathcal{P}$, we discard from the bridge all vertices except β_1, β_2. Let n_3 be the number of dotted edges in the cycle $\mathcal{C}_3$; n_3 is also odd. Therefore, $n_1 + n_2 + n_3$ is odd. On the other hand, every dotted edge enters twice, so $n_1 + n_2 + n_3$ is even. Thus, there is no vertex in the bridge $\{\beta_1, \beta_2\}$ between β_1 and β_2.

(ii) The diagram $\Gamma \cup \{\alpha_1, \alpha_2\}$ contains the bridge $\{\beta_1, \gamma, \beta_2\}$ of length 2, see Figure 8. Thus, by (i), the diagram $\Gamma \cup \{\alpha_1, \alpha_2\}$ in Figure 8 is not a Carter diagram.

(iii) By (i) the bridge consists of one edge $\{\beta_1, \alpha_2\}$, see Figure 9. Then at least one of the cycles is of length 4. Otherwise, the Carter diagram contains the extended Dynkin diagram $\widetilde{D}_5$ contradicting Proposition 2.1. As above, the dotted edge may be eliminated from $\widetilde{D}_5$ by changing the sign of one of the roots.

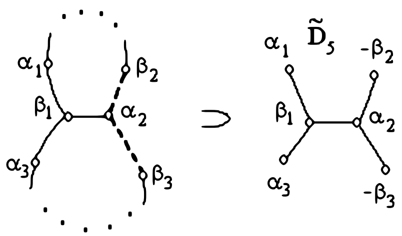


FIGURE 9.

The second cycle can be only of length 4 or 6 as in Figure 10. It cannot be a cycle of length 8, otherwise the Carter diagram contains the extended Dynkin diagram $\widetilde{E}_7$, see Figure 11. According to (i), we cannot add edges $\{\alpha_1, \gamma\}$, $\{\beta_2, \gamma\}$, where $\gamma \in \{\alpha_3, \alpha_4, \beta_3, \beta_4\}$. □

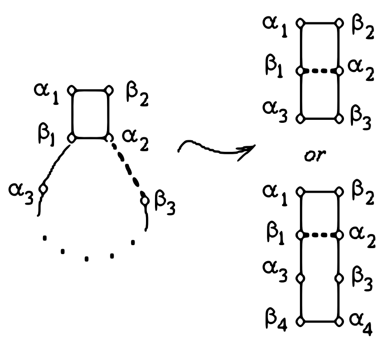


FIGURE 10.

Corollary 2.4 (On impossible quintuples of roots)**.** (i) *Let an α-set contain 3 roots $\{\alpha_1, \alpha_2, \alpha_3\}$. There does not exist two non-connected roots β and γ connected to every α_i so that the vectors of the quintuple $\{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\}$ are linearly independent.*

(ii) *Let $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ be a square in a connection diagram. There does not exist a root γ connected to all vertices of the square so that the vectors of the quintuple $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma\}$ are linearly independent.*

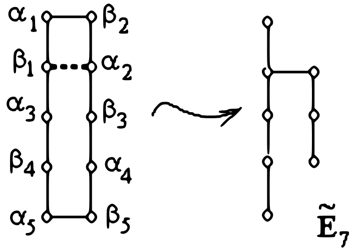


FIGURE 11.

Proof. (i) This is the particular case of Proposition 2.3(i).

(ii) Suppose a certain root γ is connected to all vertices of the square. Then we have 5 cycles: Four triangles $\{\alpha_i, \beta_j, \gamma\}$, where $i = 1, 2$ and $j = 1, 2$, and the square $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$, see Figure 12($b$). Every cycle should contain an odd number of dotted edges. Let n_1, n_2, n_3, n_4, n_5 be the numbers of dotted edges in every cycle, therefore $n_1+n_2+n_3+n_4+n_5$ is odd. On the other hand, every dotted edge enters twice, so $n_1+n_2+n_3+n_4+n_5$ is even, which is a contradiction. For example, the left square in Figure 12(b) is transformed to the right one by the reflection s_{α_1}, then the right square contains the cycle $\{\alpha_1, \beta_2, \gamma\}$ with 3 solid edges, i.e., the extended Dynkin diagram $\widetilde{A}_2$, contradicting Proposition 2.1. □

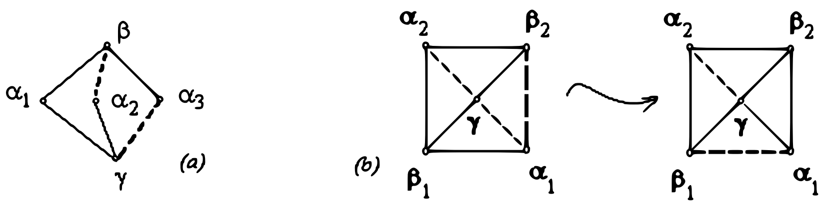


FIGURE 12. Every cycle should contain an odd number of dotted edges, a case which cannot happen

2.1.2. The Carter diagrams with cycles on 6 vertices. There are only four 6-vertex simply-laced Carter diagrams containing cycles, see Table 1. As we show in § 3.2, the diagram $D_6(b_2)$ is equivalent to $D_6(a_2)$, so $D_6(b_2)$ can be excluded from the list of Carter diagrams. The diagrams depicted in Figure 13 are not Carter diagrams. One should discard the bold vertex and apply Corollary 2.4(i), see Figure 12.

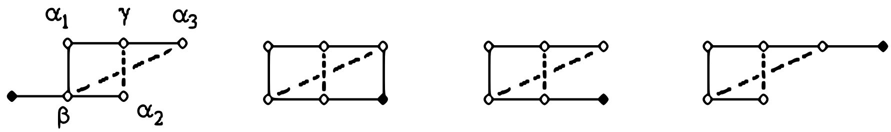


FIGURE 13. Not Carter diagrams on 6-vertices

2.1.3. The Carter diagrams with cycles on 7 vertices. There are only six 7-vertex simply-laced Carter diagrams containing cycles, see Table 1. According to § 3.2, the diagram $E_7(b_2)$ is equivalent to $E_7(a_2)$. Thus, the diagram $E_7(b_2)$ is excluded from the list of Carter diagrams. Note that the diagrams (a) and (b) depicted in Figure 14 are not Carter diagrams since each of them contains the extended Dynkin diagram $\widetilde{D}_4$ contradicting Proposition 2.1. The diagrams (c) and (d) are not Carter diagrams since for each of them there exist two cycles with the bridge of length > 1, contradicting Proposition 2.3. In order to see that (e) and (f) are not Carter diagrams, one can discard bold vertices and apply Corollary 2.4(i) as in § 2.1.2.

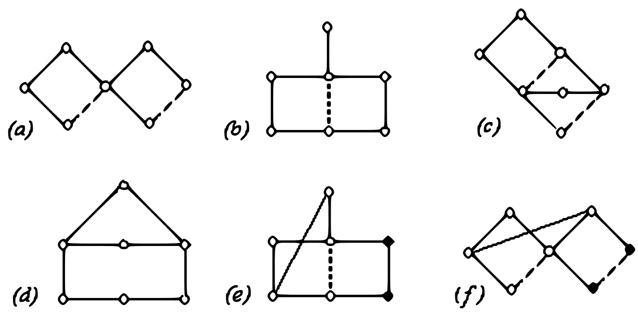


FIGURE 14.

2.1.4. The Carter diagrams with cycles on 8 vertices. There are only eleven 8-vertex simply-laced Carter diagrams containing cycles, see Table 1.

The diagrams depicted in Figure 15 are not Carter diagrams. One can discard the bold vertices to see that each of depicted diagrams contains an extended Dynkin diagram. The diagram (a) contains $\widetilde{E}_6$; (b) and (c) contain $\widetilde{D}_5$; (d) and (e) contain $\widetilde{D}_6$. Thus cases (a), (b), (c), (d), (e) contradict Proposition 2.1. For diagrams (f) and (g), see Lemma 2.5.

The diagram (h) is not a Carter diagram since there exists the bridge of length > 1, see Proposition 2.3[3].

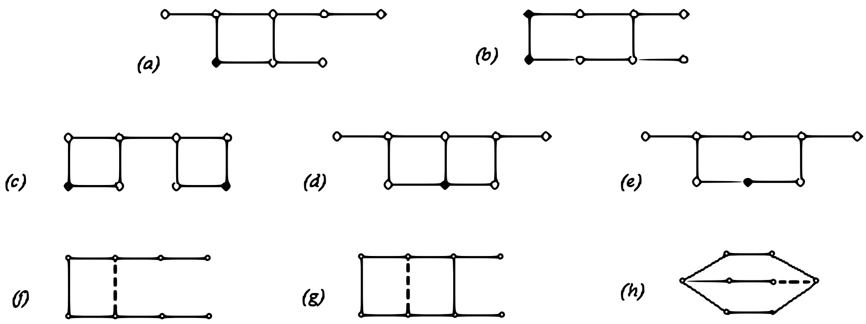


FIGURE 15. 8-vertex diagrams are not Carter diagrams

Lemma 2.5. *Diagrams* (f) *and* (g) *in Figure* 15 *are not Carter diagrams.*

Proof. In cases (f) and (g), we transform the given diagram to an equivalent one containing an extended Dynkin diagram. Let Γ be the diagram (f) in Figure 15. The corresponding roots are depicted in the diagram in Figure 16(1). Let w be the Γ-associated element:

$$w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\beta_4}.$$

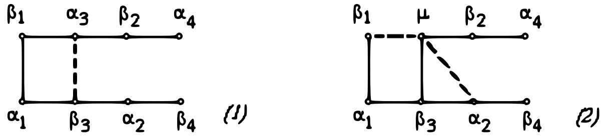


FIGURE 16.

Since $s_{\alpha_3} s_{\beta_1} s_{\beta_3} = s_{\beta_1} s_{\beta_3} s_{\mu}$, where $\mu = \alpha_3 - \beta_3 + \beta_1$, we have

$$w = s_{\alpha_1} s_{\alpha_2} s_{\alpha_4} s_{\beta_1} s_{\beta_3} s_{\mu} s_{\beta_2} s_{\beta_4}.$$

Therefore, the element w is associated with the connection diagram depicted in Figure 16(2). Discard the vertex β_3, the remaining diagram is the extended Dynkin diagram $\widetilde{E}_6$.

[3]We do not depict here the diagrams corresponding to Proposition 2.3(ii), see Figure 8. For $l = 6$, they are depicted in Figure 13; for $l = 7$, see diagrams (e), (f) from § 2.1.3.

Let Γ be the diagram (g) in Figure 15. The same diagram with corresponding roots is the diagram Γ_1 depicted in Figure 17.

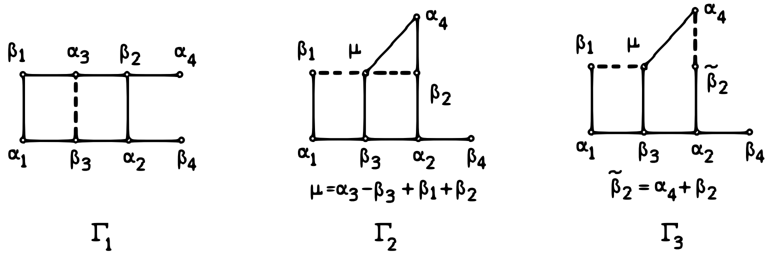


FIGURE 17. The equivalence transformation from Γ_1 to Γ_3

The Γ_1-associated element is as follows:

$$\begin{aligned} w &= s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_4} = s_{\alpha_1}s_{\alpha_2}s_{\alpha_4}(s_{\alpha_3}s_{\beta_1}s_{\beta_2}s_{\beta_3})s_{\beta_4} \\ &= s_{\alpha_1}s_{\alpha_2}s_{\alpha_4}s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\mu}s_{\beta_4}, \end{aligned} \tag{2.1}$$

where $\mu = \alpha_3 - \beta_3 + \beta_1 + \beta_2$. The last expression of w is a (Γ_2, o_2)-associated element, where the diagram Γ_2 in Figure 17 is the connection diagram, not a Carter diagram, and the order o_2 is given by (2.1). Further,

$$w = s_{\alpha_1}s_{\alpha_2}(s_{\alpha_4}s_{\beta_2})s_{\beta_1}s_{\beta_3}s_{\mu}s_{\beta_4} = s_{\alpha_1}s_{\alpha_2}s_{\widetilde{\beta}_2}s_{\alpha_4}s_{\beta_1}s_{\beta_3}s_{\mu}s_{\beta_4}, \tag{2.2}$$

where $\widetilde{\beta}_2 = \alpha_4 + \beta_2$. The obtained expression of w is a (Γ_3, o_3)-associated element, where Γ_3 it the connection diagram in Figure 17 and o_3 is the order given by (2.2). The diagram Γ_3 contains the extended Dynkin diagram $\widetilde{D}_5 = \{\alpha_1, \alpha_2, \mu, \widetilde{\beta_2}, \beta_3, \beta_4\}$, but this is impossible. □

2.1.5. The Carter diagrams with cycles on $l > 8$ vertices. The Dynkin diagram A_l does not contain any Carter diagrams with cycles, see § A.2.2. For the Dynkin diagram D_l, we refer to Carter's discussion in [Ca72, p. 13]. In this case, there are the two types of Carter diagrams (Table 1, $l > 8$):

(1) pure cycles $D_l(b_{\frac{l}{2}-1})$ for l is even, $l \leqslant n$

(2) $D_l(a_1)$, $D_l(a_2)$, $\dots$, $D_l(a_{\frac{l}{2}-1})$ for l is even, $l \leqslant n$.

In § 3.4, we will show that any pure cycle $D_l(b_{\frac{l}{2}-1})$ from (1) is equivalent to $D_l(a_{\frac{l}{2}-1})$ from (2), and hence pure cycles $D_l(b_{\frac{l}{2}-1})$ can be excluded from Carter's list.

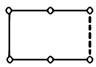
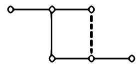
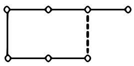
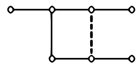
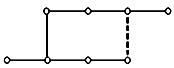
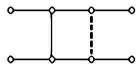
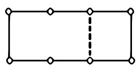
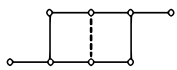
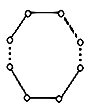
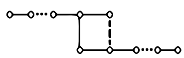
	The Carter diagram with l-cycles, $l > 4$	The equivalent Carter diagram Γ, only 4-cycles	The characteristic polynomial of the Γ-associated element
1	 $D_6(b_2)$, $l = 6$	 $D_6(a_2)$	$(t^3 + 1)^2$
2	 $E_7(b_2)$, $l = 6$	 $E_7(a_2)$	$(t^4 - t^2 + 1)(t^2 - t + 1)(t + 1)$
3	 $E_8(b_3)$, $l = 6$	 $E_8(a_3)$	$(t^4 - t^2 + 1)^2$
4	 $E_8(b_5)$, $l = 6$	 $E_8(a_5)$	$t^8 - t^7 + t^5 - t^4 + t^3 - t^2 + 1$
5	 $D_l(b_{\frac{l}{2}-1})$, l even	 $D_l(a_{\frac{l}{2}-1})$, l even	$(t^{\frac{l}{2}} + 1)^2$

TABLE 2. Pairs of equivalent Carter diagrams

3. Exclusion of long cycles

In this section, we show that Carter diagrams containing cycles of length $n > 4$ can be discarded from the list.

Theorem 3.1 (On exclusion of long cycles). *Any Carter diagram containing l-cycles, where $l > 4$, is equivalent to another Carter diagram containing only* 4*-cycles.*

In all cases we construct a certain explicit transformation of the diagram containing l-cycles, where $l > 4$, to a diagram containing only 4-cycles. The corresponding pairs of equivalent diagrams are depicted in Table 2.

Note that the coincidence of characteristic polynomials of diagrams in pairs of Table 2 is the necessary condition of equivalence of these diagrams, see [Ca72, Table 3]. As it is shown in Theorem 3.1, this condition is also sufficient for the Carter diagrams.

For convenience, we consider the equivalence $D_6(b_2) \simeq D_6(a_2)$ as a separated case, though this is a particular case of the pair $D_l(b_{\frac{l}{2}-1}) \simeq D_l(a_{\frac{l}{2}-1})$ with $l = 6$, Table 2. The idea of explicit transformation connecting elements of every pair is similar for all pairs[4].

3.1. Equivalence $E_8(b_3) \simeq E_8(a_3)$

The $E_8(a_3)$-associated element w is transformed as follows:

$$\begin{aligned} w = s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_4} &= s_{\alpha_1}s_{\alpha_4}\left(s_{\alpha_2}s_{\alpha_3}s_{\beta_3}\right)s_{\beta_1}s_{\beta_2}s_{\beta_4} \\ &= s_{\alpha_1}s_{\alpha_4}s_{\mu}s_{\alpha_2}s_{\alpha_3}s_{\beta_1}s_{\beta_2}s_{\beta_4}, \end{aligned} \tag{3.1}$$

where $\mu = \beta_3 + \alpha_3 - \alpha_2$.

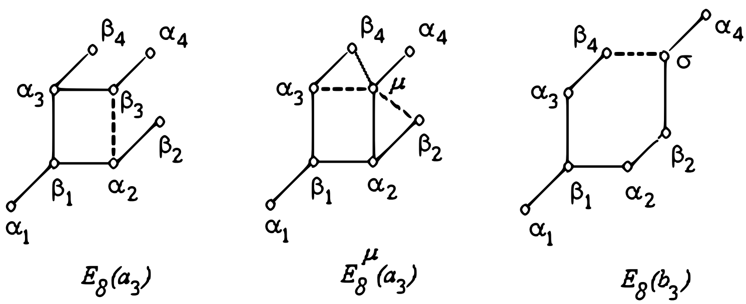


Figure 18. Equivalence $E_8(b_3) \simeq E_8(a_3)$; $E_8^{\mu}(a_3)$ is the connection diagram

The element w is $(E_8^{\mu}(a_3), o)$-associated, where $E_8^{\mu}(a_3)$ is the connection diagram in Figure 18, the order o is given by (3.1). From (3.1) we have:

$$w \overset{s_{\beta_2}s_{\beta_4}}{\simeq} s_{\alpha_1}s_{\alpha_4}\left(s_{\beta_2}s_{\beta_4}s_{\mu}\right)s_{\alpha_2}s_{\alpha_3}s_{\beta_1} = s_{\alpha_1}s_{\alpha_4}s_{\sigma}s_{\beta_2}s_{\beta_4}s_{\alpha_2}s_{\alpha_3}s_{\beta_1}, \tag{3.2}$$

where $\sigma = \mu - \beta_2 + \beta_4$.

[4]Redrawing elements of pairs as the projection of 3-dimensional cube in Figure 18–Figure 24 may give, perhaps, a hint to a geometric interpretation of these explicit transformations.

So, $\sigma = \beta_3 + \alpha_3 - \alpha_2 - \beta_2 + \beta_4$, and it is easy to see that

$$
\begin{aligned}
&(\sigma, \alpha_3) = (\alpha_3, \alpha_3) + (\alpha_3, \beta_4) + (\alpha_3, \beta_3) = 1 - \frac{1}{2} - \frac{1}{2} = 0,\\
&(\sigma, \alpha_2) = -(\alpha_2, \alpha_2) - (\alpha_2, \beta_2) + (\alpha_2, \beta_3) = -1 + \frac{1}{2} + \frac{1}{2} = 0,\\
&(\sigma, \beta_1) = -(\alpha_2, \beta_1) + (\alpha_3, \beta_1) = \frac{1}{2} - \frac{1}{2} = 0,\\
&(\sigma, \alpha_1) = 0,\\
&(\sigma, \beta_4) = (\beta_4, \beta_4) + (\alpha_3, \beta_4) = 1 - \frac{1}{2} = \frac{1}{2},\\
&(\sigma, \beta_2) = -(\beta_2, \beta_2) - (\alpha_2, \beta_2) = -1 + \frac{1}{2} = -\frac{1}{2},\\
&(\sigma, \alpha_4) = (\beta_3, \alpha_4) = -\frac{1}{2}.
\end{aligned} \tag{3.3}
$$

Relations (3.3) describe the Carter diagram $E_8(b_3)$, Figure 18. We only need to check that the element w is conjugate to a product of two involutions:

$$
\begin{aligned}
w &\simeq s_{\alpha_1} s_{\alpha_4} s_\sigma s_{\beta_2} s_{\beta_4} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} \overset{s_{\alpha_4}}{\simeq} s_{\alpha_1} s_\sigma (s_{\beta_2} s_{\beta_4} s_{\alpha_4}) s_{\alpha_2} s_{\alpha_3} s_{\beta_1}\\
&\overset{s_\sigma}{\simeq} s_{\alpha_1} (s_{\beta_2} s_{\beta_4} s_{\alpha_4})(s_{\alpha_2} s_{\alpha_3} s_\sigma) s_{\beta_1} = (s_{\beta_2} s_{\beta_4} s_{\alpha_4})(s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_\sigma) s_{\beta_1}\\
&\overset{s_{\beta_1}}{\simeq} (s_{\beta_1} s_{\beta_2} s_{\beta_4} s_{\alpha_4})(s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_\sigma).
\end{aligned} \tag{3.4}
$$

Thus, $w_1 = s_{\beta_1} s_{\beta_2} s_{\beta_4} s_{\alpha_4}$ and $w_2 = s_{\alpha_1} s_{\alpha_2} s_{\alpha_3} s_\sigma$ are two involutions, $w = w_1 w_2$, i.e., w is conjugate to the $E_8(b_3)$-associated element, which was to be proven.

3.2. Equivalences $E_7(b_2) \simeq E_7(a_2)$ and $D_6(b_2) \simeq D_6(a_2)$

These equivalences directly follow from the equivalence $E_8(b_3) \simeq E_8(a_3)$ that we see from Figure 19 and Figure 20. For the equivalence $E_7(b_2) \simeq E_7(a_2)$, we discard s_{α_1} in relations (3.1)–(3.4) as follows:

$$
\begin{aligned}
w &= s_{\alpha_2} s_{\alpha_3} s_{\alpha_4} s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\beta_4}\\
&= s_{\alpha_4} s_\mu s_{\alpha_2} s_{\alpha_3} s_{\beta_1} s_{\beta_2} s_{\beta_4} \quad (\text{where } \mu = \beta_3 + \alpha_3 - \alpha_2)\\
&\overset{s_{\beta_2} s_{\beta_4}}{\simeq} s_{\alpha_4} s_\sigma s_{\beta_2} s_{\beta_4} s_{\alpha_2} s_{\alpha_3} s_{\beta_1} \quad (\text{where } \sigma = \mu - \beta_2 + \beta_4)\\
&\overset{s_{\alpha_4}}{\simeq} s_\sigma (s_{\beta_2} s_{\beta_4} s_{\alpha_4}) s_{\alpha_2} s_{\alpha_3} s_{\beta_1} \overset{s_\sigma}{\simeq} (s_{\beta_2} s_{\beta_4} s_{\alpha_4})(s_{\alpha_2} s_{\alpha_3} s_\sigma) s_{\beta_1}\\
&\overset{s_{\beta_1}}{\simeq} (s_{\beta_1} s_{\beta_2} s_{\beta_4} s_{\alpha_4})(s_{\alpha_2} s_{\alpha_3} s_\sigma).
\end{aligned} \tag{3.5}
$$

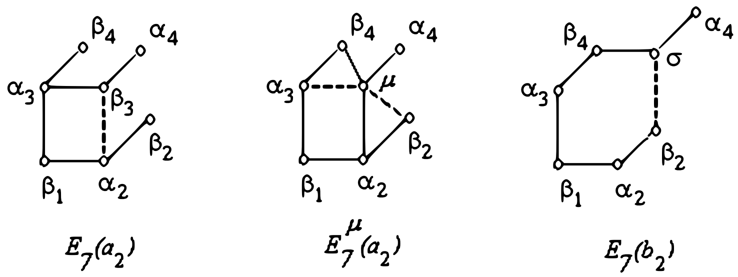


Figure 19. Equivalence $E_7(b_2) \simeq E_7(a_2)$; $E_7^\mu(a_2)$ is the connection diagram

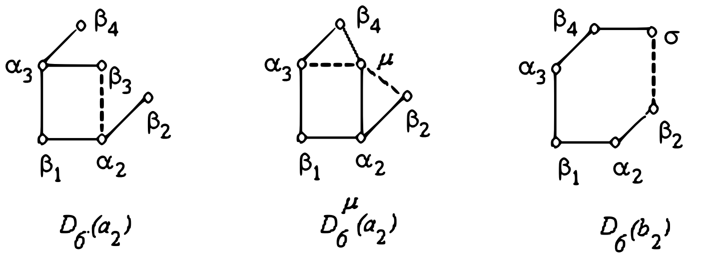


Figure 20. Equivalence $D_6(b_2) \simeq D_6(a_2)$; $D_6^\mu(a_2)$ is the connection diagram

Here, $w_1 = s_{\beta_1}s_{\beta_2}s_{\beta_4}s_{\alpha_4}$ and $w_2 \simeq s_{\alpha_2}s_{\alpha_3}s_\sigma$ are two involutions, $w \simeq w_1w_2$ and the element w is $E_7(b_2)$-associated, which was to be proven. For the equivalence $D_6(b_2) \simeq D_6(a_2)$, we discard s_{α_4} in relation (3.5):

$$\begin{aligned} w &= s_{\alpha_2}s_{\alpha_3}s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_4} = s_\mu s_{\alpha_2}s_{\alpha_3}s_{\beta_1}s_{\beta_2}s_{\beta_4} \quad (\text{where } \mu = \beta_3 + \alpha_3 - \alpha_2) \\ &\overset{s_{\beta_2}s_{\beta_4}}{\simeq} s_\sigma s_{\beta_2}s_{\beta_4}s_{\alpha_2}s_{\alpha_3}s_{\beta_1} \quad (\text{where } \sigma = \mu - \beta_2 + \beta_4) \\ &\overset{s_\sigma}{\simeq} (s_{\beta_2}s_{\beta_4})(s_{\alpha_2}s_{\alpha_3}s_\sigma)s_{\beta_1} \overset{s_{\beta_1}}{\simeq} (s_{\beta_1}s_{\beta_2}s_{\beta_4})(s_{\alpha_2}s_{\alpha_3}s_\sigma). \end{aligned} \tag{3.6}$$

Here, $w_1 = s_{\beta_1}s_{\beta_2}s_{\beta_4}$ and $w_2 = s_{\alpha_2}s_{\alpha_3}s_\sigma$ are two involutions, $w \simeq w_1w_2$ and the element w is $E_7(b_2)$-associated.

3.3. Equivalence $E_8(b_5) \simeq E_8(a_5)$

This equivalence is the most difficult.

Step 1. Let us transform the $E_8(b_5)$-associated element w as follows:

$$\begin{aligned} w &= (s_{\beta_1}s_{\beta_2}s_{\beta_4}s_\gamma)(s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}s_{\alpha_4}) \overset{s_{\alpha_4}}{\simeq} s_{\alpha_4}s_{\beta_2}s_{\beta_4}(s_{\beta_1}s_\gamma s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}) \\ &= (s_{\beta_2}s_{\beta_4}s_\mu)(s_{\beta_1}s_\gamma s_{\alpha_1}s_{\alpha_2}s_{\alpha_3}), \end{aligned} \tag{3.7}$$

where $\mu = \alpha_4 - \beta_2 + \beta_4$.

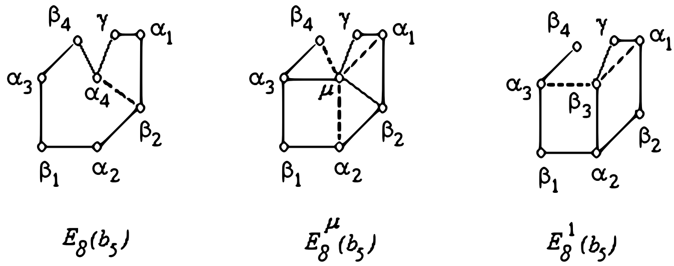


FIGURE 21. Step 1: $E_8(b_5) \Rightarrow E_8^{\mu}(b_5)$ and $E_8^{\mu}(b_5) \Rightarrow E_8^1(b_5)$; $E_8^{\mu}(b_5)$, $E_8^1(b_5)$ are connection diagrams

We have

$$(\mu, \alpha_3) = (\beta_4, \alpha_3) = -\frac{1}{2}, \quad (\mu, \beta_4) = (\beta_4, \beta_4) + (\beta_4, \alpha_4) = 1 - \frac{1}{2} = \frac{1}{2},$$
$$(\mu, \alpha_2) = -(\beta_2, \alpha_2) = \frac{1}{2}, \quad (\mu, \beta_2) = -(\beta_2, \beta_2) + (\alpha_4, \beta_2) = -1 + \frac{1}{2} = -\frac{1}{2},$$
$$(\mu, \alpha_1) = -(\beta_2, \alpha_1) = \frac{1}{2}.$$

see $E_8^{\mu}(b_5)$ in Figure 21. Further, by (3.7)

$$w \simeq (s_{\beta_2} s_{\beta_4} s_{\beta_1}) s_{\mu} s_{\alpha_2} s_{\alpha_3} (s_{\gamma} s_{\alpha_1}) = (s_{\beta_2} s_{\beta_4} s_{\beta_1})(s_{\alpha_2} s_{\alpha_3} s_{\beta_3})(s_{\gamma} s_{\alpha_1}), \quad (3.8)$$

where $\beta_3 = \mu - \alpha_2 + \alpha_3$, $\mu = \alpha_4 - \beta_2 + \beta_4$. Here,

$$\begin{aligned}
(\beta_3, \alpha_3) &= (\mu, \alpha_3) + (\alpha_3, \alpha_3) = -\frac{1}{2} + 1 = \frac{1}{2}, \\
(\beta_3, \beta_4) &= (\mu, \beta_4) + (\alpha_3, \beta_4) = \frac{1}{2} - \frac{1}{2} = 0, \\
(\beta_3, \alpha_2) &= (\mu, \alpha_2) - (\alpha_2, \alpha_2) = \frac{1}{2} - 1 = -\frac{1}{2}, \\
(\beta_3, \beta_2) &= (\mu, \beta_2) - (\alpha_2, \beta_2) = \frac{1}{2} - \frac{1}{2} = 0, \\
(\beta_3, \gamma) &= (\mu, \gamma) = -\frac{1}{2}, \quad (\beta_3, \alpha_1) = (\mu, \alpha_1) = \frac{1}{2}, \\
(\beta_3, \beta_1) &= (\alpha_3, \beta_1) - (\alpha_2, \beta_1) = -\frac{1}{2} + \frac{1}{2} = 0.
\end{aligned} \quad (3.9)$$

see $E_8^1(b_5)$ in Figure 21.

Step 2. From (3.8) we obtain

$$\begin{aligned} w &\overset{s_{\beta_2}s_{\beta_4}s_{\beta_1}}{\simeq} s_{\alpha_2}s_{\alpha_3}s_{\beta_3}(s_{\alpha_1+\gamma}s_{\gamma})s_{\beta_2}s_{\beta_4}s_{\beta_1} = s_{\alpha_2}s_{\alpha_3}s_{\beta_3}s_{\alpha_1+\gamma}(s_{\beta_2}s_{\beta_4}s_{\beta_1})s_{\gamma} \\ &= s_{\alpha_2}s_{\alpha_3}(s_{\beta_3}s_{\beta_2}s_{\beta_4}s_{\beta_1})s_{\alpha_1+\gamma+\beta_2}s_{\gamma} \overset{s_{\alpha_2}s_{\alpha_3}}{\simeq} (s_{\beta_3}s_{\beta_2}s_{\beta_4}s_{\beta_1})s_{\alpha_1+\gamma+\beta_2}s_{\alpha_2}s_{\alpha_3}s_{\gamma} \\ &= s_{\beta_2}(s_{\beta_1}s_{\beta_3}s_{\beta_4}s_{\alpha_1+\gamma+\beta_2})(s_{\alpha_2}s_{\alpha_3}s_{\gamma}), \end{aligned} \tag{3.10}$$

where

$$\begin{gathered} (\alpha_1+\gamma+\beta_2, \beta_3) = (\beta_3, \alpha_1) + (\beta_3, \gamma) = \frac{1}{2} - \frac{1}{2} = 0, \\ (\alpha_1+\gamma+\beta_2, \gamma) = (\gamma, \gamma) + (\gamma, \alpha_1) = 1 - \frac{1}{2} = \frac{1}{2}, \\ (\alpha_1+\gamma+\beta_2, \alpha_2) = (\beta_2, \alpha_2) = -\frac{1}{2}, \\ (\alpha_1+\gamma+\beta_2, \beta_2) = (\beta_2, \beta_2) + (\beta_2, \alpha_1) = 1 - \frac{1}{2} = \frac{1}{2}, \\ (\alpha_1+\gamma+\beta_2, \beta_4) = 0, \quad (\alpha_1+\gamma+\beta_2, \beta_1) = 0, \quad (\alpha_1+\gamma+\beta_2, \alpha_3) = 0. \end{gathered} \tag{3.11}$$

see $E_8^2(b_5)$ in Figure 22.

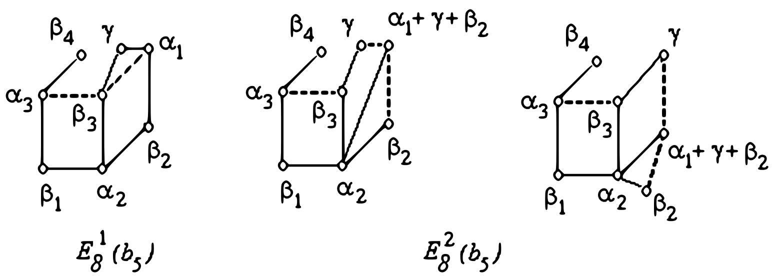


Figure 22. Step 2: $E_8^1(b_5) \Rightarrow E_8^2(b_5)$; $E_8^1(b_5$, $E_8^2(b_5)$ are connection diagrams

Step 3. Let us transform the $E_8^2(b_5)$-associated element w from (3.10) to a certain $E_8^3(b_5)$-associated element (where $E_8^2(b_5)$ and $E_8^3(b_5)$ are connection diagrams, see Figure 23):

$$\begin{aligned} w &\simeq s_{\beta_2}(s_{\beta_1}s_{\beta_3}s_{\beta_4}s_{\alpha_1+\gamma+\beta_2})(s_{\alpha_2}s_{\alpha_3}s_{\gamma}) \\ &= (s_{\beta_1}s_{\beta_3}s_{\beta_4})(s_{\beta_2}s_{\alpha_1+\gamma+\beta_2})(s_{\alpha_2}s_{\alpha_3}s_{\gamma}) \\ &= (s_{\beta_1}s_{\beta_3}s_{\beta_4})(s_{\alpha_1+\gamma+\beta_2}s_{\alpha_1+\gamma})(s_{\alpha_2}s_{\alpha_3}s_{\gamma}). \end{aligned} \tag{3.12}$$

By the orthogonality relations of Figure 22 we have

$$\begin{gathered}(\alpha_1+\gamma,\alpha_2)=(\alpha_1+\gamma+\beta_2,\alpha_2)-(\beta_2,\alpha_2)=-\frac{1}{2}+\frac{1}{2}=0,\\(\alpha_1+\gamma,\tau)=0\quad\text{for } \tau=\beta_1,\beta_3,\beta_4,\alpha_3,\\(\alpha_1+\gamma,\gamma)=-\frac{1}{2}+1=\frac{1}{2}.\end{gathered}\tag{3.13}$$

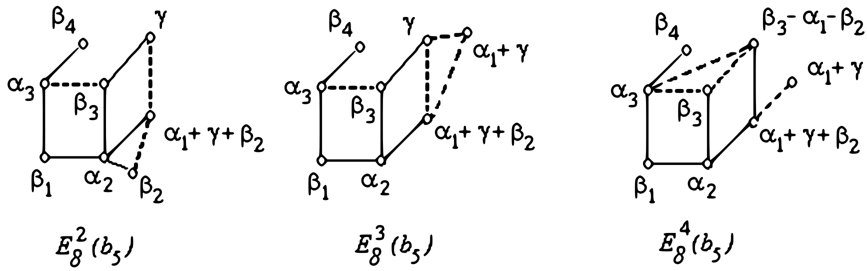


FIGURE 23. Step 3: $E_8^2(b_5) \Rightarrow E_8^3(b_5)$; Step 4: $E_8^3(b_5) \Rightarrow E_8^4(b_5)$; $E_8^2(b_5)$, $E_8^3(b_5)$, $E_8^4(b_5)$ are connection diagrams

Step 4. Now, we transform the $E_8^3(b_5)$-associated element w from (3.12) into a certain $E_8^4(b_5)$-associated element ($E_8^3(b_5)$ and $E_8^3(b_5)$ are connection diagrams, see Figure 23):

$$\begin{aligned}w &\simeq (s_{\beta_1}s_{\beta_3}s_{\beta_4})(s_{\alpha_1+\gamma+\beta_2}s_{\alpha_1+\gamma})(s_{\alpha_2}s_{\alpha_3}s_\gamma)\\&\overset{s_\gamma}{\simeq} s_\gamma s_{\beta_3}s_{\alpha_1+\gamma+\beta_2}(s_{\beta_1}s_{\beta_4})s_{\alpha_1+\gamma}(s_{\alpha_2}s_{\alpha_3})\\&= s_{\beta_3}s_{\beta_3+\gamma}s_{\alpha_1+\gamma+\beta_2}(s_{\beta_1}s_{\beta_4})s_{\alpha_1+\gamma}(s_{\alpha_2}s_{\alpha_3})\\&= s_{\beta_3}s_{\alpha_1+\gamma+\beta_2}s_{\beta_3-\alpha_1-\beta_2}s_{\alpha_1+\gamma}s_{\beta_1}s_{\beta_4}(s_{\alpha_2}s_{\alpha_3}),\end{aligned}\tag{3.14}$$

since $\beta_3-\alpha_1-\beta_2=\gamma+\beta_3-(\alpha_1+\gamma+\beta_2)$.

By orthogonality relations (3.9), (3.11) and (3.13) we have

$$\begin{aligned}&(\beta_3-\alpha_1-\beta_2,\alpha_1+\gamma)=(\gamma+\beta_3,\alpha_1+\gamma)-(\alpha_1+\gamma+\beta_2,\alpha_1+\gamma)\\&\qquad=(\gamma,\alpha_1+\gamma)-(\alpha_1+\gamma,\alpha_1+\gamma)-(\beta_2,\alpha_1+\gamma)=\frac{1}{2}-1+\frac{1}{2}=0,\end{aligned}$$

$$(\beta_3-\alpha_1-\beta_2,\tau)=(\gamma+\beta_3,\tau)-(\alpha_1+\gamma+\beta_2,\tau)=0 \text{ for } \tau=\beta_1,\beta_4,$$

$$(\beta_3-\alpha_1-\beta_2,\alpha_2)=(\beta_3,\alpha_2)-(\beta_2,\alpha_2)=-\frac{1}{2}+\frac{1}{2}=0,$$

$$(\beta_3 - \alpha_1 - \beta_2, \beta_3) = (\gamma + \beta_3, \beta_3) - (\alpha_1 + \gamma + \beta_2, \beta_3)$$
$$= (\gamma + \beta_3, \beta_3) = 1 - \frac{1}{2} = \frac{1}{2},$$
$$(\beta_3 - \alpha_1 - \beta_2, \alpha_1 + \gamma + \beta_2) = (\gamma + \beta_3, \alpha_1 + \gamma + \beta_2)$$
$$- (\alpha_1+\gamma+\beta_2, \alpha_1+\gamma+\beta_2) = (\gamma, \alpha_1 + \gamma + \beta_2) - 1 = \frac{1}{2} - 1 = -\frac{1}{2},$$
$$(\beta_3 - \alpha_1 - \beta_2, \alpha_3) = (\beta_3, \alpha_3) = \frac{1}{2}.$$

Step 5. The last step: From $E_8^4(b_5)$ to $E_8(a_5)$, see Figure 24. The $E_8^4(b_5)$-associated element w from (3.14) is transformed as follows:

$$\begin{aligned} w &\simeq s_{\beta_3} s_{\alpha_1+\gamma+\beta_2} s_{\beta_3-\alpha_1-\beta_2} s_{\alpha_1+\gamma} s_{\beta_1} s_{\beta_4} (s_{\alpha_2} s_{\alpha_3}) \\ &= s_{\beta_3} s_{\beta_1} s_{\alpha_1+\gamma+\beta_2} s_{\beta_3-\alpha_1-\beta_2} s_{\alpha_1+\gamma} s_{\beta_4} (s_{\alpha_2} s_{\alpha_3}) \\ &\overset{s_{\alpha_3}}{\simeq} (s_{\alpha_3} s_{\beta_3} s_{\beta_1}) s_{\alpha_1+\gamma+\beta_2} s_{\beta_3-\alpha_1-\beta_2} s_{\alpha_1+\gamma} s_{\beta_4} s_{\alpha_2} \\ &= s_{\beta_3} s_{\beta_1} s_{\alpha_3-\beta_3+\beta_1} s_{\alpha_1+\gamma+\beta_2} s_{\beta_3-\alpha_1-\beta_2} s_{\alpha_1+\gamma} s_{\beta_4} s_{\alpha_2}. \end{aligned} \tag{3.15}$$

By relations (3.11) we have $(\alpha_3 - \beta_3 + \beta_1, \alpha_1 + \gamma + \beta_2) = 0$. Then

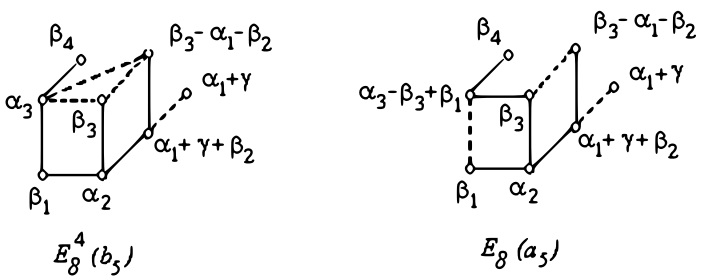


FIGURE 24. Step 5: $E_8^4(b_5) \Rightarrow E_8(a_5)$; $E_8^4(b_5)$ is the connection diagram

$$\begin{aligned} w &= s_{\beta_3} s_{\beta_1} s_{\alpha_1+\gamma+\beta_2} s_{\alpha_3-\beta_3+\beta_1} s_{\beta_3-\alpha_1-\beta_2} s_{\alpha_1+\gamma} s_{\beta_4} s_{\alpha_2} \\ &\overset{s_{\beta_4}}{\simeq} (s_{\beta_4} s_{\beta_3} s_{\beta_1} s_{\alpha_1+\gamma+\beta_2})(s_{\alpha_3-\beta_3+\beta_1} s_{\beta_3-\alpha_1-\beta_2} s_{\alpha_1+\gamma} s_{\alpha_2}), \end{aligned} \tag{3.16}$$

where[5]

$$(\alpha_3 - \beta_3 + \beta_1, \alpha_2) = -(\beta_3, \alpha_2) + (\beta_1, \alpha_2) = -\frac{1}{2} + \frac{1}{2} = 0,$$

[5] Recall that $\beta_3 = \mu - \alpha_2 + \alpha_3 = \alpha_4 - \beta_2 + \beta_4 - \alpha_2 + \alpha_3$, see (3.7), (3.8).

$$(\alpha_3 - \beta_3 + \beta_1, \tau) = (\alpha_3, \tau) - (\beta_3, \tau) + (\beta_1, \tau) = 0$$
$$\text{for } \tau = \alpha_1 + \gamma, \alpha_1 + \gamma + \beta_2,$$

$$(\alpha_3 - \beta_3 + \beta_1, \beta_3) = (\alpha_3, \beta_3) - (\beta_3, \beta_3) = \frac{1}{2} - 1 = -\frac{1}{2},$$

$$(\alpha_3 - \beta_3 + \beta_1, \beta_1) = 1 - \frac{1}{2} = \frac{1}{2},$$

$$(\alpha_3 - \beta_3 + \beta_1, \beta_4) = (\alpha_3, \beta_4) = -\frac{1}{2},$$

$$(\alpha_3 - \beta_3 + \beta_1, \beta_3 - \alpha_1 - \beta_2) = (\alpha_3, \beta_3 - \alpha_1 - \beta_2)$$
$$- (\beta_3, \beta_3 - \alpha_1 - \beta_2) = \frac{1}{2} - \frac{1}{2} = 0.$$

Thus, (3.16) is a bicolored decomposition of the $E_8(a_5)$-associated element w, see Figure 24. The equivalence $E_8(b_5) \simeq E_8(a_5)$ is proven.

3.4. Equivalence $D_l(b_{\frac{l}{2}-1}) \simeq D_l(a_{\frac{l}{2}-1})$

We consider the two cases of cycles $D_l(b_{\frac{l}{2}-1})$ differing by length l, see Figure 25.

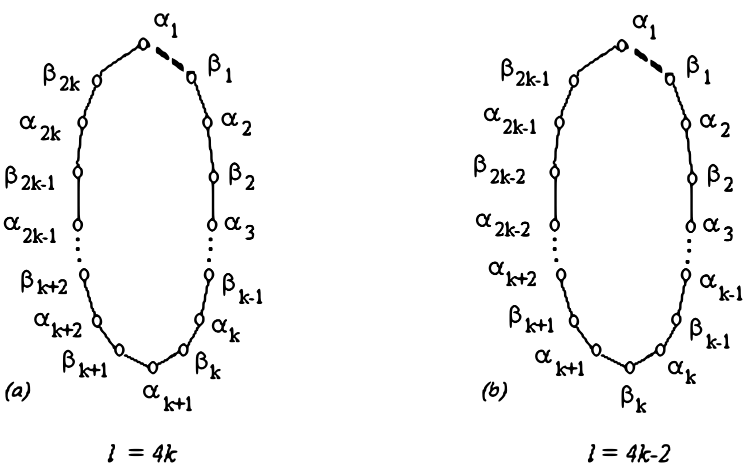


FIGURE 25. The two cases of even cycles $D_l(b_{\frac{l}{2}-1})$: 1) $l = 4k$; 2) $l = 4k - 2$

Case 1) $l = 4k$. The opposite vertices, i.e., vertices at distance $2k$, are of the same type, for example, α_1 and α_{k+1}, see Figure 25(a).

Case 2) $l = 4k-2$. The opposite vertices, i.e., vertices at distance $2k-1$, are of different types, for example, α_1 and β_k, see Figure 25(b).

3.4.1. The case $\boldsymbol{l = 4k}$. Consider the chains of vertices passing through the top vertex α_1 and with endpoints lying on the same horizontal level, see Figure 25. Let L (resp. R) be the index of the left (resp. right) end of the chain. Then the endpoints of these chains are as follows:

$$\begin{aligned}
&\{\beta_L, \beta_R\}, \quad L = 2k-i+1, R = i, \quad 1 \leqslant i \leqslant k, \text{ or} \\
&\{\alpha_L, \alpha_R\}, \quad L = 2k-i+2, R = i, \quad 2 \leqslant i \leqslant k.
\end{aligned} \tag{3.17}$$

Consider the following vectors associated with chains (3.17):

$$\begin{aligned}
\theta(\beta_L, \beta_R) &= \alpha_1 - \sum_{i=1}^{R}\beta_i - \sum_{i=2}^{R}\alpha_i + \sum_{i=L}^{2k}\beta_i + \sum_{i=L+1}^{2k}\alpha_i, \quad R+L = 2k+1, \\
\theta(\alpha_L, \alpha_R) &= \alpha_1 - \sum_{i=1}^{R-1}\beta_i - \sum_{i=2}^{R}\alpha_i + \sum_{i=L}^{2k}\beta_i + \sum_{i=L}^{2k}\alpha_i, \quad R+L = 2k+2.
\end{aligned} \tag{3.18}$$

We have the following actions on vectors (3.18):

$$\begin{aligned}
s_{\beta_1}s_{\beta_{2k}}\alpha_1 &= \theta(\beta_1, \beta_{2k}), \\
s_{\alpha_2}s_{\alpha_{2k}}\theta(\beta_1, \beta_{2k}) &= \theta(\alpha_2, \alpha_{2k}), \\
&\dots \\
s_{\beta_{L-1}}s_{\beta_R}\theta(\alpha_L, \alpha_R) &= \theta(\beta_{L-1}, \beta_R), \\
s_{\alpha_L}s_{\alpha_{R+1}}\theta(\beta_L, \beta_R) &= \theta(\alpha_L, \alpha_{R+1}).
\end{aligned} \tag{3.19}$$

Thus, $\theta(\beta_L, \beta_R)$, $\theta(\alpha_L, \alpha_R)$ from (3.18) are roots. The following orthogonality relations hold

$$\begin{gathered}
\theta(\beta_L, \beta_R) \perp \beta_i, \quad i \neq R, L, \qquad \theta(\beta_L, \beta_R) \not\perp \beta_L, \beta_R, \\
\theta(\beta_L, \beta_R) \perp \alpha_i, \quad i \neq R+1, L, \qquad \theta(\beta_L, \beta_R) \not\perp \alpha_{R+1}, \alpha_L \quad (R \neq k), \\
\theta(\beta_{k+1}, \beta_k) \perp \alpha_{k+1}, \\
\theta(\alpha_L, \alpha_R) \perp \beta_i, \quad i \neq L-1, R, \qquad \theta(\alpha_L, \alpha_R) \not\perp \beta_{L-1}, \beta_R, \\
\theta(\alpha_L, \alpha_R) \perp \alpha_i, \quad i \neq R, L, \qquad \theta(\alpha_L, \alpha_R) \not\perp \alpha_L, \alpha_R.
\end{gathered} \tag{3.20}$$

Lemma 3.2. *The following commutation relations hold:*

$$
\begin{aligned}
s_{\theta(\beta_L,\beta_R)} \prod_{i=1}^{2k} s_{\alpha_i} &= (\prod_{i=1}^{2k} s_{\alpha_i}) s_{\theta(\alpha_L,\alpha_{R+1})}, \qquad L+R = 2k+1, \quad R \leqslant k,\\
s_{\theta(\alpha_L,\alpha_R)} \prod_{i=1}^{2k} s_{\beta_i} &= (\prod_{i=1}^{2k} s_{\beta_i}) s_{\theta(\beta_{L-1},\beta_R)}, \qquad L+R = 2k+2, \quad R \leqslant k.
\end{aligned}
\tag{3.21}
$$

Proof. According to the orthogonality relations (3.20), we have

$$
\begin{aligned}
s_{\theta(\beta_L,\beta_R)} \prod_{i=1}^{2k} s_{\alpha_i} &= \Bigg(\prod_{\alpha_i \neq R+1, L} s_{\alpha_i}\Bigg) s_{\theta(\beta_L,\beta_R)} s_{\alpha_{R+1}} s_{\alpha_L}\\
&= \Bigg(\prod_{\alpha_i \neq R+1, L} s_{\alpha_i}\Bigg) s_{\alpha_{R+1}} s_{\alpha_L} s_{\theta(\beta_L,\beta_R)-\alpha_{R+1}+\alpha_L} = \Bigg(\prod_{i=1}^{2k} s_{\alpha_i}\Bigg) s_{\theta(\alpha_L,\alpha_{R+1})}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
s_{\theta(\alpha_L,\alpha_R)} \prod_{i=1}^{2k} s_{\beta_i} &= \Bigg(\prod_{\beta_i \neq R, L-1} s_{\beta_i}\Bigg) s_{\theta(\alpha_L,\alpha_R)} s_{\beta_{L-1}} s_{\beta_R}\\
&= (\prod_{\beta_i \neq R, L-1} s_{\beta_i}) s_{\beta_{L-1}} s_{\beta_R} s_{\theta(\alpha_L,\alpha_R)-\beta_R+\beta_{L-1}} = \Bigg(\prod_{i=1}^{2k} s_{\beta_i}\Bigg) s_{\theta(\beta_{L-1},\beta_R)}.
\end{aligned}
$$

□

Proposition 3.3. *Let*

$$
w = w_\beta w_\alpha = \prod_{i=1}^{2k} s_{\beta_i} \prod_{i=1}^{2k} s_{\alpha_i}
$$

be the $D_l(b_{\frac{l}{2}-1})$-associated element, where $D_l(b_{\frac{l}{2}-1})$ is the cycle with $l = 4k$, see Figure 25*. The element w is conjugate to the element*

$$
(\prod_{i=1}^{2k} s_{\beta_i}) s_{\theta(\beta_{k+1},\beta_k)} (\prod_{i=2}^{2k} s_{\alpha_i}). \tag{3.22}
$$

Proof. First, we have

$$
\begin{aligned}
w = \prod s_{\beta_i} \prod s_{\alpha_j} &\overset{s_{\alpha_1}}{\simeq} s_{\alpha_1} (s_{\beta_1} s_{\beta_{2k}}) \prod_{i \neq 1, 2k} s_{\beta_i} \prod_{j \neq 1} s_{\alpha_j}\\
&= s_{\beta_1} s_{\beta_{2k}} s_{\theta(\beta_{2k},\beta_1)} \prod_{i \neq 1, 2k} s_{\beta_i} \prod_{j \neq 1} s_{\alpha_j}.
\end{aligned}
$$

By relations (3.20), the elements $s_{\theta(\beta_1,\beta_{2k})}$ and $\prod\limits_{i\neq 1,2k} s_{\beta_i}$ commute, and we have:

$$w = s_{\beta_1}s_{\beta_{2k}}(\prod_{i\neq 1,2k} s_{\beta_i})s_{\theta(\beta_{2k},\beta_1)}\prod_{j\neq 1} s_{\alpha_j} = (\prod s_{\beta_i})s_{\theta(\beta_{2k},\beta_1)}(\prod_{j\neq 1} s_{\alpha_j}).$$

Further, we use Lemma 3.2 to prove the equivalences:

$$\begin{aligned}
w &= \prod s_{\beta_i}\Big(\prod_{j\neq 1} s_{\alpha_j}\Big)s_{\theta(\alpha_{2k},\alpha_2)} \overset{s_{\theta(\alpha_{2k},\alpha_2)}}{\simeq} s_{\theta(\alpha_{2k},\alpha_2)}\prod s_{\beta_i}\prod_{j\neq 1} s_{\alpha_j} \\
&= \Big(\prod s_{\beta_i}\Big)s_{\theta(\beta_{2k-1},\beta_2)}\prod_{j\neq 1} s_{\alpha_j} = \prod s_{\beta_i}\Big(\prod_{j\neq 1} s_{\alpha_j}\Big)s_{\theta(\alpha_{2k-1},\alpha_3)} \\
&\overset{s_{\theta(\alpha_{2k-1},\alpha_3)}}{\simeq} s_{\theta(\alpha_{2k-1},\alpha_3)}\prod s_{\beta_i}\prod_{j\neq 1} s_{\alpha_j} = \Big(\prod s_{\beta_i}\Big)s_{\theta(\beta_{2k-2},\beta_3)}\prod_{j\neq 1} s_{\alpha_j} \\
&\dots \\
&= \Big(\prod s_{\beta_i}\Big)s_{\theta(\beta_{k+1},\beta_k)}\prod_{j\neq 1} s_{\alpha_j}.
\end{aligned}$$

The relation (3.22) is proved. □

Proposition 3.4. *The conjugacy class containing elements*

$$\prod_{i=1}^{2k} s_{\beta_i}\prod_{i=1}^{2k} s_{\alpha_i} \simeq (\prod_{i=1}^{2k} s_{\beta_i})(s_{\theta(\beta_{k+1},\beta_k)}\prod_{i=2}^{2k} s_{\alpha_i}) \qquad (3.23)$$

is $D_l(a_{\frac{l}{2}-1})$-associated (as well $D_l(b_{\frac{l}{2}-1})$-associated) conjugacy class for $l = 4k$, see Figure 26.

Proof. For $i \neq k+1$, the orthogonality $\theta(\beta_{k+1},\beta_k) \perp \alpha_i$ follows from (3.20). For $i = k+1$, it is easy to check:

$$(\theta(\beta_{k+1},\beta_k),\alpha_{k+1}) = (\beta_{k+1},\alpha_{k+1}) - (\beta_k,\alpha_{k+1}) = -\frac{1}{2}+\frac{1}{2} = 0.$$

Besides, for $i \neq k, k+1$, we have $\theta(\beta_{k+1},\beta_k) \perp \beta_i$, see (3.20). Finally, for $i = k, k+1$, we have:

$$\begin{aligned}
(\theta(\beta_{k+1},\beta_k),\beta_k) &= (-\beta_k,\beta_k) + (-\alpha_k,\beta_k) = -1+\frac{1}{2} = -\frac{1}{2}, \\
(\theta(\beta_{k+1},\beta_k),\beta_{k+1}) &= (\beta_{k+1},\beta_{k+1}) + (\alpha_{k+2},\beta_{k+1}) = 1-\frac{1}{2} = \frac{1}{2}. \quad \square
\end{aligned}$$

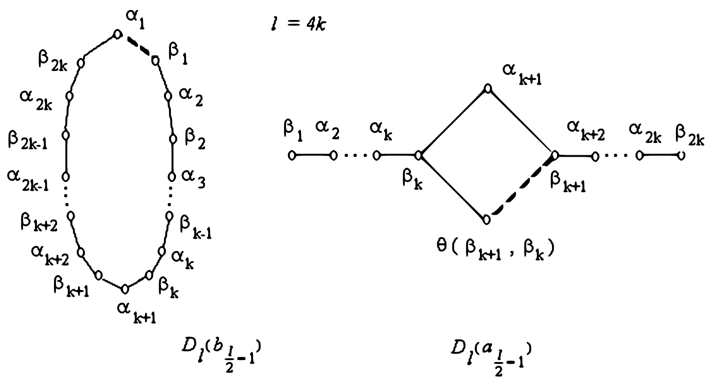


FIGURE 26. The equivalent diagrams $D_l(b_{\frac{l}{2}-1})$ and $D_l(a_{\frac{l}{2}-1})$, where $l = 4k$

3.4.2. The case $l = 4k - 2$. Similarly to the case (3.17), we consider chains

$$\begin{aligned} &\{\beta_L, \beta_R\}, \quad L = 2k - i, R = i, \quad 1 \leqslant i \leqslant k-1, \text{ or} \\ &\{\alpha_L, \alpha_R\}, \quad L = 2k - i + 1, R = i, \quad 2 \leqslant i \leqslant k. \end{aligned} \tag{3.24}$$

Next, we consider the following vectors associated with the chains (3.24):

$$\begin{aligned} \mu(\beta_L, \beta_R) =& \alpha_1 - \sum_{i=1}^{R} \beta_i - \sum_{i=2}^{R} \alpha_i + \sum_{i=L}^{2k-1} \beta_i + \sum_{i=L+1}^{2k-1} \alpha_i, \quad R + L = 2k, \\ \mu(\alpha_L, \alpha_R) =& \alpha_1 - \sum_{i=1}^{R-1} \beta_i - \sum_{i=2}^{R} \alpha_i + \sum_{i=L}^{2k-1} \beta_i + \sum_{i=L}^{2k-1} \alpha_i, \quad R + L = 2k + 1. \end{aligned} \tag{3.25}$$

As above in operations (3.19), vectors $\mu(\beta_L, \beta_R)$, $\mu(\alpha_L, \alpha_R)$ from (3.25) are roots.

Lemma 3.5. *The following commutation relations hold:*

$$\begin{aligned} s_{\mu(\beta_L,\beta_R)} \prod_{i=1}^{2k-1} s_{\alpha_i} &= \left(\prod_{i=1}^{2k-1} s_{\alpha_i} \right) s_{\mu(\alpha_L,\alpha_{R+1})}, \quad L + R = 2k, \ R \leqslant k - 1, \\ s_{\mu(\alpha_L,\alpha_R)} \prod_{i=1}^{2k-1} s_{\beta_i} &= \left(\prod_{i=1}^{2k-1} s_{\beta_i} \right) s_{\mu(\beta_{L-1},\beta_R)}, \quad L + R = 2k + 1, \ R \leqslant k. \end{aligned} \tag{3.26}$$

Proof is as in Lemma 3.2.

Proposition 3.6. *Let*

$$w = w_\beta w_\alpha = \prod_{i=1}^{2k-1} s_{\beta_i} \prod_{i=1}^{2k-1} s_{\alpha_i}$$

be the $D_l(b_{\frac{l}{2}-1})$*-associated element, where* $D_l(b_{\frac{l}{2}-1})$ *is the cycle with* $l = 4k-2$*, see Figure* 25*. The element* w *is conjugate to the element*

$$s_{\mu(\alpha_{k+1},\alpha_k)}\Bigg(\prod_{i=1}^{2k-1} s_{\beta_i}\Bigg)\Bigg(\prod_{i=2}^{2k-1} s_{\alpha_i}\Bigg). \tag{3.27}$$

Proof. As in Proposition 3.3, we have

$$\begin{aligned}
w = \prod s_{\beta_i} \prod s_{\alpha_j} &\overset{s_{\alpha_1}}{\simeq} s_{\alpha_1}(s_{\beta_1}s_{\beta_{2k-1}}) \prod_{i\neq 1,2k-1} s_{\beta_i} \prod_{j\neq 1} s_{\alpha_j} \\
&= s_{\beta_1}s_{\beta_{2k-1}}s_{\mu(\beta_{2k-1},\beta_1)} \prod_{i\neq 1,2k-1} s_{\beta_i} \prod_{j\neq 1} s_{\alpha_j} \\
&= s_{\beta_1}s_{\beta_{2k-1}}\Bigg(\prod_{i\neq 1,2k-1} s_{\beta_i}\Bigg)s_{\mu(\beta_{2k-1},\beta_1)} \prod_{j\neq 1} s_{\alpha_j} \\
&= \Bigg(\prod s_{\beta_i}\Bigg)s_{\mu(\beta_{2k-1},\beta_1)}\Bigg(\prod_{j\neq 1} s_{\alpha_j}\Bigg).
\end{aligned}$$

By Lemma 3.5, we have:

$$\begin{aligned}
w &= \prod s_{\beta_i}\Bigg(\prod_{j\neq 1} s_{\alpha_j}\Bigg)s_{\mu(\alpha_{2k-1},\alpha_2)} \overset{s_{\mu(\alpha_{2k-1},\alpha_2)}}{\simeq} s_{\mu(\alpha_{2k-1},\alpha_2)} \prod s_{\beta_i} \prod_{j\neq 1} s_{\alpha_j} \\
&= \Bigg(\prod s_{\beta_i}\Bigg)s_{\mu(\beta_{2k-2},\beta_2)} \prod_{j\neq 1} s_{\alpha_j} = \prod s_{\beta_i}\Bigg(\prod_{j\neq 1} s_{\alpha_j}\Bigg)s_{\mu(\alpha_{2k-2},\alpha_3)} \\
&\overset{s_{\mu(\alpha_{2k-2},\alpha_3)}}{\simeq} s_{\mu(\alpha_{2k-2},\alpha_3)} \prod s_{\beta_i} \prod_{j\neq 1} s_{\alpha_j} = \Bigg(\prod s_{\beta_i}\Bigg)s_{\mu(\beta_{2k-3},\beta_3)} \prod_{j\neq 1} s_{\alpha_j} \\
&\dots \\
&= s_{\mu(\alpha_{k+1},\alpha_k)} \prod s_{\beta_i} \prod_{j\neq 1} s_{\alpha_j}.
\end{aligned}$$

□

Proposition 3.7. *The conjugacy class containing elements*

$$\prod_{i=1}^{2k-1} s_{\beta_i} \prod_{i=1}^{2k-1} s_{\alpha_i} \simeq \Bigg(s_{\mu(\alpha_{k+1},\alpha_k)}\Bigg(\prod_{i=1}^{2k-1} s_{\beta_i}\Bigg) \prod_{i=2}^{2k-1} s_{\alpha_i}\Bigg). \tag{3.28}$$

is $D_l(a_{\frac{l}{2}-1})$*-associated (as well* $D_l(b_{\frac{l}{2}-1})$*-associated) conjugacy class for* $l = 4k-2$*, see Figure* 27*.*

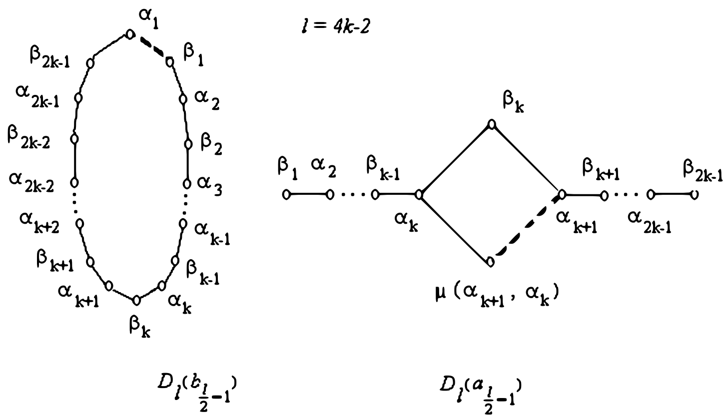


FIGURE 27. The equivalent diagrams $D_l(b_{\frac{l}{2}-1})$ and $D_l(a_{\frac{l}{2}-1})$, where $l = 4k-2$

For $i \neq k$, the orthogonality $\mu(\alpha_{k+1}, \alpha_k) \perp \beta_i$ follows from (3.25). For $i = k$, we have:

$$(\mu(\alpha_{k+1}, \alpha_k), \beta_k) = (\alpha_{k+1}, \beta_k) - (\alpha_k, \beta_k) = -\frac{1}{2} + \frac{1}{2} = 0.$$

For $i \neq k, k+1$, we have $\mu(\alpha_{k+1}, \alpha_k) \perp \alpha_i$, and, for $i = k, k+1$, we get:

$$(\mu(\alpha_{k+1}, \alpha_k), \alpha_k) = (-\beta_{k-1}, \alpha_k) + (-\alpha_k, \alpha_k) = \frac{1}{2} - 1 = -\frac{1}{2},$$
$$(\mu(\alpha_{k+1}, \alpha_k), \alpha_{k+1}) = (\beta_{k+1}, \alpha_{k+1}) + (\alpha_{k+1}, \alpha_{k+1}) = 1 - \frac{1}{2} = \frac{1}{2}.$$

Appendix A. More about cycles

A.1. The ratio of lengths of roots

Let Γ be a Dynkin diagram, and $\sqrt{t}$ be the ratio of the length of any long root to the length of any short root. The inner product between two long roots is

$$(\alpha, \beta) = \sqrt{t} \cdot \sqrt{t} \cdot \cos(\widehat{\alpha, \beta}) = \sqrt{t} \cdot \sqrt{t} \cdot (\pm \frac{1}{2}) = \pm\frac{t}{2}.$$

By Remark 1.3, we may put $(\alpha, \beta) = -\dfrac{t}{2}$. The inner product between two short roots is

$$(\alpha, \beta) = \cos(\widehat{\alpha, \beta}) = \pm\frac{1}{2}.$$

Again, by Remark 1.3, we may put $(\alpha, \beta) = -\dfrac{1}{2}$. The inner product (α, β) between roots of different lengths is

$$(\alpha, \beta) = 1 \cdot \sqrt{t} \cdot \cos(\widehat{\alpha, \beta}) = 1 \cdot \sqrt{t} \cdot \big(\pm \frac{\sqrt{t}}{2}\big) = \pm \frac{t}{2}.$$

As above, we choose the obtuse angle and put $(\alpha, \beta) = -\dfrac{t}{2}$.

We can summarize:

$$(\alpha, \beta) = \begin{cases} -\frac{1}{2} & \text{for} \quad \|\alpha\| = \|\beta\| = 1, \\ \\ -1 & \text{for} \quad \|\alpha\| = \|\beta\| = 2, \quad \text{or} \quad \|\alpha\| = 1,\ \|\beta\| = 2, \\ \\ -\frac{3}{2} & \text{for} \quad \|\alpha\| = \|\beta\| = 3, \quad \text{or} \quad \|\alpha\| = 1,\ \|\beta\| = 3, \end{cases} \qquad \text{(A.1)}$$

where all angles $\widehat{\alpha, \beta}$ are obtuse.

A.2. Cycles in the simply-laced case

A.2.1. The Carter and connection diagrams for trees.

Lemma A.1. *There is no root subset (in the root system associated with a Dynkin diagram) forming a simply-laced cycle containing only solid edges. Every cycle in the Carter diagram or in the connection diagram contains at least one solid edge and at least one dotted edge.*

Proof. Suppose a subset $S = \{\alpha_1, \dots, \alpha_n\} \subset \varPhi$ forms a cycle containing only solid edges. Consider the vector

$$v = \sum_{i=1}^{n} \alpha_i.$$

The value of the quadratic Tits form $\mathscr{B}$ (see [St08]) on v is equal to

$$\mathscr{B}(v) = \sum_{i \in \Gamma_0} 1 - \sum_{i \in \Gamma_1} 1 = n - n = 0,$$

where Γ_0 (resp. Γ_1) is the set of all vertices (resp. edges) of the diagram Γ associated with S. Therefore, $v = 0$ and elements of the root subset S are linearly dependent. □

The following proposition is true only for trees.

Proposition A.2 (Lemma 8, [Ca72])**.** *Let $S = \{\alpha_1, \dots, \alpha_n\}$ be a subset of linearly independent (not necessarily simple) roots of the root system Φ associated with a certain Dynkin diagram Γ, and Γ_S the Carter diagram or the connection diagram associated with S. If Γ_S is a tree, then Γ_S is a Dynkin diagram.*

Proof. If Γ_S is not a Dynkin diagram, then Γ_S contains an extended Dynkin diagram $\widetilde{\Gamma}$ as a subdiagram. Since Γ_S is a tree, we can turn all dotted edges to solid ones[6], see Remark 1.3.

Further, we consider the vector

$$v = \sum_{i \in \widetilde{\Gamma}_0} t_i \alpha_i, \tag{A.2}$$

where $\widetilde{\Gamma}_0$ is the set of all vertices of $\widetilde{\Gamma}$, and t_i (where $i \in \widetilde{\Gamma}_0$) are the coefficients of the nil-root, see [Kac80]. Let the remaining coefficients corresponding to $\Gamma_S \backslash \widetilde{\Gamma}$ be equal to 0. Let $\mathscr{B}$ be the positive definite quadratic Tits form (see [St08]) associated with the diagram Γ, and $(\cdot\,, \cdot)$ the symmetric bilinear form associated with $\mathscr{B}$. Let $\{\delta_i \mid i \in \widetilde{\Gamma}_0\}$ be the set of simple roots associated with vertices $\widetilde{\Gamma}_0$. For all $i, j \in \widetilde{\Gamma}_0$, we have $(\alpha_i, \alpha_j) = (\delta_i, \delta_j)$, since this value is described by edges of $\widetilde{\Gamma}$. Therefore,

$$\mathscr{B}(v) = \sum_{i,j \in \widetilde{\Gamma}_0} t_i t_j (\alpha_i, \alpha_j) = \sum_{i,j \in \widetilde{\Gamma}_0} t_i t_j (\delta_i, \delta_j) = \mathscr{B}(\sum_{i \in \widetilde{\Gamma}_0} t_i \delta_i) = 0.$$

Since $\mathscr{B}$ is a positive definite form, we have $v = 0$, i.e., vectors α_i are linearly dependent. This contradicts the definition of the set S. □

Example A.3 (multiply-laced cases)**.** Let the coefficients of linear dependence be as in the proof of Proposition A.2. The labels at vertices are coordinates of the nil-root of the corresponding extended Dynkin diagrams, [Kac80]. In all cases below, inner products are calculated in accordance with § A.1 and (A.1). In all these calculations except for $\widetilde{G}_{21}$ and $\widetilde{G}_{22}$, $t = 2$.

Case $\widetilde{F}_{41}$. We have $v = \alpha + 2\beta + 3\gamma + 2\delta + \varphi$. Then

$$\begin{aligned} \|v\| &= 1 + 4 + 9 + 4t + t - 1 \cdot 2 - 2 \cdot 3 - 3 \cdot 2t - 2 \cdot t \\ &= 6 - 3t = 0. \end{aligned}$$

[6] This fact is not true for cycles, since by Lemma A.1 we cannot eliminate all dotted edges.

Case $\widetilde{F}_{42}$. Here, $v = \alpha + 2\beta + 3\gamma + 4\delta + 2\varphi$, and

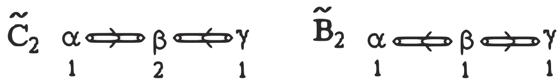
$$\begin{aligned}\|v\| = (1+4+9)t + 16 + 4 - 1\cdot 2t - 2\cdot 3t \\ - 3\cdot 4t - 4\cdot 2 = 12 - 6t = 0.\end{aligned}$$

Case $\widetilde{C}_2$. We have $v = \alpha + t\beta + \gamma$, where $t = 2$. Then

$$\|v\| = t + t^2 + t - t\cdot t - t\cdot t = 2t - t^2 = 0.$$

Case $\widetilde{B}_2$. In this case, $v = \alpha + \beta + \gamma$. Then

$$\|v\| = 1 + 1 + t - t - t = 2 - t = 0.$$



Case $\widetilde{C}_3$. Here, $v = \alpha + t\beta + t\gamma + \delta$, and

$$\|v\| = t + t + t^2 + t^2 - t^2 - t^2 - t^2 = 2t - t^2 = 0.$$

Case $\widetilde{B}_3$. We have $v = \alpha + \beta + \gamma + \delta$. Then

$$\|v\| = 1 + 1 + t + t - t - t - t = 2 - t = 0.$$

For $\widetilde{C}_n$, where $n \geqslant 4$, we have $v = \alpha_1 + t\alpha_2 + \cdots + t\alpha_n + \alpha_{n+1}$. Any new short edge adds $t^2 - t^2$, i.e., $\|v\| = 0$. For $\widetilde{B}_n$, where $n \geqslant 4$, we have $v = \alpha_1 + \alpha_2 + \cdots + \alpha_n + \alpha_{n+1}$. Any new long edge adds $t - t$, i.e., $\|v\| = 0$.
Case $\widetilde{G}_{21}$. Here, $v = \alpha + 2\beta + \gamma$. Here, $t = 3$. Then

$$\|v\| = 1 + 4 + t - 1\cdot 2 - 2\cdot t = 3 - t = 0.$$

Case $\widetilde{G}_{22}$. In this case, $v = \alpha + 2\beta + 3\gamma$. Again, $t = 3$. Then

$$\|v\| = t + 4\cdot t + 9 - 2\cdot t - 2\cdot 3\cdot t = 9 - 3t = 0. \qquad \square$$

A.2.2. There are no cycles in the root system A_n. Recall that any root in A_n is of the form $\pm(e_i - e_j)$, where $1 \leqslant i < j \leqslant n+1$. Then, up to the similarity transformation, $\alpha \longmapsto -\alpha$, see § 1.3.1, a cycle of roots is of one of the followings forms:

$$\{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, \dots, e_{i_{k-1}} - e_{i_k}, e_{i_k} - e_{i_1}\},$$
$$\{e_{i_1} - e_{i_2}, e_{i_2} - e_{i_3}, \dots, e_{i_{k-1}} - e_{i_k}, -(e_{i_k} - e_{i_1})\}.$$

In the first case, the sum of all these roots is equal to 0, i.e., these roots are linearly dependent. In the second case, the sum of the $k-1$ first roots is equal to the last one, and roots are also linearly dependent. Thus, for A_n, there are no cycles of linearly independent roots.

A.3. Cycles in the multiply-laced case

A.3.1. There are no 4-cycles with all angles obtuse. No root system R containing a 4-cycle with all angles obtuse can occur. Suppose this is possible, so a quadruple of roots $\{\alpha, \beta, \gamma, \delta\}$ yields pairs with the following values of the Tits form:

$$(\alpha, \beta) = -1, \quad (\beta, \gamma) = -\frac{1}{2}, \quad (\gamma, \delta) = -1, \quad (\delta, \alpha) = -1,$$

see Figure 28.

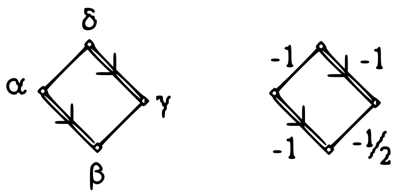


FIGURE 28.

Then we see that

$$s_\alpha = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad s_\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$s_\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad s_\delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix}.$$

Then the semi-Coxeter element $\mathbf{C} = s_\alpha s_\beta s_\gamma s_\delta$ in the Weyl group generated by the quadruple $\{s_\alpha, s_\beta, s_\gamma, s_\delta\}$, and its characteristic polynomial is as follows:

$$\mathbf{C} = s_\alpha s_\beta s_\gamma s_\delta = \begin{pmatrix} 4 & 0 & 2 & -3 \\ 4 & 0 & 1 & -2 \\ 2 & 1 & 1 & -2 \\ 1 & 0 & 1 & -1 \end{pmatrix}, \qquad \chi(\mathbf{C}) = x^4 - 4x^3 - x^2 - 4x + 1.$$

Since the maximal root of $\chi(\mathbf{C})$ is $\lambda \approx 4.419 > 1$, then the semi-Coxeter element $\mathbf{C}$ is of infinite order, but this is impossible.

A.3.2. More of impossible cases of multiply-laced cycles. We consider several patterns (of multiply-laced diagrams) that are not a part of any Carter diagram. First of all, the arrows on the double edges connecting roots of different lengths should be directed face to face, otherwise we have 3 different lengths of roots, as depicted in Figure 29:

$$\|\delta\| > \|\gamma\| = \|\beta\| > \|\alpha\|.$$

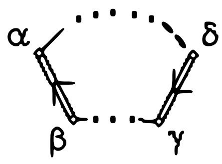


FIGURE 29.

Further, two double edges connecting roots of different lengths cannot be adjacent, as depicted in Figure 30. Otherwise, the root subset contains the extended Dynkin diagram $\widetilde{B}_2$ or $\widetilde{C}_2$ which cannot occur.

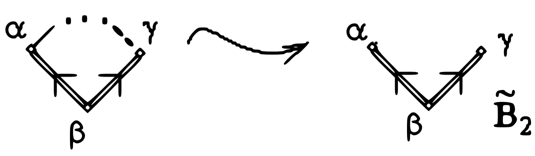


FIGURE 30.

For cycles of length 5 or more, the diagram contains the extended Dynkin diagram of type $\widetilde{B}_n$ or $\widetilde{C}_n$ which cannot happen. If the acute angle (resp. the dotted edge) lies on the part corresponding to $\widetilde{B}_n$ (or $\widetilde{C}_n$), this obstacle can be easily eliminated by changing certain roots with their opposites; the procedure of eliminating the acute angle may be applied to any tree regardless of whether it contains roots of different lengths or not.

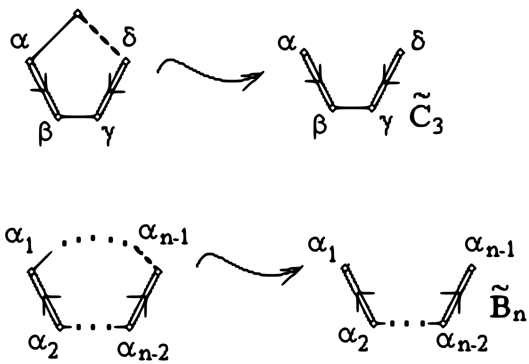


FIGURE 31.

There are no "kites", i.e., cycles of length 4 with an additional fifth edge, since any such subset contains the extended Dynkin diagram $\widetilde{CD}_2$ or $\widetilde{DD}_2$ which cannot be, see Figure 32. One should note that every cycle ins the Carter diagram contains, by definition, an even number of vertices, so the connection like $\{\varphi, \alpha\}$ or $\{\varphi, \beta\}$ forming a triangle cannot occur.

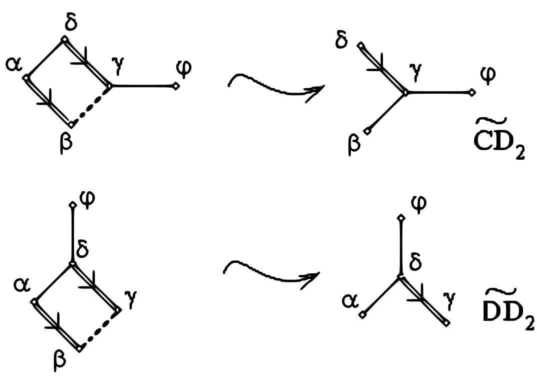


FIGURE 32.

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