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RESEARCH ARTICLE

Modules in which every surjective endomorphism has a δ-small kernel

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Abstract. In this paper, we introduce the notion of δ-Hopfian modules. We give some properties of these modules and provide a characterization of semisimple rings in terms of δ-Hopfian modules by proving that a ring R is semisimple if and only if every R-module is δ-Hopfian. Also, we show that for a ring R, $\delta(R) = J(R)$ if and only if for all R-modules, the conditions δ-Hopfian and generalized Hopfian are equivalent. Moreover, we prove that δ-Hopfian property is a Morita invariant. Further, the δ-Hopficity of modules over truncated polynomial and triangular matrix rings are considered.

Introduction

Throughout rings will have unity and modules will be unitary. Let M denote a right module over a ring R. The study of modules by properties of their endomorphisms has long been of interest. The concept of Hopfian groups was introduced by Baumslag in 1963 ([2]). In [8], Hiremath generalized this concept to general module theoretic setting. A right R-module M is called Hopfian, if any surjective endomorphism of M is an isomorphism. Later, the dual concept of Hopfian modules (co-Hopfian modules) was introduced. Hopfian and co-Hopfian modules (rings) have been investigated by several authors [4], [7], [14], [15] and [17]. Direct finiteness (or

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Dedekind finiteness) evolved from the concepts of "finite projections" in operator algebras and "finite idempotents" in Baer rings (see [5, p. 74] and [9, p.10]). A module M is called *Dedekind-finite*, if $X = 0$ is the only module for which $M \cong M \oplus X$. Equivalently, $fg = 1$ implies $gf = 1$ for each $f, g \in \mathrm{End}(M)$.

In [4], a proper generalization of Hopfian modules, called generalized Hopfian modules, was given. A right R-module M is called generalized Hopfian, if any surjective endomorphism of M has a small kernel. Recall that a submodule N of a module M is called small, denoted by $N \ll M$, if $N + X \neq M$ for all proper submodules X of M. In [4, Corollary 1.4], it is shown that the concepts of Dedekind finite modules, Hopfian modules and generalized Hopfian modules coincide for every (quasi-)projective module.

A submodule N of a module M is called δ-small in M, written $N \ll_\delta M$, provided $N + K \neq M$ for any proper submodule K of M with M/K singular (see [18]). In this paper, we introduce and study the notion of δ- Hopfian modules, which is a generalization of Hopfian modules and generalized Hopfian modules. We replace "small kernel" by "δ-small kernel". We discuss the following questions: When does a module have the property that every of its surjective endomorphisms has a δ-small kernel? Further, how can δ-Hopfian modules be used to characterize the base ring itself?

We summarize the contents of this article as follows. In Section 2, we give some equivalent properties and characterizations of δ-Hopfian modules. We characterize semisimple rings in terms of δ-Hopfian modules and show that a ring R is semisimple if and only if every R-module is δ-Hopfian. We prove that for a ring R, $\delta(R) = J(R)$ if and only if for all R-modules, the conditions δ-Hopfian and generalized Hopfian are equivalent. It is shown that a direct sum of δ-Hopfian modules and their endomorphism rings need not have the same property. Also, we prove that δ-Hopfian property is a Morita invariant.

In Section 3, we consider the δ-Hopfian property of $M[x]$ (as an $R[x]$-module) and $M[x]/(x^{n+1})$ (as an $R[x]/(x^{n+1})$-module). We characterize the structures of maximal submodules, essential submodules of $M[x]/(x^{n+1})$. Also, we show that

$$\delta(M[x]/(x^{n+1})) = \mathrm{Rad}(M) + Mx + Mx^2 + \cdots + Mx^n.$$

Moreover, we prove that if $M[x]/(x^{n+1})$ is δ-Hopfian as an $R[x]/(x^{n+1})$-module, then M is δ-Hopfian, but the converse is not true.

In Section 3, we characterize the generalized triangular matrix rings which are right δ-Hopfian and prove that if M is an (S,R)-bimodule, and

$T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$, then $\delta(T_T) = \begin{pmatrix} H & M \\ 0 & \delta(R_R) \end{pmatrix}$, where,

$$H = \delta(S_S) \cap \{I \colon I \text{ is a maximal right ideal of S with } \operatorname{ann}_S(M) \subseteq I\}.$$

At the end of the paper, some open problems are given.

We now fix our notations and state a few well known preliminary results that will be needed. Let M be a right R-module. For submodules N and K of M, $N \leqslant K$ denotes that N is a submodule of K, $\operatorname{Rad}(M)$ denotes the Jacobson radical of M and $\operatorname{End}(M)$ denotes the ring of endomorphisms of M. By $N \leqslant^{\text{ess}} M$, we mean that N is an essential submodule of M. Also, for a module M and a set Λ, let $M^{(\Lambda)}$ denote the direct sum of $|\Lambda|$ copies of M, where $|\Lambda|$ is the cardinality of Λ. The symbols $J(R)$, $M_n(R)$ and $T_n(R)$ denote the Jacobson radical of R, the full ring of n-by-n matrices over R, and the ring of n-by-n upper triangular matrices over R, respectively. As in [18], we define $\delta(M)$ to be

$$\operatorname{Rej}_{\mathbf{P}}(M) = \cap\{N \leqslant M \colon M/N \in \mathbf{P}\},$$

where $\mathbf{P}$ is the class of all singular simple modules.

Recall that a ring R is said to satisfy the rank condition if a right R-epimorphism $R^m \to R^n$ can exist only when $m \geqslant n$ (see [10]).

Lemma 1 ([18, Lemma 1.2]). *Let N be a submodule of M. The following are equivalent:*

(1) $N \ll_\delta M$.
(2) *If $X + N = M$, then $M = X \oplus Y$ for a projective semisimple submodule Y with $Y \subseteq N$;*
(3) *If $X + N = M$ with M/X Goldie torsion, then $X = M$.*

Lemma 2 ([18, Lemma 1.3]). *Let M be a module.*

(1) *For submodules N, K, L of M with $K \subseteq N$, we have*
    (a) *$N \ll_\delta M$ if and only if $K \ll_\delta M$ and $N/K \ll_\delta M/K$.*
    (b) *$N + L \ll_\delta M$ if and only if $N \ll_\delta M$ and $L \ll_\delta M$.*
(2) *If $K \ll_\delta M$ and $f \colon M \to N$ is a homomorphism, then $f(K) \ll_\delta N$.*
(3) *Let $K_1 \leqslant M_1 \leqslant M$, $K_2 \leqslant M_2 \leqslant M$, and $M = M_1 \oplus M_2$. Then $K_1 \oplus K_2 \ll_\delta M_1 \oplus M_2$ if and only if $K_1 \ll_\delta M_1$ and $K_2 \ll_\delta M_2$.*

Lemma 3 ([18, Lemma 1.5, Theorem 1.6]). *Let R be a ring and M an R-module. Then $\delta(M) = \sum\{L \leqslant M \colon L \ll_\delta M\}$ and $\delta(R)$ equals the intersection of all essential maximal right ideals of R.*

Lemma 4 ([3, 5, 10]). *Let* $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$, *where* R *and* S *are rings and* M *is an (S,R)-bimodule.*

(1) *Every right ideal of* T *has the form* $\begin{pmatrix} I & N \\ 0 & J \end{pmatrix}$ *such that* $I \leqslant S_S$, $J \leqslant R_R$ *and* $IM \subseteq N$.

(2) *Let* $Q = \begin{pmatrix} I & N \\ 0 & J \end{pmatrix}$. *Then* Q *is a maximal right ideal of* T *if and only if* $N = M$ *and either* $I = S$ *and* J *is a maximal right ideal of* R *or* $J = R$ *and* I *is a maximal right ideal of* S.

(3) *The right ideal* $Q = \begin{pmatrix} I & N \\ 0 & J \end{pmatrix}$ *of* T *is essential in* T_T *if and only if* $N \leqslant^{\text{ess}} M_R$, $J \leqslant^{\text{ess}} R_R$ *and* $I \cap (\text{ann}_S(M)) \leqslant^{\text{ess}} \text{ann}_S(M)$.

1. δ-Hopfian modules

Motivated by the definition of generalized Hopfian modules, we introduce the key definition of this paper.

Definition 1. Let M be an R-module. We say that M is δ-Hopfian (δH for short) if any surjective R-endomorphism of M has a δ-small kernel in M.

The next result gives several equivalent conditions for a δH module.

Theorem 1. *Let* M *be an* R*-module. The following statements are equivalent:*

(1) M *is* δH;

(2) *For any epimorphism* $f \colon M \to M$, *if* $N \ll_\delta M$, *then* $f^{-1}(N) \ll_\delta M$;

(3) *If* $N \leqslant M$ *and there is an* R*-epimorphism* $M/N \to M$, *then* $N \ll_\delta M$;

(4) *If* M/N *is nonzero and singular for some* $N \leqslant M$, *then* $f(N) \neq M$, *for each* R*-surjective endomorphism* f *of* M;

(5) *There exists a fully invariant* δ*-small submodule* N *of* M *such that* M/N *is* δH;

(6) *If* $f \colon M \to M \oplus X$ *is an epimorphism, where* X *is a module, then* X *is projective and semisimple.*

Proof. (1) $\Rightarrow$ (2) Assume that $f \colon M \to M$ is an epimorphism and $N \ll_\delta M$. Let $f^{-1}(N) + K = M$ for some $K \leqslant M$, where M/K is singular. Hence $N + f(K) = M$. As M/K is singular and $M/f(K)$ is an image of M/K, $M/f(K)$ is singular. Hence $N + f(K) = M$ and $N \ll_\delta M$, giving

$f(K) = M$. So $K + \mathrm{Ker}(f) = M$. Since M is δH, $\mathrm{Ker}(f) \ll_\delta M$. Hence M/K is singular implies that $K = M$. Thus $f^{-1}(N) \ll_\delta M$.

(2) $\Rightarrow$ (3) Let $f\colon M/N \to M$ be an epimorphism. It is clear that $N \leqslant (f\pi)^{-1}(0)$, where $\pi\colon M \to M/N$ is the canonical epimorphism. By (2), $(f\pi)^{-1}(0) \ll_\delta M$, Hence by Lemma 2, $N \ll_\delta M$.

(3) $\Rightarrow$ (4) Let N be a proper submodule of M such that M/N is singular and f a surjective endomorphism of M with $f(N) = M$. Then $N + \mathrm{Ker}(f) = M$. Hence $\mathrm{Ker}(f) \ll_\delta M$ by (3), and so $N = M$, a contradiction.

(4) $\Rightarrow$ (1) Let $f\colon M \to M$ be an epimorphism. If $M = N + \mathrm{Ker}(f)$, with M/N is singular, then $M = f(M) = f(N)$. Hence $N = M$ by (4). Thus $\mathrm{Ker}(f) \ll_\delta M$.

(1) $\Rightarrow$ (5) Take $N = 0$.

(5) $\Rightarrow$ (1) Let M/N be δH for some fully invariant δ-small submodule N of M. If $f\colon M \to M$ is an epimorphism, then $\bar{f}\colon M/N \to M/N$ with $\bar{f}(m+N) = f(m) + N$ $(m \in M)$ is an epimorphism. As M/N is δH, $\ker(\bar{f}) \ll_\delta M/N$. Since $(\mathrm{Ker}(f) + N)/N \subseteq \ker(\bar{f}) \ll_\delta M/N$, $\mathrm{Ker}(f) + N \ll_\delta M$ by Lemma 2. Hence $\mathrm{Ker}(f) \ll_\delta M$ by Lemma 2, and so M is δH.

(1) $\Rightarrow$ (6) Let $f\colon M \to M \oplus X$ be an epimorphism, $\pi\colon M \oplus X \to M$ the natural projection. It is clear that $\mathrm{Ker}(\pi f) = f^{-1}(0 \oplus X)$. By (1), M is δH. Hence $\mathrm{Ker}(\pi f) \ll_\delta M$. Since f is an epimorphism, $f[f^{-1}(0 \oplus X)] = 0 \oplus X$. Hence by Lemma 2, $0 \oplus X = f(\mathrm{Ker}(\pi f)) \ll_\delta M \oplus X$. Therefore $X \ll_\delta X$ by Lemma 2. So, by Lemma 1, X is projective and semisimple.

(6) $\Rightarrow$ (1) Let f be a surjective endomorphism of M and $\mathrm{Ker}(f) + L = M$ for some $L \leqslant M$, where M/L is singular. Since

$$\begin{aligned} M/\mathrm{Ker}(f) \cap L &= \mathrm{Ker}(f)/(\mathrm{Ker}(f) \cap L) \oplus L/(\mathrm{Ker}(f) \cap L) \\ &\cong M/L \oplus M/\mathrm{Ker}(f) \cong M/L \oplus M, \end{aligned}$$

the epimorphism $M \to M \oplus M/L$ exists. By (6), M/L is semisimple and projective. As M/L is singular, $M/L = 0$. Thus $M = L$ and $\mathrm{Ker}(f) \ll_\delta M$. □

Corollary 1. *Let M be a δH module, $f \in \mathrm{End}(M)$ an epimorphism and $N \leqslant M$. Then $N \ll_\delta M$ if and only if $f(N) \ll_\delta M$ if and only if $f^{-1}(N) \ll_\delta M$. Moreover $\delta(M) = \sum_{N \ll_\delta M} f(N) = \sum_{N \ll_\delta M} f^{-1}(N)$.*

Proposition 1. *Let M be a δH module. If N is a direct summand of M, then N is δH.*

Definition 2. Let M and N be two R-modules. M is called δ-Hopfian (δH, for short) relative to N, if for each epimorphism $f\colon M \to N$, $\mathrm{Ker}(f) \ll_\delta M$.

In view of the above definition, an R-module M is δH if and only if M is δH relative to M.

In the following, we characterize the δ-Hopfian modules in terms of their direct summands and factor modules.

Proposition 2. *Let M and N be two R-modules. Then the following statements are equivalent:*

(1) *M is δH relative to N;*
(2) *for each $L \leqslant^\oplus M$, L is δH relative to N;*
(3) *for each $L \leqslant M$, M/L is δH relative to N.*

Proof. (1) ⇒ (2) Let $L \leqslant^\oplus M$ say $M = L \oplus K$, where $K \leqslant M$ and $f\colon L \to M$ an epimorphism. Let $\pi\colon M \to L$ be the natural projection. Then $f\pi\colon M \to N$ is an epimorphism and so $\mathrm{Ker}(f\pi) \ll_\delta M$ by (1). It is clear that $\mathrm{Ker}(f\pi) = \mathrm{Ker}(f) \oplus K$. Thus $\mathrm{Ker}(f\pi) = \mathrm{Ker}(f) \oplus K \ll_\delta M$. By Lemma 2(3), $\mathrm{Ker}(f) \ll_\delta L$.

(2) ⇒ (1) Take $L = M$.

(1) ⇒ (3) Let $L \leqslant M$ and $f\colon M/L \to N$ be an epimorphism. Then $f\pi\colon M \to N$ is an epimorphism, where $\pi\colon M \to M/L$ is the natural homomorphism. As $\mathrm{Ker}(f\pi) = \pi^{-1}(\mathrm{Ker}(f))$ and $\mathrm{Ker}(f\pi) \ll_\delta M$, $\pi(\mathrm{Ker}(f\pi)) = \mathrm{Ker}(f) \ll_\delta M/L$ by Lemma 2. Therefore M/L is δH relative to N.

(3) ⇒ (1) Take $L = 0$. □

In the following, we present some characterizations of projective δH modules.

Theorem 2. *Let M be a projective R-module. Then the following statements are equivalent:*

(1) *M is δH;*
(2) *If $f \in \mathrm{End}(M)$ has a right inverse, then $\mathrm{Ker}(f)$ is semisimple and projective;*
(3) *If $f \in \mathrm{End}(M)$ has a right inverse in $\mathrm{End}(M)$, then $\mathrm{Ker}(f) \ll_\delta M$;*
(4) *If $f \in \mathrm{End}(M)$ has a right inverse g, then $(1 - gf)M \ll_\delta M$.*

Proof. Let $f \in \mathrm{End}(M)$ be an epimorphism. Then there exists $g \in \mathrm{End}(M)$ such that $fg = 1 \in \mathrm{End}(M)$. It is clear that $\mathrm{Ker}(f) = (1-gf)M$ and $M = \mathrm{Ker}(f) \oplus (gf)M$.

(1) $\Rightarrow$ (2) Let $f \in \text{End}(M)$ have a right inverse in $\text{End}(M)$. Then $fg = 1$ for some $g \in \text{End}(M)$. Thus f is an epimorphism and so $\text{Ker}(f) \ll_\delta M$. As $\text{Ker}(f) = (1 - gf)M$ is a direct summand of M, it is semisimple and projective by Lemma 1.

(2) $\Rightarrow$ (3) Let $f \in \text{End}(M)$ have a right inverse in $\text{End}(M)$. Then by (2), $\text{Ker}(f)$ is semisimple and projective. We show that $\text{Ker}(f) \ll_\delta M$. Let $\text{Ker}(f) + L = M$ for some $L \leqslant M$. Since $\text{Ker}(f)$ is semisimple, $(\text{Ker}(f) \cap L) \oplus T = \text{Ker}(f)$ for some $T \leqslant \text{Ker}(f)$. Therefore $T \oplus L = M$. As T is semisimple and projective, $\text{Ker}(f) \ll_\delta M$, by Lemma 1.

(3) $\Rightarrow$ (4) Let $f \in \text{End}(M)$ have a right inverse g. Hence by (3), $\text{Ker}(f) = (1 - gf)M \ll_\delta M$.

(4) $\Rightarrow$ (1) Let $f \in \text{End}(M)$ be an epimorphism. As M is projective, $f \in \text{End}(M)$ has a right inverse g and $\text{Ker}(f) = (1 - gf)M$. Therefore by (4), $\text{Ker}(f) \ll_\delta M$ and M is δH. □

Next, we characterize the class of rings R for which every (free) R-module is δH.

Theorem 3. *Let R be a ring. Then the following statements are equivalent:*
(1) *Every R-module is δH;*
(2) *Every projective R-module is δH;*
(3) *Every free R-module is δH;*
(4) *R is semisimple.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) They are clear.

(3) $\Rightarrow$ (4) By (3), $R^{(\mathbb{N})}$ is δH. As $R^{(\mathbb{N})} \cong R^{(\mathbb{N})} \oplus R^{(\mathbb{N})}$, by Theorem 1, $R^{(\mathbb{N})}$ is semisimple. Hence R is semisimple.

(4) $\Rightarrow$ (1) Let M be an R-module. Hence M is projective and for each surjective endomorphism f of M, $\text{Ker}(f)$ is semisimple and projective. Hence by Theorem 2, M is δH. □

It is clear that every generalized Hopfian module is δH. The following example shows that the converse is not true, in general. Also, it shows that a δH module need not be Dedekind-finite.

Example 1. Let R be a semisimple ring. Then by Theorem 3, $M = R^{(\mathbb{N})}$ is a δH R-module. Since $R^{(\mathbb{N})} \cong R^{(\mathbb{N})} \oplus R^{(\mathbb{N})}$ and $R^{(\mathbb{N})} \neq 0$, M is not a gH (Dedekind-finite) module (see [4, Corollary 1.4]).

The following lemma gives a source of examples of δH modules.

Lemma 5. *Let M be a projective and semisimple R-module. Then M is δH.*

Proof. If M be a projective and semisimple R-module, then $M \ll_\delta M$, by [12, Lemma 2.9] and so every surjective endomorphism of M has a δ-small kernel. □

In the following, it is shown that every δH R-module is gH if and only if $\delta(R) = J(R)$.

Theorem 4. *Let R be a ring. Then the following statements are equivalent:*
(1) *The class of δH R-modules coincide with the class of gH R-modules;*
(2) *Every projective δH R-module is gH;*
(3) *Every maximal right ideal of R is essential in R_R;*
(4) *R has no non-zero semisimple projective R-module;*
(5) *$\delta(R) = J(R)$.*

Proof. (1) $\Rightarrow$ (2) Is clear.

(2) $\Rightarrow$ (3) Let $\mathbf{m}$ be a maximal right ideal of R. It is clear that either $\mathbf{m}$ is essential in R_R or a direct summand of R_R. If $\mathbf{m}$ is a direct summand of R_R, then $M = (R/\mathbf{m})^{(\mathbb{N})}$ is projective and semisimple. Hence M is δH by Lemma 5. Therefore by (2), M is gH. As $M \cong M \oplus M$, $M = 0$, by [4, Theorem 1.1]. This is a contradiction, and so $\mathbf{m}$ is essential in R_R.

(3) $\Rightarrow$ (4) Is clear.

(4) $\Rightarrow$ (1) Let M be a δH module and $f\colon M \to M \oplus X$ an epimorphism. Since M is δH, X is projective and semisimple by Theorem 1. Therefore $X = 0$, by (4), and so M is gH, by [4, Theorem 1.1].

(3) $\Rightarrow$ (5) Is clear.

(5) $\Rightarrow$ (3) Let R be a ring such that $\delta(R) = J(R)$. If $\mathbf{m}$ is a maximal right ideal of R such that $\mathbf{m} \leqslant^{\oplus} R_R$, say $R = \mathbf{m} \oplus \mathbf{m}'$ for some right ideal $\mathbf{m}'$ of R, then $\mathbf{m}' \subseteq \mathrm{Soc}(R) \subseteq \delta(R) \subseteq J(R) \subseteq \mathbf{m}$, a contradiction. Therefore every maximal right ideal of R is essential in R_R. □

Lemma 6. *Let R be a domain, which is not a division ring. Then $\delta(R) = J(R)$.*

Proof. Let $x \in \delta(R)$. Then $xR \ll_\delta R$. We show that $xR \ll R$. Let $xR + K = R$ for some $K \leqslant R_R$. By Lemma 1, there exists $Y \leqslant xR$ such that $Y \oplus K = R$. As R is a domain, $Y = R$ or $K = R$. If $Y = R$, then $xR = R$. Hence $\delta(R) = R$, therefore R is semisimple by [18, Corollary 1.7]. Hence R is a division ring, a contradiction. Therefore $K = R$ and so $xR \ll R$. It implies that $\delta(R) = J(R)$. □

A direct sum of δH modules need not be a δH module, as the following example shows.

Example 2. ([4, Remark 1.5], [10, Page 19, Exercise 18]) Let R be the K-algebra generated over a field K by $\{s, t, u, v, w, x, y, z\}$ with relations

$$sx + uz = 1, \quad sy + uw = 0, \quad tx + vz = 0 \quad \text{and} \quad ty + vw = 1.$$

Then R is a domain which is not a division ring. Hence by Lemma 6, $\delta(R) = J(R)$. By [4, Remark 1.5], R is gH, however R^2 is not gH. Therefore R is δH, but R^2 is not δH.

The next result gives a condition that a direct sum of two δH modules is δH.

Proposition 3. *Let M_1 and M_2 be two R-modules. If for every $i \in \{1, 2\}$, M_i is a fully invariant submodule of $M = M_1 \oplus M_2$, then M is δH if and only if M_i is δH for each $i \in \{1, 2\}$.*

Proof. The necessity is clear from Proposition 1. For the sufficiency, let $f = (f_{ij})$ be a surjective endomorphism of M, where $f_{ij} \in \mathrm{Hom}(M_i, M_j)$ and $i, j \in \{1, 2\}$. By assumption, $\mathrm{Hom}(M_i, M_j) = 0$ for every $i, j \in \{1, 2\}$ with $i \neq j$. Since f is an epimorphism, f_{ii} is a surjective endomorphism of M_i for each $i \in \{1, 2\}$. As M_i is δH for each $i \in \{1, 2\}$, $\mathrm{Ker}(f_{ii}) \ll_\delta M_i$. Since $\mathrm{Ker}(f) = \mathrm{Ker}(f_{11}) \oplus \mathrm{Ker}(f_{22})$, $\mathrm{Ker}(f) \ll_\delta M$ by Lemma 2(3). Hence M is δH. □

In the following example, it is shown that the δH property of a module dose not inherit by its endomorphism ring.

Example 3. Let M be an infinite dimensional vector space over a division ring K. Then by Theorem 3, M is δH. Since $S \cong S^2$ by [5, Example 5.16] and S is not a semisimple ring, $S = \mathrm{End}(M)$ is not δH, by Theorem 1.

Theorem 5. *Let M be a quasi-projective R-module. Then M is δH if and only if M/N is δH for any small submodule N of M.*

Proof. Let M be δH, $N \leqslant M$ and $f\colon M/N \to M/N$ be an epimorphism. Since M is quasi-projective, there exists a homomorphism $g\colon M \to M$ such that $\pi g = f\pi$, where $\pi\colon M \to M/N$ is the natural epimorphism. As $N \ll M$, g is an epimorphism, by [16, 19.2]. Therefore $\mathrm{Ker}(g) \ll_\delta M$. Since $\pi g = f\pi$, $g(N) \leqslant N$ and $\mathrm{Ker}(f) = (g^{-1}(N))/N$. As $N \ll M$ (and so $N \ll_\delta M$) and g is an epimorphism, $g^{-1}(N) \ll_\delta M$, by Theorem 1(2). Therefore $\mathrm{Ker}(f) \ll_\delta M/N$, by Lemma 2. Therefore M/N is δH. The converse is clear by taking $N = 0$. □

Theorem 6. *Let M be an R-module. If M satisfies a.c.c or d.c.c on non δ-small submodules, then M is a δH module.*

Proof. Let M be a module that satisfies a.c.c. on non δ-small submodules and $f\colon M \to M$ an epimorphism. If $\mathrm{Ker}(f)$ is not δ-small in M, then $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(f^2) \subseteq \mathrm{Ker}(f^3) \subseteq \dots$ is an ascending chain of non δ-small submodules of M. Hence there exists $n \geqslant 1$ such that $\mathrm{Ker}(f^n) = \mathrm{Ker}(f^{n+i})$ for each $i \geqslant 1$. By a usual argument, $\mathrm{Ker}(f) = 0$, a contradiction. Therefore $\mathrm{Ker}(f) \ll_\delta M$ and so M is δH.

Assume that M satisfies d.c.c on non δ-small submodules and M is not δH. Hence there exists an epimorphism $f\colon M \to M$ such that $K = \mathrm{Ker}(f)$ is not a δ-small submodule of M. Therefore then each submodule L of M, which contains K, is not a δ-small submodule of M. As M is not δH, it is not Artinian. Hence $M/K \cong M$ is not Artinian and there is a descending chain $L_1/K \supset L_2/K \supset L_3/K \supset \dots$ of submodules of M/K. Thus $L_1 \supset L_2 \supset L_3 \supset \dots$ is a descending chain of non δ-small submodule of M, a contradiction. □

Proposition 4. *Let R be a ring. If $R/\delta(R)$ is a semisimple ring, then every finitely generated right R-module M is δH.*

Proof. Assume that $R/\delta(R)$ is a semisimple ring, and M is a finitely generated right R-module. Hence $\delta(M) = \delta(R)M$ by [18, Theorem 1.8]. Therefore $M/\delta(M)$ is semisimple as an $R/\delta(R) - module$, and so it is semisimple as R-module. Therefore $M/\delta(M)$ is δH, by Theorem 6. As M is finitely generated, $\delta(M) \ll_\delta M$, and so M is δH, by Theorem 1(5). □

The following result shows δH property is preserved under Morita equivalences.

Theorem 7. *δ-Hopfian is a Morita invariant property.*

Proof. Let R and S be Morita equivalent rings with inverse category equivalences $\alpha\colon \text{Mod-}R \to \text{Mod-}S$, $\beta\colon \text{Mod-}S \to \text{Mod-}R$. Let M be a δH R-module. We show that $\alpha(M)$ is a δH S-module. Assume that $\phi\colon \alpha(M) \to \alpha(M) \oplus X$ be an S-module epimorphism where X is a right S-module. Since any category equivalence preserves epimorphisms and direct sums, we have $\beta(\phi)\colon \beta\alpha(M) \to \beta\alpha(M) \oplus \beta(X)$, as an epimorphism of right R-modules. As $\beta\alpha(M) \cong M$, we have an epimorphism $M \to M \oplus \beta(X)$ of R-modules. Therefore $\beta(X)$ is semisimple and projective as an R-module, by Theorem 1. Since any category equivalence preserves semisimple and projective properties, X is semisimple and projective as an S-module. Therefore $\alpha(M)$ is δH. □

Corollary 2. *Let $n \geqslant 2$. Then the following statements are equivalent for a ring R:*

(1) *Every n-generated R-module is* δH*;*

(2) *Every cyclic $M_n(R)$-module is* δH.

Proof. Let $P = R^n$ and $S = \mathrm{End}(P)$. Then, it is known that $\mathrm{Hom}_R(P, -)\colon N_R \to \mathrm{Hom}_R({}_SP_R, N_R)$ defines a Morita equivalence between Mod-R and Mod-S with the inverse equivalence $- \otimes_S P\colon M_S \to M \otimes P$. Moreover, if N is an n-generated R-module, then $\mathrm{Hom}_R(P, N)$ is a cyclic S-module and for any cyclic S-module M, $M \otimes_S P$ is an n-generated R-module. By Theorem 7, a Morita equivalence preserves the δH property of modules. Therefore, every cyclic S-module is δH if and only if every n-generated R-module is δH. □

In the following, we characterize the rings R for which every finitely generated free R-module is δH.

Corollary 3. *Let R be a ring. Then the following statements are equivalent:*

(1) *Every finitely generated free R-module is* δH*;*

(2) *Every finitely generated projective R-module is* δH*;*

(3) *$M_n(R)$ is* δH *(as an $M_n(R)$-module) for each $n \geqslant 1$.*

Proof. (1) ⇒ (2) It is clear from Proposition 1.

(2) ⇒ (1) It is clear.

(1) ⇔ (3) Let n be a positive integer and $S = M_n(R)$. By the proof of Corollary 2 and Theorem 7, if R^n is δH, then $\mathrm{Hom}_R(R^n, R^n)$ is δH as an S-module. Conversely, if S is δH as an S-module, then $S \otimes_S R^n$ is δH as an R-module. □

2. Polynomial extensions of δ-Hopfian modules

Let M be an R-module. In this section we will briefly recall the definitions of the modules $M[X]$ and $M[x]/(x^{n+1})$ from [13] and [17]. The elements of $M[X]$ are formal sums of the form $m_0 + m_1x + \cdots + m_nx^n$ with $m_i \in M$ and $n \in \mathbb{N}$. We denote this sum by $\sum_{i=0}^{n} m_ix^i$(m_0x^0 is to be understood as the element of M). Addition is defined by adding the corresponding coefficients. The $R[x]$-module structure is given by

$$\left(\sum_{i=0}^{k} m_ix^i\right)\left(\sum_{i=0}^{t} r_ix^i\right) = \sum_{i=0}^{k+t} m'_ix^i,$$

where $m'_p = \sum_{i+j=p} m_i r_j$, $r_j \in R$ and $m_i \in M$. Any nonzero element β of $M[x]$ can be written uniquely as $\sum_{i=k}^{l} m_i x^i$ with $l \geqslant k \geqslant 0$, $m_i \in M$, $m_k \neq 0$ and $m_l \neq 0$. In this case, we refer to k as the order of β, l as the degree of β, m_k as the initial coefficient of β, and m_l as the leading coefficient of β.

Let n be any non-negative integer and

$$I_{n+1} = \{0\} \cup \{\beta |\ 0 \neq \beta \in R[x],\ \text{order of } \beta \geqslant n+1\}.$$

Then I_{n+1} is a two-sided ideal of $R[x]$. The quotient ring $R[x]/I_{n+1}$ will be called the truncated polynomial ring, truncated at degree $n+1$. Since R has an identity element, I_{n+1} is the ideal generated by x^{n+1}. Even when R does not have an identity element, we will denote the ring $R[x]/I_{n+1}$ by $R[x]/(x^{n+1})$. Any element of $R[x]/(x^{n+1})$ can be uniquely written as $\sum_{i=0}^{t} r_i x^i$, with $r_i \in R$. Let $D_{n+1} = \{0\} \cup \{\beta |\ 0 \neq \beta \in M[x],\ \text{order of } \beta \geqslant n+1\}$. Then D_{n+1} is an $R[x]$-submodule of $M[x]$. Since $I_{n+1}M[x] \subseteq D_{n+1}$, we can see that $R[x]/(x^{n+1})$ acts on $M[x]/D_{n+1}$. We denote the module $M[x]/D_{n+1}$ by $M[x]/x^{n+1}$. The action of $R[x]/(x^{n+1})$ on $M[x]/x^{n+1}$ is given by

$$\left(\sum_{i=0}^{n} m_i x^i\right)\left(\sum_{i=0}^{n} r_i x^i\right) = \sum_{i=0}^{n} m'_i x^i,$$

where $m'_p = \sum_{i+j=p} m_i r_j$, $r_j \in R$ and $m_i \in M$. Any nonzero element β of $M[x]/D_{n+1}$ can be written uniquely as $\sum_{i=k}^{n} m_i x^i$ with $n \geqslant k \geqslant 0$, $m_i \in M$ and $m_k \neq 0$. In this case, k is called the order of β and m_k the initial coefficient of β.

Proposition 5. *Let M be an R-module. If $M[x]$ is δH as an $R[x]$-module, then M is δH.*

Proof. Let $M[x]$ be δH as an $R[x]$-module and f a surjective endomorphism of M. Assume that $\mathrm{Ker}(f) + K = M$, for some $K \leqslant M$, with M/K singular. Define $\bar{f}\colon M[x] \to M[x]$ by $\bar{f}(\sum_{j=0}^{n} m_j x^j) = \sum_{j=0}^{n} f(m_j)x^j$. It is easy to see that $\bar{f}$ is a surjective endomorphism of $M[x]$ and $\mathrm{Ker}(\bar{f}) = \mathrm{Ker}(f)[x]$. Hence $\mathrm{Ker}(\bar{f}) + K[x] = M[x]$. We show that $M[x]/K[x]$ is a singular $R[x]$-module. Let $\beta = m_0 + m_1 x + \cdots + m_n x^n \in M[x]$. As M/K is singular, for each $0 \leqslant i \leqslant n$, there exists $I_i \leqslant^{\text{ess}} R$ such that $m_i I_i \subseteq K$. Put $I = \cap_{i=0}^{n} I_i$. Therefore $I \leqslant^{\text{ess}} R_R$ and $\beta I \subseteq K[x]$. We claim that $I[x] \leqslant^{\text{ess}} R[x]$. Let $\alpha = r_0 + r_1 x + \cdots + r_t x^t \in R[x]$, where $t \in \mathbb{N}$. If $r_0 \neq 0$, then $t_0 \in R$ exists such that $0 \neq r_0 t_0 \in I$ (because $I \leqslant^{\text{ess}} R_R$). Now, if $r_1 t_0 \neq 0$, then there exists $t_1 \in R$ such that $0 \neq r_1 t_0 t_1 \in I$. Continuing this process, we get $r \in R$ such that $0 \neq \alpha r \in I[x]$ and $I[x] \leqslant^{\text{ess}} R[x]$. As $\beta I \subseteq K[x]$, $\beta I[x] \subseteq K[x]$. Thus $M[x]/K[x]$ is singular. Since $M[x]$ is δH, $\text{Ker}(\bar{f}) \ll_\delta M[x]$. Therefore $K[x] = M[x]$ and so $M = K$. This implies that $\text{Ker}(f) \ll_\delta M$ and M is δH. □

The following example shows that the converse of proposition 5 is not correct.

Example 4. Let R be a semisimple ring, where $R[x]$ is not semisimple, and $M = R^{(\mathbb{N})}$. As $M \cong M \oplus M$ and $M[x] \cong M \otimes_R R[x]$, we have $M[x] \cong M[x] \oplus M[x]$. Since R is semisimple, M is δH, by Theorem 3. As $M[x]$ is not semisimple, $M[x]$ is not δH, by Theorem 1.

Remark 1. Let M be an R-module and N a submodule of $M[x]/(x^{n+1})$ as an $R[x]/(x^{n+1})$-module, where $n \geqslant 0$. Define

$$N_i = \{0\} \cup \{\text{initial coefficients of elements of order } i \text{ in } N\}$$

for each $1 \leqslant i \leqslant n$. By [17], $N_i \leqslant M$ and $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n$.

Definition 3. Let M be an R-module and N a submodule of $M[x]/(x^{n+1})$ as an $R[x]/(x^{n+1})$-module, where $n \geqslant 0$. Then we say that $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n$ is the adjoint chain of N.

In the following, we show that for each submodule N of $M[x]/(x^{n+1})$ as an $R[x]/(x^{n+1})$-module, its adjoint chain plays an important role to find its properties. By the definition of N_i $(1 \leqslant i \leqslant n)$, it is clear that N_i is uniquely determined by N.

Lemma 7. *Let N be a submodule of $M[x]/(x^{n+1})$ as an $R[x]/(x^{n+1})$-module, where $n \geqslant 0$. Then $N \leqslant^{\text{ess}} M[x]/(x^{n+1})$ if and only if $N_n \leqslant^{\text{ess}} M_R$.*

Proof. Let $N \leqslant^{\text{ess}} M[x]/(x^{n+1})$ and $0 \neq m \in M$. Then there exists $r_0 + r_1x + \cdots + r_nx^n \in R[x]/(x^{n+1})$ such that $0 \neq m(r_0 + r_1x + \cdots + r_nx^n) \in N$. Let s be the order of $m(r_0 + r_1x + \cdots + r_nx^n)$. Hence $0 \neq mr_sx^s \in N_s \subseteq N_n$. Therefore $N_n \leqslant^{\text{ess}} M$.

Conversely, assume that $N_n \leqslant^{\text{ess}} M$ and $m_sx^s + m_{s+1}x^{s+1} + \cdots + m_nx^n \in M[x]/(x^{n+1})$ of order s. Since $m_s \neq 0$ and $N_n \leqslant^{\text{ess}} M$, there exists $r \in R$ such that $0 \neq m_sr \in N_n$. Therefore $0 \neq (m_sx^s + m_{s+1}x^{s+1} + \cdots + m_nx^n)(rx^{n-s}) = m_srx^n$. Clearly $m_srx^n \in N$ (by the definition of N_n). Therefore $N \leqslant^{\text{ess}} M[x]/(x^{n+1})$. □

Lemma 8. *Let N be a submodule of $M[x]/(x^{n+1})$ as an $R[x]/(x^{n+1})$-module ($n \geqslant 0$) and $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n$ the adjoint chain of N. Then N is a maximal submodule of $M[x]/(x^{n+1})$ if and only if N_0 is a maximal submodule of M and $N_i = M$ for each $1 \leqslant i \leqslant n$. Moreover, if N is a maximal submodule of $M[x]/(x^{n+1})$, then $N = N_0 + Mx + \cdots + Mx^n$.*

Proof. Let N be a maximal submodule of $M[x]/(x^{n+1})$. Let $N_0 \subseteq N_0'$ for some $N_0' \leqslant M$. Set

$$N' = \{m_0 + m_1x + \cdots + m_nx^n \in M[x]/(x^{n+1})\colon m_0 \in N_0', m_i \in M\}.$$

It is clear that $N' \leqslant M[x]/(x^{n+1})$ and $N \subseteq N'$. Since N is maximal, $N = N'$ or $N' = M[x]/(x^{n+1})$. Therefore $N_0 = N_0'$ or $N_0' = M$, where N_0' is the first component of the adjoint chain of N'. This implies that N_0 is maximal in M. Also, maximality of N gives $N_i = M$ for each $1 \leqslant i \leqslant n$.

Conversely, assume that N_0 is a maximal submodule of M and $N_i = M$ for each $1 \leqslant i \leqslant n$. If there exists $N' \lneqq M[x]/(x^{n+1})$ such that $N \subseteq N'$, then $N_0 \subseteq N_0'$ and $N_i \subseteq N_i'$ for each $1 \leqslant i \leqslant n$, where $N_0' \subseteq N_1' \subseteq \cdots \subseteq N_n'$ is the adjoint chain of N'. As $N' \neq M[x]/(x^{n+1})$, $N_0 = N_0'$ and $N_i' = M$ for each $1 \leqslant i \leqslant n$. Therefore $N = N'$ and N is a maximal submodule of $M[x]/(x^{n+1})$.

Now, it is clear that, if N is a maximal submodule of $M[x]/(x^{n+1})$, then $N = N_0 + Mx + \cdots + Mx^n$. □

Now, we are ready to determine the $\delta(M[x]/(x^{n+1}))$ for a module M.

Theorem 8. *Let M be an R-module. Then $\delta(M[x]/(x^{n+1})) = \mathrm{Rad}(M) + Mx + Mx^2 + \cdots + Mx^n$.*

Proof. By Lemmas 7 and 8, every maximal submodule of $M[x]/(x^{n+1})$ is essential. Hence

$$\begin{aligned}
&\delta(M[x]/(x^{n+1}))\\
&\qquad = \bigcap\{N \leqslant M[x]/(x^{n+1})\colon (M[x]/(x^{n+1}))/N \text{ is simple and singular}\}\\
&\qquad = \bigcap\{N\colon N \text{ is maximal in } M[x]/(x^{n+1})\}\\
&\qquad = \mathrm{Rad}(M) + Mx + Mx^2 + \cdots + Mx^n.
\end{aligned}$$ □

It is known that, if M is an R-module and $K \ll M$, then $K[x]/(x^{n+1}) \ll M[x]/(x^{n+1})$ as an $R[x]/(x^{n+1})$-module, by [17, Lemma 2.1]. However, it is not true for δ-small submodules, as the following example shows.

Example 5. Let F be a field and $R = T_2(F)$, the ring of upper triangular matrices over F. Then $\delta(R_R) = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $J(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $I = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. Then $I \ll_\delta R$, because $I \subseteq \delta(R_R)$ and $\delta(R_R) \ll_\delta R$. However $I[x]/(x^{n+1})$ is not a δ-small right ideal of $R[x]/(x^{n+1})$, because $I[x]/(x^{n+1}) \not\subseteq \delta(R[x]/(x^{n+1})) = J(R) + Rx + \cdots + Rx^n$, by Theorem 8.

In [17, Theorem 2.2], it is shown that, if M is a gH R-module, then $M[x]/(x^{n+1})$ is gH as an $R[x]/(x^{n+1})$-module, however it is not true that, if M is a δH R-module, then $M[x]/(x^{n+1})$ is δH as an $R[x]/(x^{n+1})$-module, as the following example shows.

Example 6. Let R be a semisimple ring and $M = R^{(\mathbb{N})}$. Then M is δH by Theorem 3. Define $f\colon M \to M$ by $f((r_1, r_2, \ldots, r_n, \ldots)) = (r_2, r_3, \ldots, r_n, \ldots)$. Then f is an epimorphism and

$$\mathrm{Ker}(f) = \{(r, 0, 0, 0, \ldots) \in R^{(\mathbb{N})} : r \in R\}.$$

It is clear that $\alpha\colon M[x]/(x^{n+1}) \to M[x]/(x^{n+1})$ defined by

$$\alpha\left(\sum_{j=0}^{n} m_j x^j\right) = \sum_{j=0}^{n} f(m_j) x^j$$

is an $R[x]/(x^{n+1})$-epimorphism and $\mathrm{Ker}(\alpha) = (\mathrm{Ker}(\alpha))[x]/(x^{n+1})$. If $\mathrm{Ker}(\alpha) \ll_\delta M[x]/(x^{n+1})$, then

$$\mathrm{Ker}(\alpha) \subseteq \delta(M[x]/(x^{n+1})) = \mathrm{Rad}(M) + Mx + Mx^2 + \cdots + Mx^n$$

by Theorem 8. But $\mathrm{Rad}(M) = 0$ and $\mathrm{Ker}(\alpha) \not\subseteq \delta(M[x]/(x^{n+1}))$. Therefore $\mathrm{Ker}(\alpha)$ is not a δ-small submodule of $M[x]/(x^{n+1})$ and $M[x]/(x^{n+1})$ is not a δH module.

Theorem 9. *Let M be an R-module. If $M[x]/(x^{n+1})$ is δH as an $R[x]/(x^{n+1})$-module, then M is δH.*

Proof. Assume that $M[x]/(x^{n+1})$ is δH as an $R[x]/(x^{n+1})$-module and $f\colon M \to M$ an R-epimorphism. Define $\alpha\colon M[x]/(x^{n+1}) \to M[x]/(x^{n+1})$ by

$$\alpha\left(\sum_{j=0}^{n} m_j x^j\right) = \sum_{j=0}^{n} f(m_j) x^j.$$

Then α is an $R[x]/(x^{n+1})$-epimorphism and $\mathrm{Ker}(\alpha) = (\mathrm{Ker}(f))[x]/(x^{n+1})$. We show that $\mathrm{Ker}(f) \ll_\delta M$. Let H be a submodule of M such that $\mathrm{Ker}(f) + H = M$ with M/H singular. Hence

$$M[x]/(x^{n+1}) = (\mathrm{Ker}(f))[x]/(x^{n+1}) + H[x]/(x^{n+1}).$$

We claim that $\frac{M[x]/(x^{n+1})}{H[x]/(x^{n+1})}$ is singular as $R[x]/(x^{n+1})$-module. Let

$$m = m_0 + m_1x + \cdots + m_nx^n \in M[x]/(x^{n+1}).$$

For each $0 \leqslant j \leqslant n$, there exists $I_j \leqslant^{\mathrm{ess}} R$ such that $m_jI_j \subseteq H$. Put $I = \cap_{i=1}^n I_j$. Then $I \leqslant^{\mathrm{ess}} R$ and so $I[x]/(x^{n+1}) \leqslant^{\mathrm{ess}} R[x]/(x^{n+1})$, by Lemma 7. As $m_jI \subseteq H$ for each $0 \leqslant j \leqslant n$, $m(I[x]/(x^{n+1})) \subseteq H[x]/(x^{n+1})$. Therefore $\frac{M[x]/(x^{n+1})}{H[x]/(x^{n+1})}$ is singular. As $\mathrm{Ker}(\alpha) \ll_\delta M[x]/(x^{n+1})$, $H[x]/(x^{n+1}) = M[x]/(x^{n+1})$, and so $H = M$. Therefore $\mathrm{Ker}(f) \ll_\delta M$ and M is δH. $\square$

3. Triangular matrix extensions

Throughout this section T will denote a 2-by-2 generalized (or formal) triangular matrix ring $\begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$, where R and S are rings and M is an (S,R)-bimodule.

Proposition 6. *Assume that M is an (S,R)-bimodule, and $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then $\delta(T_T) = \begin{pmatrix} H & M \\ 0 & \delta(R_R) \end{pmatrix}$, where*

$$H = \delta(S_S) \cap \{I\colon I \textit{ is a maximal right ideal of } S \textit{ with } \mathrm{ann}_S(M) \subseteq I\}.$$

Proof. By Lemma 4, every maximal essential right ideal of T has the form $\begin{pmatrix} S & M \\ 0 & J \end{pmatrix}$, where J is a maximal essential right ideal of R or $\begin{pmatrix} I & M \\ 0 & R \end{pmatrix}$, where I is a maximal right ideal with $I \cap \mathrm{ann}_S(M) \leqslant^{\mathrm{ess}} (\mathrm{ann}_S(M))_S$. Therefore $\delta(T_T) = \begin{pmatrix} K & M \\ 0 & \delta(R_R) \end{pmatrix}$, where

$$K = \{I \leqslant S\colon I \text{ is a maximal right ideal of } S \\ \text{with } I \cap \mathrm{ann}_S(M) \leqslant^{\mathrm{ess}} (\mathrm{ann}_S(M))_S\}.$$

We prove $K = H$. If $\mathrm{ann}_S(M) = 0$, then it is clear that $K = H = \delta(S_S)$. Assume that $\mathrm{ann}_S(M) \neq 0$. Let $x \in K$ and I be a maximal right ideal of S. If $\mathrm{ann}_S(M) \subseteq I$, then $\mathrm{ann}_S(M) = I \cap \mathrm{ann}_S(M) \leqslant^{\mathrm{ess}} (\mathrm{ann}_S(M))_S$. Therefore $x \in K$ implies that $x \in I$. If $I \leqslant^{\mathrm{ess}} S_S$, then $I \cap \mathrm{ann}_S(M) \leqslant^{\mathrm{ess}} (\mathrm{ann}_S(M))_S$. Hence $x \in I$. Therefore $x \in H$ and so $K \subseteq H$. For the reverse of the inclusion, let $x \in H$. Let I be a maximal right ideal of S such that $I \cap \mathrm{ann}_S(M) \leqslant^{\mathrm{ess}} (\mathrm{ann}_S(M))_S$. If $\mathrm{ann}_S(M) \subseteq I$, then $x \in I$. Assume that $\mathrm{ann}_S(M) \not\subseteq I$. Hence $\mathrm{ann}_S(M) + I = S$ and so $S/I \cong \mathrm{ann}_S(M)/(I \cap \mathrm{ann}_S(M))$. As $\mathrm{ann}_S(M) \neq 0$ and $\mathrm{ann}_S(M) \cap I \leqslant^{\mathrm{ess}} (\mathrm{ann}_S(M))_S$, we have $\mathrm{ann}_S(M) \cap I \neq 0$ and S/I is singular. Therefore $I \leqslant^{\mathrm{ess}} S_S$. Thus $x \in \delta(S)$ gives $x \in I$; hence $K = H$. □

The next result gives a characterization for the δH condition for a 2-by-2 generalized triangular matrix ring.

Theorem 10. *Assume that M is an (S,R)-bimodule, and $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. Then the following statements are equivalent:*

(1) *T_T is δH.*
(2) *(i) S_S is δH and if $a \in S$ has the right inverse b, then $1 - ba \in I$, for each maximal right ideal of R with $\mathrm{ann}_S(M) \subseteq I$.*
(ii) R_R is δH.

Proof. (1) ⇒ (2) Let $a \in S$ have the right inverse $b \in S$. Then $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Since T is δH, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} b & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in \delta(T_T)$, by Theorem 2. Thus

$$1 - ba \in \delta(S_S) \cap \{I \colon I \text{ is a maximal right ideal of S with } \mathrm{ann}_S(M) \subseteq I\},$$

by Proposition 6. Hence S_S is δH, by Theorem 2 and $1 - ba \in I$, for each maximal right ideal of S with $\mathrm{ann}_S(M) \subseteq I$.

(ii) It is similar to the proof of (i).

(2) ⇒ (1) Let $\begin{pmatrix} a & m \\ 0 & p \end{pmatrix}\begin{pmatrix} b & n \\ 0 & q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, where $a, b \in S$, $p, q \in R$ and $m, n \in M$. Hence $ab = 1$ and $pq = 1$. By (1) and Proposition 6, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} b & n \\ 0 & q \end{pmatrix}\begin{pmatrix} a & m \\ 0 & p \end{pmatrix} \in \delta(T_T)$. Hence by Theorem 2, T_T is δH. □

Theorem 11. *Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$, where M is an (S,R)-bimodule. If M is a faithful left R-module, then T_T is δH if and only if S_S is Dedekind finite and R_R is δH.*

Proof. By Proposition 10, $\delta(T_T) = \begin{pmatrix} H & M \\ 0 & \delta(R_R) \end{pmatrix}$, where

$$H = \delta(S_S) \cap \{I \colon I \text{ is a maximal right ideal of S with } \operatorname{ann}_S(M) \subseteq I\}.$$

Since $\operatorname{ann}_S(M) = 0$, $H = J(S)$. Let T_T be δH. If $ab = 1$ $(a, b \in S)$, then similar to the proof of Theorem 10, $1 - ba \in J(S)$, and so $ba = 1$. This implies that S is Dedekind-finite. Also, from Theorem 10, R_R is δH. The converse can be concluded from Theorem 10. □

Since M_R is always a faithful left S-module for $S = \operatorname{End}(M_R)$, we have the following corollary. It is known that an R-module M is Dedekind-finite if and only if $\operatorname{End}_R(M)$ is a Dedekind-finite ring.

Corollary 4. *Let* $T = \begin{pmatrix} \operatorname{End}(M_R) & M \\ 0 & R \end{pmatrix}$. *Then* T_T *is* δH *if and only if* M *is Dedekind-finite and* R_R *is* δH.

Theorem 12. *Assume* R *is a ring. Then the following are equivalent:*

(1) R_R *is Dedekind-finite;*

(2) $T_n(R)$ *is* δH, *for every positive integer* n.

Proof. (1) ⇒ (2) We proceed by induction on n. Note that $T_{n+1}(R) = \begin{pmatrix} R & M \\ 0 & T_n(R) \end{pmatrix}$, where $M = (R, R, \dots, R)$ (n-tuple). For $n = 2$, if R is Dedekind-finite, then $T_2(R)$ is δH, by Theorem 11. Now, assume that R is Dedekind-finite and $T_n(R)$ is δH. Hence by Theorem 11, $T_{n+1}(R)$ is δH.

(2) ⇒ (1) It is clear from Theorem 11. □

Theorem 13. *Let* R *be a ring and* $U(R)$ *the countably upper triangular matrix ring over* R. *Then* R *is Dedekind-finite if and only if* $U(R)$ *is* δH.

Proof. It is clear that $U(R) \cong \begin{pmatrix} R & M \\ 0 & U(R) \end{pmatrix}$, where $M = (R, R, \dots)$. Now, the result is clear from Theorem 11. □

Motivated by [1, Proposition 2.14], we have the following theorem.

Theorem 14. *Let* ${}_SM_R$ *be a nonzero* (S, R)*-bimodule such that* M_R^n *is* δH *for all* $n \geqslant 1$. *Then either* M_R *is semisimple and projective or one of the rings* R *or* S *satisfies the rank condition.*

Proof. Assume that M_R is not projective or semisimple. Let $T = \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$ and $I = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$. Since M_R is not projective or semisimple, I_T is not projective or semisimple. By hypothesis, we can conclude that I_T^n is δH for each $n \geqslant 1$. Now we will show that the ring T satisfies the rank condition. Assume that T does not satisfy the rank condition and $f\colon T^p \to T^q$ is an epimorphism with $q > p$. Thus $f(I^p) = f(T^p I) = f(T^p)I = T^q I = I^q$. Hence $f\colon I^p \to I^p \oplus I^{q-p}$ is an epimorphism. Since I_T^n is δH for each $n \geqslant 1$, I^{q-p} is semisimple and projective as T-module, by Theorem 1. But then I should be projective and semisimple, which is not. Hence T satisfies the rank condition. Therefore one of the rings R or S satisfies the rank condition, by [6, Proposition 4.1]. □

Open Problems. (1) What is the structure of rings whose finitely generated right modules are δH?

(2) Does Theorem 5 hold for δ-small submodules? (That is, let M be a quasi-projective R-module. Then M is δH if and only if so is M/N for any δ-small submodule N of M).

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