Algebra and Discrete Mathematics
Volume **25** (2018). Number 1, pp. 73–97

RESEARCH ARTICLE

Gram matrices and Stirling numbers of a class of diagram algebras, I

N. Karimilla Bi and M. Parvathi

Communicated by R. Wisbauer

ABSTRACT. In this paper, we introduce Gram matrices for the signed partition algebras, the algebra of $\mathbb{Z}_2$-relations and the partition algebras. The nondegeneracy and symmetic nature of these Gram matrices are establised. Also, $(s_1, s_2, r_1, r_2, p_1, p_2)$-Stirling numbers of the second kind for the signed partition algebras, the algebra of $\mathbb{Z}_2$-relations are introduced and their identities are established. Stirling numbers of the second kind for the partition algebras are introduced and their identities are established.

1. Introduction

An extensive study of partition algebras is made by Martin [7–12] and these algebras arose naturally as Potts models in statistical mechanics and in the work of V. Jones [3].

A new class of algebras, called the signed partition algebras, are introduced in [6] which are a generalization of partition algebras and signed Brauer algebras [13]. The study of the structure of such finite-dimensional algebras is important for it may be possible to find presumably new examples of subfactors of a hyper finite Π_1-factor along the lines of [16].

In this paper, we introduce a new class of matrices $G^k_{2s_1+s_2}$, $\overrightarrow{G}^k_{2s_1+s_2}$ and G^k_s of $A^{\mathbb{Z}_2}_k(x)$ (the algebra of $\mathbb{Z}_2$-relations), $\overrightarrow{A}^{\mathbb{Z}_2}_k$ (signed partition

2010 MSC: 16Z05.
Key words and phrases: Gram matrices, partition algebras, signed partition algebras and the algebra of $\mathbb{Z}_2$-relations.

algebras) and $A_k(x)$ (partition algebras) respectively which will be called as Gram matrices since by Theorem 3.8 in [1] the Gram matrices $G^{\lambda,\mu}_{2s_1+s_2}$ associated to the cell modules of $W[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ (for $\lambda = ([s_1],\Phi)$, $\mu = [s_2]$ if $s_1, s_2 \neq 0; \lambda = (\Phi,\Phi)$, $\mu = [s_2]$ if $s_1 = 0, s_2 \neq 0; \lambda = ([s_1],\Phi)$, $\mu = \Phi$ if $s_1 \neq 0, s_2 = 0$; $\lambda = (\Phi,\Phi)$, $\mu = \Phi$ if $s_1 = s_2 = 0$, $0 \leqslant s_1 \leqslant k$, $0 \leqslant s_2 \leqslant k$ and $0 \leqslant s_1 + s_2 \leqslant k$) and $\overrightarrow{G}^{\lambda,\mu}_{2s_1+s_2}$ associated to the cell modules of $\overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ (for $\lambda = ([s_1],\Phi)$, $\mu = [s_2]$ if $s_1, s_2 \neq 0; \lambda = (\Phi,\Phi)$, $\mu = [s_2]$ if $s_1 = 0, s_2 \neq 0; \lambda = ([s_1],\Phi)$, $\mu = \Phi$ if $s_1 \neq 0, s_2 = 0; \lambda = (\Phi,\Phi)$, $\mu = \Phi$ if $s_1 = s_2 = 0, 0 \leqslant s_1 \leqslant k, 0 \leqslant s_2 \leqslant k-1$ and $0 \leqslant s_1 + s_2 \leqslant k-1$) defined in [5] coincides with the matrices $G^k_{2s_1+s_2}$ and $\overrightarrow{G}^k_{2s_1+s_2}$ respectively.

In this paper, $(s_1, s_2, r_1, r_2, p_1, p_2)$-Stirling numbers of the second kind for the algebra of $\mathbb{Z}_2$-relations and signed partition algebras are introduced and their identities are established. Stirling numbers of second kind corresponding to the partition algebras are also introduced and their identities are established.

2. Preliminaries

2.1. Partition algebras

We recall the definitions in [2] required in this paper. For $k \in \mathbb{N}$, let $\underline{k} = \{1,2,\cdots,k\}, \underline{k}' = \{1',2',\cdots,k'\}$. Let $R_{\underline{k}\cup\underline{k}'}$ be the set of all partitions of $\underline{k}\cup\underline{k}'$ or equivalence relation on $\underline{k}\cup\underline{k}'$. Every equivalence class of $\underline{k}\cup\underline{k}'$ is called as connected component.

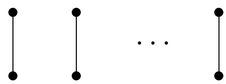
Any $d \in R_{\underline{k}\cup\underline{k}'}$ can be represented as a simple graph on two rows of k-vertices, one above the other with k vertices in the top row, labeled $1,2,\cdots,k$ left to right and k vertices in the bottom row labeled $1',2',\cdots,k'$ left to right with vertex i and vertex j connected by a path if i and j are in the same block of the set partition d. The graph representing d is called k-partition diagram and it is not unique. Two k-partition diagrams are said to be equivalent if they give rise to the same set partition of $2k$-vertices.

Any connected component C of $d, d \in R_{\underline{k}\cup\underline{k}'}$ containing an element of $\{1,2,\cdots,k\}$ and an element of $\{1',2',\cdots,k'\}$ is called a *through class.* Any connected component containing elements only, either from $\{1,2,\cdots,k\}$ or $\{1',2',\cdots,k'\}$ is called a *horizontal edge.*

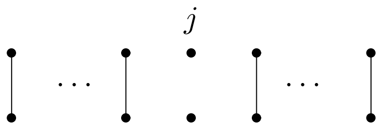
The number of through classes in d is called a *propagating number* and it is denoted by $\sharp^p(d)$. We shall define multiplication of two k-partition diagrams d' and d'' as follows:

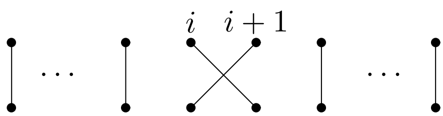
- Place d' above d''
- Identify the bottom dots of d' with the top dots of d''
- $d' \circ d''$ is the resultant diagram obtained by using only the top row of d' and bottom row of d'', replace each connected component which lives entirely in the middle row by the variable x. i.e., $d' \circ d'' = x^l d'''$ where l is the number of connected components that lie entirely in the middle row.

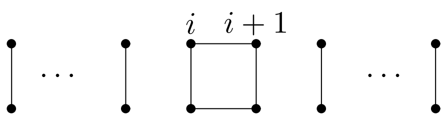
This product is associative and is independent of the graph we choose to represent the k-partition diagram. Let $\mathbb{K}(x)$ be the field and x be an indeterminate. The partition algebra $A_k(x)$ is defined to be the $\mathbb{K}(x)$-span of the k-partition diagrams, which is an associative algebra with identity 1 where

$1 =$ 

By convention $A_0(x) = \mathbb{K}(x)$. For $1 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant k$, the following are the generators of the partition algebras.

$P_j =$ 

$s_i =$ 

$\beta_i =$ 

The above generators satisfy the relations given in Theorem 1.11 of [2].

2.2. The algebra of $\mathbb{Z}_2$-relations

Definition 2.1 ([15]). Let the group $\mathbb{Z}_2$ act on the set X. Then the action of $\mathbb{Z}_2$ on X can be extended to an action of $\mathbb{Z}_2$ on R_X, where R_X denote the set of all equivalence relations on X, given by

$$g.d = \{(gp, gq) \mid (p, q) \in d\}$$

where $d \in R_X$ and $g \in \mathbb{Z}_2$. (It is easy to see that the relation $g.d$ is again an equivalence relation).

An equivalence relation d on X is said to be a $\mathbb{Z}_2$-stable equivalence relation if $p \sim q$ in d implies that $gp \sim gq$ in d for all g in $\mathbb{Z}_2$. We denote $\underline{k}$ for the set $\{1, 2, \cdots, k\}$. We shall only consider the case when $\mathbb{Z}_2$ acts freely on X; let $X = \underline{k} \times \mathbb{Z}_2$ and the action is defined by $g.(i, x) = (i, gx)$ for all $1 \leqslant i \leqslant k$. Let $R_{\underline{k}}^{\mathbb{Z}_2}$ be the set of all $\mathbb{Z}_2$-stable equivalence relations on $\underline{k} \times \mathbb{Z}_2$.

Notation 2.2 ([15])**.** (i) $R_{\underline{k}}^{\mathbb{Z}_2}$ denotes the set of all $\mathbb{Z}_2$-stable equivalence relation on $\underline{k} \times \mathbb{Z}_2$. Each $d \in R_{\underline{k}}^{\mathbb{Z}_2}$ can be represented as a simple graph on row of $2k$ vertices.

- The vertices $(1, e), (1, g), \cdots, (k, e), (k, g)$ is arranged from left to right in a single row.
- If $(i, g) \sim (j, g') \in R_{\underline{k}}^{\mathbb{Z}_2}$ then $((i, g), (j, g'))$ is an edge which is obtained by joining the vertices (i, g) and (j, g') by a line for $g, g' \in \mathbb{Z}_2$.

We say that the two graphs are equivalent if they give rise to the same set partition of the $2k$ vertices $\{(1, e), (1, g), \cdots, (k, e), (k, g)\}$.

We may regard each element d in $R_{\underline{k} \cup \underline{k}'}^{\mathbb{Z}_2}$ as a $2k$-partition diagram by arranging the $4k$ vertices $(i, g), i \in \underline{k} \cup \underline{k}', g \in \mathbb{Z}_2$ of d in two rows in such a way that $(i, g)\,((i', g))$ is in the top(bottom) row of d if $1 \leqslant i \leqslant k(1' \leqslant i' \leqslant k')$ for all $g \in \mathbb{Z}_2$ and if $(i, g) \sim (j, g')$ then $((i, g), (j, g'))$ is an edge which is obtained by joining the vertices (i, g) and (j, g') by a line where $g, g' \in \mathbb{Z}_2$.

(ii) $R_{\underline{k}}^{\mathbb{Z}_2}$ can be identified as a subset of $R_{\underline{2k}}$ by identifying (r, e) with $2r - 1$, $\forall\, 1 \leqslant r \leqslant k$ and $(r, g), g \neq e$ with $2r\ \forall\, 1 \leqslant r \leqslant k$.

(iii) The diagrams d^+ and d^- are obtained from the diagram d by restricting the vertex set to

$$\{(1, e), (1, g), \cdots, (k, e), (k, g)\} \quad \text{and} \quad \{(1', e), (1', g), \cdots (k', e), (k', g)\}$$

respectively. The diagrams d^+ and d^- are also $\mathbb{Z}_2$-stable equivalence relation with $d^+ \in R_{\underline{k}}^{\mathbb{Z}_2}$ and $d^- \in R_{\underline{k}'}^{\mathbb{Z}_2}$.

Definition 2.3. ([15]) Let $d \in R_{\underline{k} \cup \underline{k}'}^{\mathbb{Z}_2}$. Then the equation

$$R^d = \{(i, j) \mid \text{there exists } g, h \in \mathbb{Z}_2 \text{ such that } ((i, g), (j, h)) \in d\}$$

defines an equivalence relation on $\underline{k} \cup \underline{k}'$.

Remark 2.4 ([15]). For every connected component C of $R^{\mathbb{Z}_2}_{\underline{k}\cup\underline{k}'}$, C^d will be a connected component in R^d as in Definition 2.3.

For $d \in R^{\mathbb{Z}_2}_{\underline{k}\cup\underline{k}'}$ and for every $\mathbb{Z}_2$-stable equivalence class or a connected component C in d there exists a unique subgroup denoted by H^d_C for $C^d \in R^d$ where

(i) $H^d_C = \{e\}$ if $(i, e) \not\sim (i, g)\ \forall\ i \in C^d, C$ is called an $\{e\}$-class or $\{e\}$-component and the $\{e\}$-component C will always occur as a pair in d which is denoted by C^e, C^g.

(ii) $H^d_C = \mathbb{Z}_2$ if $(i, e) \sim (i, g)\ \forall\ i \in C^d, C$ is called a $\mathbb{Z}_2$-class or $\mathbb{Z}_2$-component which is denoted by $C^{\mathbb{Z}_2}$ and the number of vertices in the $\mathbb{Z}_2$-component $C^{\mathbb{Z}_2}$ will always be even.

Proposition 2.5 ([15]). *The linear span of $R^{\mathbb{Z}_2}_{\underline{k}\cup\underline{k}'}$ is a subalgebra of $A_{2k}(x)$. We denote this subalgebra by $A^{\mathbb{Z}_2}_k(x)$, called the algebra of $\mathbb{Z}_2$-relations.*

Definition 2.6 ([15]). For $0 \leqslant 2s_1 + s_2 \leqslant 2k$, define $I^{2k}_{2s_1+s_2}$ as follows:

$$I^{2k}_{2s_1+s_2} = \left\{ d \in R^{\mathbb{Z}_2}_{\underline{k}\cup\underline{k}'} \mid \sharp^p(d) = 2s_1 + s_2 \right\}$$

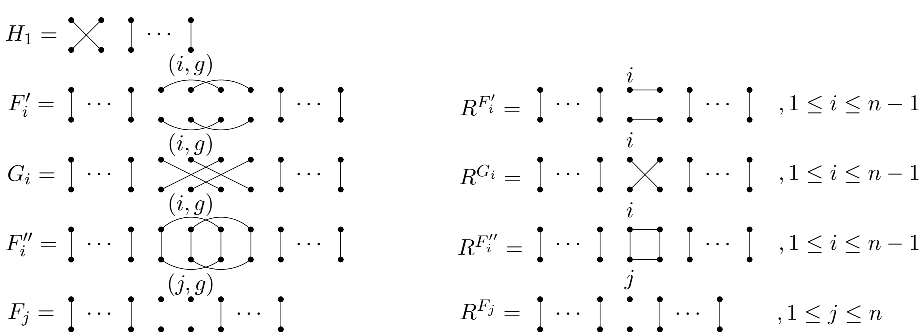
i.e., d has s_1 number of pairs of $\{e\}$-through classes and s_2 number of $\mathbb{Z}_2$-through classes.

For $0 \leqslant s \leqslant 2k$ define, $I^{2k}_s = \bigcup_{2s_1+s_2 \leqslant s} I^{2k}_{2s_1+s_2}$ then it is clear that

$$R^{\mathbb{Z}_2}_{\underline{k}\cup\underline{k}'} = \bigcup_{0\leqslant s\leqslant 2k} I^{2k}_s = \bigcup_{0\leqslant 2s_1+s_2\leqslant 2k} I^{2k}_{2s_1+s_2}.$$

2.3. Signed partition algebras

Definition 2.7 ([6], Definition 3.1.1). Let the signed partition algebra $\overrightarrow{A}^{\mathbb{Z}_2}_k(x)$ be the subalgebra of $A_{2k}(x)$ generated by $H_1, F'_i, F''_i, G_i, F_j$ for $1 \leqslant i \leqslant k-1$ and $1 \leqslant j \leqslant k$ where



The subalgebra of the signed partition algebra generated by F_i', G_i, F_i'', F_j, $1 \leqslant i \leqslant k-1$, $1 \leqslant j \leqslant k$ is isomorphic on to the partition algebra $A_{2k}(x^2)$. Also, $R^{G_i} = s_i, R^{F_i''} = \beta_i, R^{F_j} = p_j, R^{F_i'} = p_i p_{i+1} \beta_i p_{i+1} p_i$ where s_i, β_i, p_j are as in § 2.1.

We will obtain a basis for the signed partition algebra defined in Definition 2.7.

Definition 2.8 ([6], Definition 3.1.2)**.** Let $d \in R^{\mathbb{Z}_2}_{\underline{k} \cup \underline{k}'}$. For $0 \leqslant 2s_1 + s_2 \leqslant 2k-1$ and $0 \leqslant s_1, s_2 \leqslant k-1$, define

$$\overrightarrow{I}^{2k}_{2s_1+s_2} = \left\{ d \in I^{2k}_{2s_1+s_2} \;\middle|\; \begin{array}{ll} (i) & s_1+s_2+r_1+r_2 \leqslant k-1 \text{ and} \\ & s_1+s_2+r_1'+r_2' \leqslant k-1, \text{ or} \\ (ii) & s_1+s_2+r_1+r_2 \leqslant k \text{ and} \\ & s_1+s_2+r_1'+r_2' \leqslant k-1 \text{ then } r_1 \neq 0, \text{ or} \\ (iii) & s_1+s_2+r_1+r_2 \leqslant k-1 \text{ and} \\ & s_1+s_2+r_1'+r_2' \leqslant k \text{ then } r_1' \neq 0, \text{ or} \\ (iv) & s_1+s_2+r_1+r_2 \leqslant k \text{ and} \\ & s_1+s_2+r_1'+r_2' \leqslant k \text{ then } r_1 \neq 0 \text{ and } r_1' \neq 0. \end{array} \right\},$$

where

(a) $s_1 = \natural\{(C^e, C^g) : C^d$ is a through class of R^d and $H^d_C = \{e\}\}$,

(b) $s_2 = \natural\{C^{\mathbb{Z}_2} : C^d$ is a through class of R^d and $H^d_C = \mathbb{Z}_2\}$,

(c) $r_1\,(r_1')$ is the number of horizontal edges C^d in the top(bottom) row of R^d such that $H^d_C = \{e\}$

(d) $r_2\,(r_2')$ is the number of horizontal edges C^d in the top(bottom) row of R^d such that $H^d_C = \mathbb{Z}_2$

(e) $\sharp^p\left(R^d\right) = s_1 + s_2$.

Also, $\overrightarrow{I}^{2k}_{2k} = I^{2k}_{2k}$.

For $0 \leqslant s \leqslant 2k$, put $\overrightarrow{I}^{2k}_{s} = \bigcup_{2s_1+s_2 \leqslant s} \overrightarrow{I}^{2k}_{2s_1+s_2}$.

Proposition 2.9. 1) *The linear span of* $\overrightarrow{I}^{2k}_{s}$, $0 \leqslant s \leqslant 2k$ *is the signed partition algebra* $\overrightarrow{A}^{\mathbb{Z}_2}_{k}$.

2) *The linear span of* I^{2k}_{s} *is an ideal of* $\overrightarrow{A}^{\mathbb{Z}_2}_{k}$.

Remark 2.10. The algebra generated by $\{R^{F_i'}, R^{G_i}, R^{F_i''}, R^{F_j}\}_{\substack{1 \leqslant i \leqslant k-1 \\ 1 \leqslant j \leqslant k}}$ is isomorphic to the partition algebra $A_k(x)$.

Also, let I^k_s be the set of all k-partition diagrams R^d in $A_k(x)$ such that $\sharp^p(R^d) \leqslant s$ where $d \in I^{2k}_{2s_1+0} \subseteq A_{2k}(x^2)$.

Definition 2.11 ([5], Definition 4.2)**.** Define,

(i) $M^k[(s,(s_1,s_2))] = \Big\{(d,P) \mid d \in R^{\mathbb{Z}_2}_{\underline{k}}, P \in R^{\mathbb{Z}_2}_{\underline{s_1+s_2}}$ and $d \setminus P \in R^{\mathbb{Z}_2}_{\underline{k-s_1-s_2}}, |d| \geqslant 2s_1+s_2, P$ is a $\mathbb{Z}_2$-stable subset of d with $|P| = s$ where $s = 2s_1+s_2, P = \bigcup_{i=1}^{s_1}(P^e_i \cup P^g_i)\bigcup_{j=1}^{s_2} P^{\mathbb{Z}_2}_j$ such that $H^d_{P_i^{\{e\}}} = \{e\}, 1 \leqslant i \leqslant s_1,\ H^d_{P_j^{\mathbb{Z}_2}} = \mathbb{Z}_2, 1 \leqslant j \leqslant s_2\Big\}$.

(ii) $\overrightarrow{M}^k[(s,(s_1,s_2))] = \Big\{(d,P) \in M^k[(s,(s_1,s_2))] \Big| s_1+s_2+r_1+r_2 \leqslant k-1$ and if $s_1+s_2+r_1+r_2 = k$ then $s_1 = k$ or $r_1 \neq 0$ where $2r_1$ is the number of $\{e\}$-connected components in $d \setminus P$ and r_2 is the number of $\mathbb{Z}_2 -$ connected components in $d \setminus P\Big\}$.

We shall now introduce an ordering for the connected components in P. Suppose that

$$P = \bigcup_{1\leqslant i\leqslant s_1}(P^e_i \cup P^g_i) \cup \bigcup_{1\leqslant j\leqslant s_2} P^{\mathbb{Z}_2}_j$$

then $R^P = \bigcup_{1\leqslant i\leqslant s_1} R^{P_i^{\{e\}}} \cup \bigcup_{1\leqslant j\leqslant s_2} R^{P_j^{\mathbb{Z}_2}}$.

Let $a_{11},\cdots,a_{1s_1}$ be the minimal vertices of the connected components $R^{P_1^{\{e\}}},\cdots,R^{P_{s_1}^{\{e\}}}$ in R^P and $b_{11},\cdots,b_{1s_2}$ be the minimal vertices of the connected components $R^{P_1^{\mathbb{Z}_2}},\cdots,R^{P_{s_2}^{\mathbb{Z}_2}}$ in R^P then

$$P^e_i < P^e_j \text{ and } P^g_i < P^g_j \iff R^{P_i^{\{e\}}} < R^{P_j^{\{e\}}} \iff a_{1i} < a_{1j} \in R^P$$

and

$$P^{\mathbb{Z}_2}_l < P^{\mathbb{Z}_2}_f \iff R^{P_l^{\mathbb{Z}_2}} < R^{P_f^{\mathbb{Z}_2}} \iff b_{1l} < b_{1f} \in R^P.$$

Since $\overrightarrow{M}^k[(s,(s_1,s_2))] \subseteq M^k[(s,(s_1,s_2))]$, the above ordering can be used for the connected components P when $(d,P) \in \overrightarrow{M}^k[(s,(s_1,s_2))]$.

Lemma 2.12 ([5], Lemma 4.3)**.** *Let $M^k[(s,(s_1,s_2))]$ and $\overrightarrow{M}^k[(s,(s_1,s_2))]$ be as in Definition 2.11.*

(i) *Each $d \in I^{2k}_{2s_1+s_2}$ can be associated with a pair of elements $(d^+,P),(d^-,Q) \in M^k[(s,(s_1,s_2))]$ and an element $((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ where $(d^+,P),(d^-,Q) \in M^k[(s,(s_1,s_2))]$ and $((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$.*

(ii) *Each $d \in \overrightarrow{I}^{2k}_{2s_1+s_2}$ can be associated with a pair of elements $(d^+,P),(d^-,Q) \in \overrightarrow{M}^k[(s,(s_1,s_2))]$ and an element $((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ where $(d^+,P),(d^-,Q) \in \overrightarrow{M}^k[(s,(s_1,s_2))]$ and $((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$.*

Definition 2.13 ([5], Definition 4.6). (i) Define a map

$$\phi^s_{s_1,s_2} : M^k[(s,(s_1,s_2))] \times M^k[(s,(s_1,s_2))] \to R[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}]$$

as follows:

$$\phi^s_{s_1,s_2}\big((d',P),(d'',Q)\big) = x^{l(P\vee Q)}((f,\sigma_1),\sigma_2);$$

(ii) define a map

$$\overrightarrow{\phi}^s_{s_1,s_2} : \overrightarrow{M}^k[(s,(s_1,s_2))] \times \overrightarrow{M}^k[(s,(s_1,s_2))] \to R[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}]$$

as follows:

$$\overrightarrow{\phi}^s_{s_1,s_2}\big((d',P),(d'',Q)\big) = x^{l(P\vee Q)}((f,\sigma_1),\sigma_2)$$

Case (i): if

(a) no two connected components of Q in d'' have non-empty intersection with a common connected component of d' in $d'.d''$, or vice versa;

(b) no connected component of Q has non-empty intersection only with the connected components excluding the connected components of P in $d'.d''$. Similarly, no connected component in P has non-empty intersection only with a connected component excluding the connected components of Q in $d'.d''$.

Here $l(P \vee Q)$ denotes the number of connected components in $d'.d''$ excluding the union of all the connected components of P and Q and $d'.d'' \in R^{\mathbb{Z}_2}_{\underline{k}\cup\underline{k}'}$ is the smallest d in $R^{\mathbb{Z}_2}_{\underline{k}\cup\underline{k}'}$ such that $d' \cup d'' \subset d$. The permutation $((f,\sigma_1),\sigma_2)$ is obtained as follows. If there is a unique connected component in $d'.d''$ containing P^e_i and $Q^{g'}_j$ then, define $\sigma_1(i) = j$ and

$$f(i) = \begin{cases} \overline{1}, & \text{if } g' = g; \\ \overline{0}, & \text{if } g' = e. \end{cases}$$

Also, if there is a unique connected component in $d'.d''$ containing $P^{\mathbb{Z}_2}_l$ and $Q^{\mathbb{Z}_2}_f$ then, define $\sigma_2(l) = f$).

Case (ii): Otherwise,

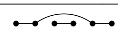
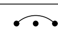
$$\phi^s_{s_1,s_2}\big((d',P),(d'',Q)\big) = 0 \quad \text{and} \quad \overrightarrow{\phi}^s_{s_1,s_2}\big((d',P),(d'',Q)\big) = 0.$$

Definition 2.14. Let $(d,P) \in M^k[(s,(s_1,s_2))]$ such that $|d\backslash P| = 2r_1+r_2$ where $M^k[(s,(s_1,s_2))]$ be as in Definition 2.11.

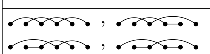
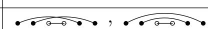
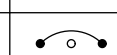
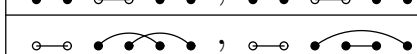
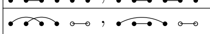
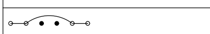
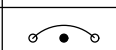
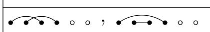
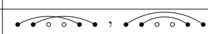
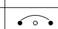
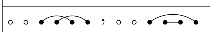
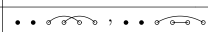
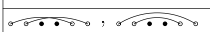
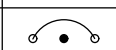
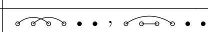
Let $\{P^g_{1i}, g \in \mathbb{Z}_2\}_{1\leqslant i\leqslant s_1} \cup \{P^{\mathbb{Z}_2}_{2j}\}_{1\leqslant j\leqslant s_2}$ be the connected components in P and $\{P^{g'}_{3l}, g' \in \mathbb{Z}_2\}_{1\leqslant l\leqslant r_1} \cup \{P^{\mathbb{Z}_2}_{4m}\}_{1\leqslant m\leqslant r_2}$ be the connected components in $d \setminus P$. Define a map $\phi : M^k[(s,(s_1,s_2))] \to P(k)$ as $\phi((d,P)) = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ where $\alpha_1 \vdash k_1$, $\alpha_2 \vdash k_2$, $\alpha_3 \vdash k_3$, $\alpha_4 \vdash k_4$ with $k_1+k_2+k_3+k_4 = k$, $\alpha_1 = (\alpha_{11},\alpha_{12},\cdots,\alpha_{1s_1})$, $\alpha_2 = (\alpha_{21},\alpha_{22},\cdots,\alpha_{2s_2})$, $\alpha_3 = (\alpha_{31},\alpha_{32},\cdots,\alpha_{3r_1})$ and $\alpha_4 = (\alpha_{41},\alpha_{42},\cdots,\alpha_{4r_2})$ such that $|P_{1i}| = \alpha_{1i}$, $|P_{2j}| = \alpha_{2j}$, $|P_{3l}| = \alpha_{3l}$, $|P_{4m}| = \alpha_{4m}$ respectively for all $1 \leqslant i \leqslant s_1$, $1 \leqslant j \leqslant s_2$, $1 \leqslant l \leqslant r_1$ and $1 \leqslant m \leqslant r_2$.

Example 2.15. The following example illustrates the use of $2s_1+s_2$ instead of $s = 2s_1+s_2$ to denote the number of through classes for the diagrams in algebra of $\mathbb{Z}_2$-relations and signed partition algebras.

For $s_1 = 0$ and $s_2 = 2$,

$(d,P) \in \overrightarrow{M}^3[(2,(0,2))]$	partition of (d,P)	$R^{(d,P)}$	partition of $R^{(d,P)}$	$\overrightarrow{M}^3[(2,(0,2))]$ $(d,P) \in$	partition of (d,P)	$R^{(d,P)}$	partition of $R^{(d,P)}$
•—•—•—• •—•	$(\Phi,(2,1),\Phi,\Phi)$	•—• •	$(2,1)$		$(\Phi,(2,1),\Phi,\Phi)$		$(2,1)$
•—• •—•—•—•	$(\Phi,(2,1),\Phi,\Phi)$	• •—•	$(2,1)$	•—• •—• ∘ ∘	$(\Phi,(1,1),1,\Phi)$	• • ∘	$(1^2,1)$
•—• ∘ ∘ •—•	$(\Phi,(1,1),1,\Phi)$	• ∘ •	$(1^2,1)$	∘ ∘ •—• •—•	$(\Phi,(1,1),1,\Phi)$	∘ • •	$(1^2,1)$

For $s_1 = 1$ and $s_2 = 0$,

$(d,P) \in \overrightarrow{M}^3[(2,(1,0))]$	partition of (d,P)	$R^{(d,P)}$	partition of $R^{(d,P)}$	$(d,P) \in \overrightarrow{M}^3[(2,(1,0))]$	partition of (d,P)	$R^{(d,P)}$	partition of $R^{(d,P)}$
	$(3,\Phi,\Phi,\Phi)$	•—•—•	$(3,\Phi)$		$(2,\Phi,\Phi,1)$		$(2,1)$
					$(2,\Phi,\Phi,1)$	∘ •—•	$(2,1)$
	$(2,\Phi,\Phi,1)$	•—• ∘	$(2,1)$	• • ∘—∘—∘—∘	$(1,\Phi,\Phi,2)$	• ∘—∘	$(1,2)$
	$(1,\Phi,\Phi,2)$		$(2,1)$	∘—∘—∘—∘ • •	$(1,\Phi,\Phi,2)$	∘—∘ •	$(1,2)$
	$(2,\Phi,1,\Phi)$	•—• ∘	$(2,1)$		$(2,\Phi,1,\Phi)$		$(2,1)$
	$(2,\Phi,1,\Phi)$	∘ •—•	$(2,1)$		$(1,\Phi,2,\Phi)$	• ∘—∘	$(1,2)$
	$(1,\Phi,2,\Phi)$		$(1,2)$		$(1,\Phi,2,\Phi)$	∘—∘ •	$(1,2)$
• • ∘—∘ ∘ ∘ ; • • ∘ ∘ ∘—∘ • • ∘ ∘ ∘ ∘	$(1,\Phi,1,1)$	• ∘ ∘	$(1,1^2)$	∘—∘ • • ∘ ∘ ; ∘ ∘ • • ∘—∘ ∘ ∘ • • ∘ ∘	$(1,\Phi,1,1)$	∘ • ∘	$(1,1^2)$
∘—∘ ∘ ∘ • • ; ∘ ∘ ∘—∘ • •, ∘ ∘ ∘ ∘ • •	$(1,\Phi,1,1)$	∘ ∘ •	$(1,1^2)$				

In the above diagrams, connected components with thick dots (hollow dots) belongs to $P(d \setminus P)$. In partition algebra, for any d whose top row is (d,P) and the bottom row is (d',P') with $|P| = s$ then the number of possible ways to permute the through classes in d will be $s!$ ways.In case of signed partition algebras, for $(d,P),(d',P') \in M^k[(s,(s_1,s_2))]$ with $|P| = |P'| = 2s_1+s_2 = s$, then the number of diagram d's whose top row is (d,P) and bottom row is (d',P') will be $2^{s_1}\ s_1!\ s_2!$. Since $\{e\}$-connected components($\mathbb{Z}_2$-connected components) in P can be joined only to $\{e\}$-connected components($\mathbb{Z}_2$-connected components) in P'.

Moreover, By Definition 2.8 we know that

$$\overrightarrow{I}_s^{2k} = \bigcup_{2s_1+s_2 \leqslant s} \overrightarrow{I}_{2s_1+s_2}^{2k}.$$

Let $\overrightarrow{L}_s^{2k}$ be the linear span of $\overrightarrow{I}_s^{2k}$ for every $0 \leqslant s \leqslant 2k$ then $\overrightarrow{L}_s^{2k}$ is an ideal of $\overrightarrow{I}_s^{2k}$ and the quotient $\overrightarrow{L}_s^{2k}/\overrightarrow{L}_{s-1}^{2k}$ = linear span of $\{d \mid \sharp^p(d) = s\}$. For example, $\overrightarrow{I}_2^6 = \overrightarrow{I}_{2\times1+0}^6 \cup \overrightarrow{I}_{2\times0+2}^6 \cup \overrightarrow{I}_{2\times0+1}^6 \cup \overrightarrow{I}_{2\times0+0}^6$ and $\overrightarrow{I}_1^6 = \overrightarrow{I}_{2\times0+1}^6 \cup \overrightarrow{I}_{2\times0+0}^6$ then the quotient ring $\overrightarrow{L}_2^6/\overrightarrow{L}_1^6$ splits into a direct sum of four ideals A_1, A_2, A_3, A_4 where

A_1 is the linear span of

$$\left\{ d\left(\frac{((0,id),id) + ((0,id),\sigma_2)}{2}\right) \;\middle|\; d = \widetilde{U}_{(d,P)}^{(d,P)} \right\}_{\widetilde{U}_{(d,P)}^{(d,P)} \in J_{2\times0+2}^6},$$

A_2 is the linear span of

$$\left\{ d\left(\frac{((0,id),id) - ((0,id),\sigma_2)}{2}\right) \;\middle|\; d = \widetilde{U}_{(d,P)}^{(d,P)} \right\}_{\widetilde{U}_{(d,P)}^{(d,P)} \in J_{2\times0+2}^6},$$

B_1 is the linear span of

$$\left\{ d\left(\frac{((0,id),id) + ((0,\sigma_1),id)}{2}\right) \;\middle|\; d = \widetilde{U}_{(d,P)}^{(d,P)} \right\}_{\widetilde{U}_{(d,P)}^{(d,P)} \in J_{2\times1+0}^6},$$

B_2 is the linear span of

$$\left\{ d\left(\frac{((0,id),id) - ((0,\sigma_1),id)}{2}\right) \;\middle|\; d = \widetilde{U}_{(d,P)}^{(d,P)} \right\}_{\widetilde{U}_{(d,P)}^{(d,P)} \in J_{2\times1+0}^6}.$$

Here $\sigma_1^2 = \mathrm{Id}$, $\sigma_2^2 = \mathrm{Id}$ and $0(i) = 0$ for every i.

3. Gram matrices and $(s_1, s_2, r_1, r_2, p_1, p_2)$-Stirling numbers

In this section, we introduce a new class of matrices $G_{2s_1+s_2}^k$, $\overrightarrow{G}_{2s_1+s_2}^k$ and G_s^k of the algebra of $\mathbb{Z}_2$-relations, signed partition algebras and partition algebras respectively which will be called as Gram matrices since by Theorem 3.8 in [1] the Gram matrices $G_{2s_1+s_2}^{\lambda,\mu}$ associated to the cell modules of

$$W[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$$

(for $\lambda = ([s_1], \Phi)$, $\mu = [s_2]$ if $s_1, s_2 \neq 0$; $\lambda = (\Phi, \Phi)$, $\mu = [s_2]$ if $s_1 = 0$, $s_2 \neq 0$; $\lambda = ([s_1], \Phi)$, $\mu = \Phi$ if $s_1 \neq 0$, $s_2 = 0$; $\lambda = (\Phi, \Phi)$, $\mu = \Phi$ if $s_1 = s_2 = 0$, $0 \leqslant s_1 \leqslant k$, $0 \leqslant s_2 \leqslant k$ and $0 \leqslant s_1 + s_2 \leqslant k$)

and

$$\overrightarrow{G}^{\lambda,\mu}_{2s_1+s_2}\overrightarrow{W}[(s,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$$

(for $\lambda = ([s_1], \Phi)$, $\mu = [s_2]$ if $s_1, s_2 \neq 0; \lambda = (\Phi, \Phi)$, $\mu = [s_2]$ if $s_1 = 0, s_2 \neq 0; \lambda = ([s_1], \Phi)$, $\mu = \Phi$ if $s_1 \neq 0, s_2 = 0$; $\lambda = (\Phi, \Phi)$, $\mu = \Phi$ if $s_1 = s_2 = 0$, $0 \leqslant s_1 \leqslant k$, $0 \leqslant s_2 \leqslant k-1$ and $0 \leqslant s_1 + s_2 \leqslant k-1$) defined in Definition 6.3 of [5] coincides with the matrices $G^k_{2s_1+s_2}$ and $\overrightarrow{G}^k_{2s_1+s_2}$ respectively.

In this paper, $(s_1, s_2, r_1, r_2, p_1, p_2)$-Stirling numbers of the second kind for the algebra of $\mathbb{Z}_2$-relations and signed partition algebras are introduced and their identities are established. Stirling numbers of second kind corresponding to the partition algebras are also introduced and their identities are established.

We begin by calculating the size of the Gram matrices before explaining the entries of the Gram matrices.

Definition 3.1. Put

(a) $\Omega^{r_1,r_2}_{s_1,s_2} = \Big\{ [\alpha_1]^1 [\alpha_2]^2 [\alpha_3]^3 [\alpha_4]^4 \Big| \alpha_1 \vdash k_1, \alpha_2 \vdash k_2, \alpha_3 \vdash k_3, \alpha_4 \vdash k_4$ with $\alpha_1 \in \mathbb{P}(k_1, s_1), \alpha_2 \in \mathbb{P}(k_2, s_2), \alpha_3 \in \mathbb{P}(k_3, r_1), \alpha_4 \in \mathbb{P}(k_4, r_2)$ such that $k_1 + k_2 + k_3 + k_4 = k \Big\}$ where $\alpha_1 = (\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1s_1})$, $\alpha_2 = (\alpha_{21}, \alpha_{22}, \cdots, \alpha_{2s_2})$, $\alpha_3 = (\alpha_{31}, \alpha_{32}, \cdots, \alpha_{3r_1})$ and $\alpha_4 = (\alpha_{41}, \alpha_{42}, \cdots, \alpha_{4r_2})$.

(b) $\overrightarrow{\Omega}^{r_1,r_2}_{s_1,s_2} = \Big\{ [\alpha_1]^1 [\alpha_2]^2 [\alpha_3]^3 [\alpha_4]^4 \in \Omega^{r_1,r_2}_{s_1,s_2} \Big| s_1 + s_2 + r_1 + r_2 \leqslant k-1$ and if $s_1 + s_2 + r_1 + r_2 = k$ then $r_1 \neq 0$ or $s_1 = k \Big\}$.

(c) $\Omega^r_s = \{[\alpha_1]^1[\alpha_2]^2 \mid \alpha_1 \in \mathbb{P}(k_1, s), \alpha_2 \in \mathbb{P}(k_2, r)$ such that $k_1 + k_2 = k\}$.

Definition 3.2. Let $\alpha = [\alpha_1]^1[\alpha_2]^2[\alpha_3]^3[\alpha_4]^4 \in \Omega^{r_1,r_2}_{s_1,s_2}$. We shall draw a graph corresponding to the partition $\alpha = [\alpha_1]^1[\alpha_2]^2[\alpha_3]^3[\alpha_4]^4$ on the vertices $(i, e), (i, g)$ for all $1 \leqslant i \leqslant k$ and $1' \leqslant i \leqslant k'$ arranged in two rows of each having k-vertices labeled from left to right. The edges are drawn as follows:

(a) Draw an edge connecting the vertices

$$\Big(\Big(\sum_{n=1}^{i-1}|\alpha_{1n}|\Big)+1,e\Big),\Big(\Big(\sum_{n=1}^{i-1}|\alpha_{1n}|\Big)+2,e\Big),\cdots,\Big(\Big(\sum_{n=1}^{i}|\alpha_{1n}|\Big),e\Big),$$
$$\Big(\Big(\Big(\sum_{n=1}^{i-1}|\alpha_{1n}|\Big)+1\Big)',e\Big),\Big(\Big(\Big(\sum_{n=1}^{i-1}|\alpha_{1n}|\Big)+2\Big)',e\Big),\Big(\Big(\sum_{n=1}^{i}|\alpha_{1n}|\Big)',e\Big)$$

and denote it by P^e_{1i} for $1\leqslant i\leqslant s_1$. Since the diagram has to be a $\mathbb{Z}_2$-stable diagram there should be a copy of the connected component which is obtained by connecting the vertices

$$\Big(\Big(\sum_{n=1}^{i-1}|\alpha_{1n}|\Big)+1,g\Big),\Big(\Big(\sum_{n=1}^{i-1}|\alpha_{1n}|\Big)+2,g\Big),\cdots,\Big(\Big(\sum_{n=1}^{i}|\alpha_{1n}|\Big),g\Big),$$
$$\Big(\Big(\Big(\sum_{n=1}^{i-1}|\alpha_{1n}|\Big)+1\Big)',g\Big),\Big(\Big(\Big(\sum_{n=1}^{i-1}|\alpha_{1n}|\Big)+2\Big)',g\Big),\Big(\Big(\sum_{n=1}^{i}|\alpha_{1n}|\Big)',e\Big)$$

and denote it by P^g_{1i} for $1\leqslant i\leqslant s_1$. The connected components P^e_{1i} and P^g_{1i} for $1\leqslant i\leqslant s_1$ are called $\{e\}$-through classes.

(b) Draw an edge connecting the vertices

$$\Big(\Big(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{m=1}^{j-1}|\alpha_{2m}|\Big)+1,e\Big),\Big(\Big(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{m=1}^{j-1}|\alpha_{2m}|\Big)+1,g\Big),$$
$$\cdots,\Big(\Big(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{m=1}^{j}|\alpha_{2m}|\Big),e\Big),$$
$$\Big(\Big(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{m=1}^{j}|\alpha_{2m}|\Big),g\Big),\Big(\Big(\Big(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{m=1}^{j-1}|\alpha_{2m}|\Big)+1\Big)',e\Big),$$
$$\Big(\Big(\Big(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{m=1}^{j-1}|\alpha_{2m}|\Big)+1\Big)',g\Big),\cdots,\Big(\Big(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{m=1}^{j}|\alpha_{2m}|\Big)',e\Big),$$
$$\Big(\Big(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{m=1}^{j}|\alpha_{2m}|\Big)',g\Big)$$

and denote it by $P^{\mathbb{Z}_2}_{2j}$ for $1\leqslant j\leqslant s_2$.

The connected components $P^{\mathbb{Z}_2}_{2j}$ for $1\leqslant j\leqslant s_2$ are called $\mathbb{Z}_2$-through classes.

(c) Draw edges connecting the vertices

$$\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{f=1}^{l-1}|\alpha_{3f}|\right)+1,e\right),\\ \cdots,\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{f=1}^{l}|\alpha_{3f}|\right),e\right)$$

in the top row and

$$\left(\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{f=1}^{l-1}|\alpha_{3f}|\right)+1\right)',e\right),\\ \cdots,\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{f=1}^{l}|\alpha_{3f}|\right)',e\right)$$

in the bottom row and denote it by P_l^e and $P_l^{'e}$ respectively. Since the diagram has to be $\mathbb{Z}_2$-stable diagram there will be copy of the above connected components obtained by connecting the vertices

$$\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{f=1}^{l-1}|\alpha_{3f}|\right)+1,g\right),\\ \cdots,\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{f=1}^{l}|\alpha_{3f}|\right),g\right)$$

in the top row

$$\left(\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{f=1}^{l-1}|\alpha_{3f}|\right)+1\right)',g\right),\\ \cdots,\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{f=1}^{l}|\alpha_{3f}|\right)',g\right)$$

and denote it by P_l^g and $P_l^{'g}$ respectively.

The connected components $P_l^e, P_l^{'e}, P_l^g$ and $P_l^{'g}$ for $1 \leqslant l \leqslant r_1$ are called $\{e\}$-horizontal edges.

(d) Draw edges connecting the vertices

$$\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{l=1}^{r_1}|\alpha_{3l}|+\sum_{t=1}^{m-1}|\alpha_{4t}|\right)+1,e\right),$$
$$\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{l=1}^{r_1}|\alpha_{3l}|+\sum_{t=1}^{m-1}|\alpha_{4t}|\right)+1,g\right),$$
$$\cdots,\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{l=1}^{r_1}|\alpha_{3l}|+\sum_{t=1}^{m}|\alpha_{4t}|\right),e\right),$$
$$\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{l=1}^{r_1}|\alpha_{3l}|+\sum_{t=1}^{m}|\alpha_{4t}|\right),g\right)$$

in the top row and

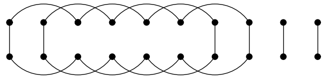
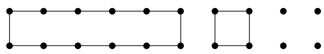
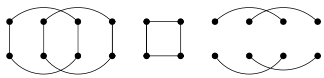
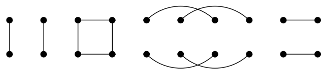
$$\left(\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{l=1}^{r_1}|\alpha_{3l}|+\sum_{t=1}^{m-1}|\alpha_{4t}|\right)+1\right)',e\right),$$
$$\left(\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{l=1}^{r_1}|\alpha_{3l}|+\sum_{t=1}^{m-1}|\alpha_{4t}|\right)+1\right)',g\right),$$
$$\cdots,\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{l=1}^{r_1}|\alpha_{3l}|+\sum_{t=1}^{m}|\alpha_{4t}|\right)',e\right),$$
$$\left(\left(\sum_{i=1}^{s_1}|\alpha_{1i}|+\sum_{j=1}^{s_2}|\alpha_{2j}|+\sum_{l=1}^{r_1}|\alpha_{3l}|+\sum_{t=1}^{m}|\alpha_{4t}|\right)',g\right)$$

in the bottom row and it is denoted by $P_m^{\mathbb{Z}_2}$ and $P_m^{'\mathbb{Z}_2}$ for $1 \leqslant m \leqslant r_2$.

The connected components $P_m^{\mathbb{Z}_2}, P_m^{'\mathbb{Z}_2}$ for $1 \leqslant m \leqslant r_2$ are called $\mathbb{Z}_2$-horizontal edges.

The diagram obtained above is called standard diagram and it is denoted by U^α where $\alpha = [\alpha_1]^1[\alpha_2]^2[\alpha_3]^3[\alpha_4]^4 \in \Omega_{s_1,s_2}^{r_1,r_2}$.

Example 3.3. The following are some examples of standard diagrams of U^α type in signed partition algebras $\overrightarrow{A}_5^{\mathbb{Z}_2}$ with their corresponding partitions.

		corresponding partition	α_1	α_2	α_3	α_4
d_1		(α_1)	$(4,1)$	Φ	Φ	Φ
d_2		(α_2, α_3)	Φ	$(3,1)$	(1)	Φ
d_3		$(\alpha_1, \alpha_2, \alpha_3)$	(2)	(1)	(2)	Φ
d_4		$(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$	(1)	(1)	(2)	(1)

Remark 3.4. Let $d \in I^{2k}_{2s_1+s_2}$. By Lemma 2.12, for any $d \in I^{2k}_{2s_1+s_2}$ we can associate a pair $(d^+, P), (d^-, Q) \in M^k[(s,(s_1,s_2))]$ and an element $((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ and vice versa and it is denoted by $U^{(d^+,P)}_{(d^-,Q)}((f,\sigma_1),\sigma_2)$. If $d^+ = d^-, P = Q$ and $((f,\sigma_1),\sigma_2) = ((0,id),id) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ then without loss of generality we can write such d as $\widetilde{U}^{(d,P)}_{(d,P)}$.

Definition 3.5. Let $\alpha = [\alpha_1]^1 [\alpha_2]^2 [\alpha_3]^3 [\alpha_4]^4 \in \Omega^{r_1,r_2}_{s_1,s_2}$. Define,

$$\mathrm{St}^c(U^\alpha) = \left\{ \sigma \in \mathbb{Z}_2 \wr \mathfrak{S}_k \;\middle|\; \sigma U^\alpha \sigma^{-1} = U^\alpha \right\}$$

where U^α is the standard diagram corresponding to the partition α as in Definition 3.2.

Note 1. (i) Let $U^{\overrightarrow{\alpha}}$ denote the standard diagram in signed partition algebra corresponding to the partition $\overrightarrow{\alpha} \in \overrightarrow{\Omega}^{r_1,r_2}_{s_1,s_2}$ and R^{U^α} denote the standard diagram in partition algebra corresponding to the partition $R^\alpha \in \Omega^r_s$ which can be defined as in Definition 3.2, $\mathrm{St}^c\left(U^{\overrightarrow{\alpha}}\right)$ and $\mathrm{St}^c\left(R^{U^\alpha}\right)$ can also be defined as in Definition 3.5 for the signed partition algebras $\overrightarrow{A}^{\mathbb{Z}_2}_k(x)$ and the partition algebras $A_k(x)$.

(ii) All other diagrams $U^{(d,P)}_{(d,P)}, \overrightarrow{U}^{(d,P)}_{(d,P)}$, and $R^{U^{(d,P)}_{(d,P)}}$ whose underlying partition is same as the underlying partition of $U^\alpha, U^{\overrightarrow{\alpha}}$ and R^{U^α} respectively can be obtained as follows:

$$U^{(d,P)}_{(d,P)} = \tau U^\alpha \tau^{-1}, \quad \overrightarrow{U}^{(d,P)}_{(d,P)} = \overrightarrow{\tau} U^{\overrightarrow{\alpha}} \overrightarrow{\tau}^{-1} \quad \text{and} \quad R^{U^{(d,P)}_{(d,P)}} = \rho R^{U^\alpha} \rho$$

where $\tau, \in \mathbb{Z}_2 \wr \mathfrak{S}_k$ and $\rho \in \mathfrak{S}_k$ are the coset representatives of $\mathrm{St}^c(U^\alpha)$, $\mathrm{St}^c(U^{\overrightarrow{\alpha}})$ and $\mathrm{St}^c(R^{U^\alpha})$ respectively. Also, $U^\alpha, U^{\overrightarrow{\alpha}}$ and R^{U^α} are the standard diagrams as in Definition 3.2.

Notation 3.6. (a) For $0 \leqslant r_1, r_2 \leqslant k - s_1 - s_2$ and $0 \leqslant s_1, s_2 \leqslant k$, put

$$J^{2k}_{2s_1+s_2} = \bigcup_{0 \leqslant r_1+r_2 \leqslant k-s_1-s_2} \mathbb{J}^{2r_1+r_2}_{2s_1+s_2}$$

and

$$\mathbb{J}^{2r_1+r_2}_{2s_1+s_2} = \bigcup_{\alpha=[\alpha_1]^1[\alpha_2]^2[\alpha_3]^3[\alpha_4]^4 \in \Omega^{r_1,r_2}_{s_1,s_2}} \mathbb{J}^{2r_1+r_2,\alpha}_{2s_1+s_2}$$

where

$$\begin{aligned} \mathbb{J}^{2r_1+r_2,\alpha}_{2s_1+s_2} = \Big\{ d \in I^{2k}_{2s_1+s_2} \;\Big|\; & d = \widetilde{U}^{(d,P)}_{(d,P)} \text{ with } d^+ = (d,P), \\ & d^- = (d,P),\ \eta_e\Big(\widetilde{U}^{(d,P)}_{(d,P)}\Big) = s_1,\ \eta_{\mathbb{Z}_2}\Big(\widetilde{U}^{(d,P)}_{(d,P)}\Big) = s_2, \\ & \widetilde{U}^{(d,P)}_{(d,P)} \text{ has } r_1 \text{ number of pairs of } \{e\}\text{-horizontal edges}, \\ & r_2 \text{ number of } \mathbb{Z}_2\text{-horizontal edges}, \\ & (d,P) \in M^k[(s,(s_1,s_2))] \text{ as in Definition 2.10}, \\ & \|P\| = 2s_1 + s_2 \text{ and } \alpha \text{ is the underlying partition of } (d,P) \\ & \text{as in Definition 2.13} \Big\}. \end{aligned}$$

Also,

$$\begin{aligned} \left|\mathbb{J}^{2r_1+r_2,\alpha}_{2s_1+s_2}\right| &= \text{index of } \mathrm{St}^c(U^\alpha) = f^{2r_1+r_2,\alpha}_{2s_1+s_2}, \\ \left|\mathbb{J}^{2r_1+r_2}_{2s_1+s_2}\right| &= \sum_{\alpha=[\alpha_1]^1[\alpha_2]^2[\alpha_3]^3[\alpha_4]^4 \in \Omega^{r_1,r_2}_{s_1,s_2}} \text{index of } \mathrm{St}^c(U^\alpha) = f^{2r_1+r_2}_{2s_1+s_2}, \\ \left|J^{2k}_{2s_1+s_2}\right| &= \sum_{0 \leqslant r_1+r_2 \leqslant k-s_1-s_2} \left|\mathbb{J}^{2r_1+r_2}_{2s_1+s_2}\right|. \end{aligned}$$

$\left|J^{2k}_{2s_1+s_2}\right|$ will define the size of the Gram matrix in the algebra of $\mathbb{Z}_2$-relation and it is denoted by $f_{2s_1+s_2}$.

(b) For $0 \leqslant r_1 \leqslant k - s_1 - s_2$, $0 \leqslant r_2 \leqslant k - s_1 - s_2 - 1$, $0 \leqslant s_1 \leqslant k$, $0 \leqslant s_2 \leqslant k-1$, and $0 \leqslant s_1 + s_2 + r_1 + r_2 \leqslant k - 1$,

(i) if $r_1 \neq 0$ then $\overrightarrow{\mathbb{J}}^{2r_1+r_2,\alpha}_{2s_1+s_2} = \mathbb{J}^{2r_1+r_2,\alpha}_{2s_1+s_2}$;

(ii) if $r_1 = 0$ then

$$\overrightarrow{\mathbb{J}}^{2r_1+r_2,\alpha}_{2s_1+s_2} = \{d \in \mathbb{J}^{2r_1+r_2,\alpha}_{2s_1+s_2} \mid \text{either } s_1 = k \text{ or } s_1+s_2+r_2 \leqslant k-1\},$$

$$\overrightarrow{\mathbb{J}}^{2r_1+r_2}_{2s_1+s_2} = \bigcup_{\alpha=[\alpha_1]^1[\alpha_2]^2[\alpha_3]^3[\alpha_4]^4 \in \overrightarrow{\Omega}^{r_1,r_2}_{s_1,s_2}} \overrightarrow{\mathbb{J}}^{2r_1+r_2,\alpha}_{2s_1+s_2},$$

and

$$\overrightarrow{J}^{2k}_{2s_1+s_2} = \bigcup_{\substack{0\leqslant r_1\leqslant k-s_1-s_2 \\ 0\leqslant r_2\leqslant k-s_1-s_2-1 \\ 0\leqslant r_1+r_2\leqslant k-s_1-s_2}} \overrightarrow{\mathbb{J}}^{2r_1+r_2}_{2s_1+s_2};$$

$$\left|\overrightarrow{\mathbb{J}}^{2r_1+r_2,\alpha}_{2s_1+s_2}\right| = \text{index of } \mathrm{St}^c(U^{\alpha}) = \overrightarrow{f}^{2r_1+r_2,\alpha}_{2s_1+s_2},$$

$$\left|\overrightarrow{\mathbb{J}}^{2r_1+r_2}_{2s_1+s_2}\right| = \sum_{\alpha=[\alpha_1]^1[\alpha_2]^2[\alpha_3]^3[\alpha_4]^4 \in \overrightarrow{\Omega}^{r_1,r_2}_{s_1,s_2}} \text{index of } \mathrm{St}^c(U^{\overrightarrow{\alpha}}) = \overrightarrow{f}^{2r_1+r_2}_{2s_1+s_2},$$

$$\left|\overrightarrow{J}^{2k}_{2s_1+s_2}\right| = \sum_{\substack{0\leqslant r_1\leqslant k-s_1-s_2 \\ 0\leqslant r_2\leqslant k-s_1-s_2-1 \\ 0\leqslant r_1+r_2\leqslant k-s_1-s_2}} \left|\overrightarrow{\mathbb{J}}^{2r_1+r_2}_{2s_1+s_2}\right|.$$

$\left|\overrightarrow{J}^{2k}_{2s_1+s_2}\right|$ will define the size of the Gram matrix in signed partition algebras and it is denoted by $\overrightarrow{f}_{2s_1+s_2}$.

(c) For $0 \leqslant r \leqslant k-s$, $0 \leqslant s \leqslant k$ put

$$J^k_s = \bigcup_{0\leqslant r\leqslant k-s} \mathbb{J}^r_s \quad \text{and} \quad \mathbb{J}^r_s = \bigcup_{\alpha=[\alpha_1]^1[\alpha_2]^2 \in \Omega^r_s} \mathbb{J}^{r,\alpha}_s,$$

where $\mathbb{J}^{r,\alpha}_s = \Big\{ R^d \in I^k_s \;\Big|\; R^d = U^{(R^d)^+}_{(R^d)^-}, \left(R^d\right)^+$ and $\left(R^d\right)^-$ are the same, $\sharp^p(U^{(R^d)^+}_{(R^d)^-}) = s$, $U^{(R^d)^+}_{(R^d)^-}$ has r number of horizontal edges and α is the underlying partition of $R^d \Big\}$.

For the sake of simplicity we write, $U^{(R^d)^+}_{(R^d)^-} = U^{R^d}_{R^d}$. Also, $|\mathbb{J}^{r,\alpha}_s| =$ index of $\mathrm{St}^c\left(U^{R^\alpha}\right) = f^{r,\alpha}_s$, $|\mathbb{J}^r_s| = \sum_{R^\alpha=[\alpha_1]^1[\alpha_2]^2 \in \Omega^r_s} \text{index of } \mathrm{St}^c\left(U^{R^\alpha}\right) = f^r_s$ and $\left|J^k_s\right| = \sum_{0\leqslant r\leqslant k-s} |\mathbb{J}^r_s|$.

$\left|J^k_s\right|$ will define the size of the Gram matrix in the partition algebra and it is denoted by f_s.

Definition 3.7. (a) The diagrams in $J^{2k}_{2s_1+s_2}$ are indexed as follows:

$$\left\{ \left(\widetilde{U}^{(d,P)}_{(d,P)}\right)^{r_1,r_2}_{i,\alpha} \;\middle|\; 1 \leqslant i \leqslant f^{2r_1+r_2,\alpha}_{2s_1+s_2}, \alpha \in \Omega^{r_1,r_2}_{s_1,s_2} \right\}_{\substack{0\leqslant r_1,r_2\leqslant k-s_1-s_2 \\ 0\leqslant r_1+r_2\leqslant k-s_1-s_2}}.$$

$(i,\alpha,r_1,r_2) < (j,\beta,r_1',r_2')$,

(i) if $2r_1+r_2 < 2r_1'+r_2'$

(ii) if $2r_1+r_2 = 2r_1'+r_2'$ and $r_1+r_2 < r_1'+r_2'$

(iii) if $2r_1+r_2 = 2r_1'+r_2', r_1+r_2 = r_1'+r_2'$ and $\alpha < \beta$ (lexicographical ordering)

(iv) if $2r_1+r_2 = 2r_1'+r_2', r_1+r_2 = r_1'+r_2'$ and $\alpha = \beta$ then it can be indexed arbitrarily.

where

r_1 is the number of pairs of $\{e\}$-horizontal edges in $\left(\widetilde{U}^{(d,P)}_{(d,P)}\right)^{r_1,r_2}_{i,\alpha}$,

r_1' is the number of pairs of $\{e\}$-horizontal edges in $\left(\widetilde{U}^{(d,P)}_{(d,P)}\right)^{r_1',r_2'}_{j,\beta}$,

r_2 is the number of $\mathbb{Z}_2$-horizontal edges in $\left(\widetilde{U}^{(d,P)}_{(d,P)}\right)^{r_1,r_2}_{i,\alpha}$,

r_2' is the number of $\mathbb{Z}_2$-horizontal edges in $\left(\widetilde{U}^{(d,P)}_{(d,P)}\right)^{r_1',r_2'}_{j,\beta}$,

$\alpha[\beta]$ is the partition corresponding to the diagram

$$\left(\widetilde{U}^{(d,P)}_{(d,P)}\right)^{r_1,r_2}_{i,\alpha} \left(\left(\widetilde{U}^{(d,P)}_{(d,P)}\right)^{r_1',r_2'}_{j,\beta}\right)$$

and $\alpha,\beta \in \Omega^{r_1,r_2}_{s_1,s_2}$.

(b) Since $\overrightarrow{J}^{2k}_{2s_1+s_2} \subset J^{2k}_{2s_1+s_2}$, we shall use the index defined above in (i) to index the diagrams of $\overrightarrow{J}^{2k}_{2s_1+s_2}$.

(c) The diagrams in J^k_s are indexed as follows:

$$\left\{ \left(U^{R^d}_{R^d}\right)^{r}_{i,\alpha} \;\middle|\; 1 \leqslant i \leqslant f^{r,\alpha}_{s} \text{ and } \alpha \in \Omega^r_s \right\}_{0\leqslant r\leqslant k-s}$$

$(i,r,\alpha) < (j,r',\beta)$,

(1) if $r < r'$,

(2) if $r = r'$ and $\alpha < \beta$ (lexicographic ordering)

(3) if $r = r', \alpha = \beta$, then it can be indexed arbitrarily

where

$r(r')$ is the number of horizontal edges in $\left(U^{R^d}_{R^d}\right)^{r}_{i,\alpha}\left(\left(U^{R^d}_{R^d}\right)^{r'}_{j,\beta}\right)$,

$\alpha(\beta)$ is the partition corresponding to the diagram $\left(U^{R^d}_{R^d}\right)^{r}_{i,\alpha}\left(\left(U^{R^d}_{R^d}\right)^{r'}_{j,\beta}\right)$

and $\alpha, \beta \in \Omega_s^r$.

Now, $(d, P) \mapsto U_{(d,P)}^{(d,P)}$ gives a bijection of $M^k[(s,(s_1,s_2))]$ and $J_{2s_1+s_2}^{2k}$.

Note 2. For the sake of simplicity, we shall write $\left(\widetilde{U}_{(d,P)}^{(d,P)}\right)_{i,\alpha}^{r_1,r_2}$ as $d_{i,\alpha}^{r_1,r_2}$ and $\left(U_{R^d}^{R^d}\right)_{i,\alpha}^{r}$ as $R^{d_{i,\alpha}^r}$.

We shall now explain the entries of the Gram matrices.

Definition 3.8. (a) For $0 \leqslant s_1 + s_2 \leqslant k$, define $G_{2s_1+s_2}^k$ (*Gram matrices of the algebra of $\mathbb{Z}_2$-relations*) as follows:

$$G_{2s_1+s_2}^k = \left(A_{2r_1+r_2, 2r_1'+r_2'}\right)_{\substack{0 \leqslant r_1+r_2 \leqslant k-s_1-s_2 \\ 0 \leqslant r_1'+r_2' \leqslant k-s_1-s_2}}$$

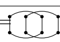
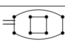
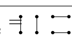
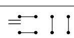
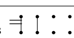
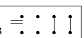
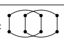
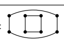
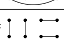
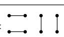
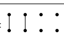
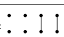
where $A_{2r_1+r_2,2r_1'+r_2'}$ denotes the block matrix whose entries are

$$a_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} = \begin{cases} x^{l(P_i \vee P_j)} & \text{if } \sharp^p\left(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r_1',r_2'}\right) = 2s_1 + s_2, \\ 0 & \text{if } \sharp^p\left(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r_1',r_2'}\right) < 2s_1 + s_2, \end{cases}$$

where $1 \leqslant i \leqslant \left|\mathbb{J}_{2s_1+s_2}^{2r_1+r_2,\alpha}\right|$, $1 \leqslant j \leqslant \left|\mathbb{J}_{2s_1+s_2}^{2r_1'+r_2',\beta}\right|$, $l(P_i \vee P_j) = l\left(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r_1',r_2'}\right)$, $l(P_i \vee P_j)$ denotes the number of connected components in $d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r_1',r_2'}$ excluding the union of all the connected components of P_i and P_j or equivalently, $l\left(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r_1',r_2'}\right)$ is the number of loops which lie in the middle row when $d_{i,\alpha}^{r_1,r_2}$ is multiplied with $d_{j,\beta}^{r_1',r_2'}$, $d_{i,\alpha}^{r_1,r_2} \in \mathbb{J}_{2s_1+s_2}^{2r_1+r_2,\alpha}$ and $d_{j,\beta}^{r_1',r_2'} \in \mathbb{J}_{2s_1+s_2}^{2r_1'+r_2',\beta}$ respectively.

For example,

$G_{2\times 1+0}^2 =$

	$d_{1,\alpha_1}^{0,0} =$ 	$d_{2,\alpha_1}^{0,0} =$ 	$d_{3,\alpha_2}^{1,0} =$ 	$d_{4,\alpha_2}^{1,0} =$ 	$d_{5,\alpha_3}^{1,0} =$ 	$d_{6,\alpha_3}^{1,0} =$ 
$d_{1,\alpha_1}^{0,0} =$ 	1	0	0	0	1	1
$d_{2,\alpha_1}^{0,0} =$ 	0	1	0	0	1	1
$d_{3,\alpha_2}^{0,0} =$ 	0	0	x	0	x	0
$d_{4,\alpha_2}^{0,0} =$ 	0	0	0	x	0	x
$d_{5,\alpha_3}^{0,0} =$ 	1	1	x	0	x^2	0
$d_{6,\alpha_3}^{0,0} =$ 	1	1	0	x	0	x^2

where $\alpha_1 = (2, \Phi, \Phi, \Phi)$, $\alpha_2 = (1, \Phi, \Phi, 1)$ and $\alpha_3 = (1, \Phi, 1, \Phi)$.

(b) For $0 \leqslant s_1 \leqslant k$, $0 \leqslant s_2 \leqslant k-1$, $0 \leqslant s_1+s_2 \leqslant k-1$, define $\overrightarrow{G}^k_{2s_1+s_2}$ (*Gram matrices of signed partition algebra*) as follows:

$$\overrightarrow{G}^k_{2s_1+s_2} = \left(\overrightarrow{A}_{2r_1+r_2, 2r_1'+r_2'}\right)^{0 \leqslant r_1+r_2, r_1'+r_2' \leqslant k-1-s_1-s_2}_{0 \leqslant r_1, r_1' \leqslant k-s_1-s_2,\ 0 \leqslant r_2, r_2' \leqslant k-s_1-s_2-1}$$

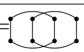
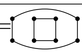
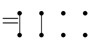
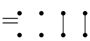
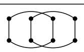
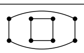
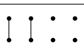
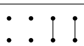
where $\overrightarrow{A}_{2r_1+r_2, 2r_1'+r_2'}$ denotes the block matrix whose entries are

$$a_{(i,\alpha,r_1,r_2),(j,\beta,r_1',r_2')} = \begin{cases} x^{l(P_i \vee P_j)} & \text{if } \sharp^p\left(d^{r_1,r_2}_{i,\alpha}.d^{r_1',r_2'}_{j,\beta}\right) = 2s_1+s_2, \\ 0 & \text{if } \sharp^p\left(d^{r_1,r_2}_{i,\alpha}.d^{r_1',r_2'}_{j,\beta}\right) < 2s_1+s_2, \end{cases}$$

where $1 \leqslant i \leqslant \left|\overrightarrow{\mathbb{J}}^{2r_1+r_2,\alpha}_{2s_1+s_2}\right|$, $1 \leqslant j \leqslant \left|\overrightarrow{\mathbb{J}}^{2r_1'+r_2',\beta}_{2s_1+s_2}\right|$, $l(P_i \vee P_j) = l\left(d^{r_1,r_2}_{i,\alpha}.d^{r_1',r_2'}_{j,\beta}\right)$, $l(P_i \vee P_j)$ denotes the number of connected components in $d^{r_1,r_2}_{i,\alpha}.d^{r_1',r_2'}_{j,\beta}$ excluding the union of all the connected components of P_i and P_j or equivalently, $l\left(d^{r_1,r_2}_{i,\alpha}.d^{r_1',r_2'}_{j,\beta}\right)$ is the number of loops which lie in the middle row when $d^{r_1,r_2}_{i,\alpha}$ is multiplied with $d^{r_1',r_2'}_{j,\beta}$, $d^{r_1,r_2}_{i,\alpha} \in \overrightarrow{\mathbb{J}}^{2r_1+r_2,\alpha}_{2s_1+s_2}$ and $d^{r_1',r_2'}_{j,\beta} \in \overrightarrow{\mathbb{J}}^{2r_1'+r_2',\beta}_{2s_1+s_2}$ respectively.

For example,

$\overrightarrow{G}^2_{2\times 1+0} =$

	$d^{0,0}_{1,\alpha_1} =$ 	$d^{0,0}_{2,\alpha_1} =$ 	$d^{1,0}_{5,\alpha_3} =$ 	$d^{1,0}_{6,\alpha_3} =$ 
$d^{0,0}_{1,\alpha_1} =$ 	1	0	1	1
$d^{0,0}_{2,\alpha_1} =$ 	0	1	1	1
$d^{0,0}_{5,\alpha_3} =$ 	1	1	x^2	0
$d^{0,0}_{6,\alpha_3} =$ 	1	1	0	x^2

where $\alpha_1 = (2, \Phi, \Phi, \Phi)$, $\alpha_2 = (1, \Phi, \Phi, 1)$ and $\alpha_3 = (1, \Phi, 1, \Phi)$.

(c) For $0 \leqslant s \leqslant k$, define G^k_s (*Gram matrices of partition algebra*) as follows:

$$G^k_s = \left(A_{r,r'}\right)_{0 \leqslant r, r' \leqslant k-s}$$

where $A_{r,r'}$ denotes the block matrix whose entries are $a_{(i,\alpha,r),(j,\beta,r')}$ with

$$a_{(i,\alpha,r),(j,\beta,r')} = \begin{cases} x^{l\left(R^{d_i}R^{d_j}\right)} & \text{if } \sharp^p\left(R^{d^r_{i,\alpha}}.R^{d^{r'}_{j,\beta}}\right) = s, \\ 0 & \text{otherwise i.e., } \sharp^p\left(R^{d^r_{i,\alpha}}.R^{d^{r'}_{j,\beta}}\right) < s, \end{cases}$$

where $1 \leqslant i \leqslant \left|\mathbb{J}_s^{r,\alpha}\right|$, $1 \leqslant j \leqslant \left|\mathbb{J}_s^{r',\beta}\right|$, $l(R^{d_i} R^{d_j}) = l(R^{d^r_{i,\alpha}}.R^{d^{r'}_{j,\beta}})$, $l(R^{d^r_{i,\alpha}} R^{d^{r'}_{j,\beta}})$ denotes the number of connected components which lie in the middle row while multiplying $R^{d^r_{i,\alpha}}$ with $R^{d^{r'}_{j,\beta}}$, $R^{d^r_{i,\alpha}} \in \mathbb{J}_s^{r,\alpha}$ and $R^{d^{r'}_{j,\beta}} \in \mathbb{J}_s^{r',\beta}$ respectively. For example,

$G_1^2 =$

	$R^{d^0_{1,\alpha}} =$ 	$R^{d^1_{5,\beta}} =$ 	$R^{d^1_{6,\beta}} =$ 
$R^{d^0_{1,\alpha}} =$ 	1	1	1
$R^{d^1_{5,\beta}} =$ 	1	x	0
$R^{d^1_{6,\beta}} =$ 	1	0	x

We establish the non-singularity of the Gram matrices over the field $\mathbb{K}(x)$ where x is an indeterminate.

Lemma 3.9. (i) *The following statements hold:*

(a) *For the algebra of $\mathbb{Z}_2$-relations, $l\big(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}\big) < l\big(d^{r_1,r_2}_{i,\alpha}.d^{r_1,r_2}_{i,\alpha}\big)$ for all $(j,\beta,r'_1,r'_2) < (i,\alpha,r_1,r_2)$, where $l\big(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}\big)$ is the number of loops which lie in the middle row when $d^{r_1,r_2}_{i,\alpha}$ is multiplied with $d^{r'_1,r'_2}_{j,\beta}$ where $d^{r_1,r_2}_{i,\alpha}, d^{r'_1,r'_2}_{j,\beta} \in J^{2k}_{2s_1+s_2}$ and $J^{2k}_{2s_1+s_2}$ is as in Notation 3.6(a).*

(b) *For the signed partition algebras, $l\big(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}\big) < l\big(d^{r_1,r_2}_{i,\alpha}.d^{r_1,r_2}_{i,\alpha}\big)$ for all $(j,\beta,r'_1,r'_2) < (i,\alpha,r_1,r_2)$, where $l\big(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}\big)$ is the number of loops which lie in the middle row when $d^{r_1,r_2}_{i,\alpha}$ is multiplied with $d^{r'_1,r'_2}_{j,\beta}$ where $d^{r_1,r_2}_{i,\alpha}, d^{r'_1,r'_2}_{j,\beta} \in \overrightarrow{J}^{2k}_{2s_1+s_2}$ and $\overrightarrow{J}^{2k}_{2s_1+s_2}$ is as in Notation 3.6(b).*

(c) *For the partition algebras, $l(R^{d^r_{i,\alpha}}.R^{d^{r'}_{j,\beta}}) < l(R^{d^r_{i,\alpha}}.R^{d^r_{i,\alpha}})$ for all $(j,\beta,r') < (i,\alpha,r)$, where $l\big(R^{d^r_{i,\alpha}}.R^{d^{r'}_{j,\beta}}\big)$ is the number of loops which lie in the middle row when $R^{d^r_{i,\alpha}}$ is multiplied with $R^{d^{r'}_{j,\beta}}$ where $R^{d^r_{i,\alpha}}, R^{d^{r'}_{j,\beta}} \in J^k_s$ and J^k_s is as in Notation 3.6(c).*

(ii) $\det G^k_{2s_1+s_2}$, $\det \overrightarrow{G}^k_{2s_1+s_2}$ *and* $\det G^k_s$ *are non-zero polynomials with leading coefficient 1.*

Proof. (i)(a) A loop consists of at least one horizontal edge from the bottom row of $d^{r_1,r_2}_{i,\alpha}$ and one from the top row of $d^{r'_1,r'_2}_{j,\beta}$, hence the number of loops in the middle component of the product $d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}$ is always less than the minimum of number of loops in $\big(d^{r_1,r_2}_{i,\alpha}.d^{r_1,r_2}_{i,\alpha}\big)$ and $\big(d^{r'_1,r'_2}_{j,\beta}.d^{r'_1,r'_2}_{j,\beta}\big)$.

Thus, $l(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}) \leqslant l(d^{r_1,r_2}_{i,\alpha}.d^{r_1,r_2}_{i,\alpha})$ and $l(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}) \leqslant l(d^{r'_1,r'_2}_{j,\beta}.d^{r'_1,r'_2}_{j,\beta})$, $\forall\ i,j$. If $(j,\beta,r'_1,r'_2) < (i,\alpha,r_1,r_2)$

Case 1: $2r'_1 + r'_2 < 2r_1 + r_2$ where $r_1(r'_1)$ is the number of pairs of $\{e\}$ horizontal edges and $r_2(r'_2)$ is the number of $\mathbb{Z}_2$-horizontal edges in $d^{r_1,r_2}_{i,\alpha}(d^{r'_1,r'_2}_{j,\beta})$ respectively, then

$$l(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}) \leqslant l(d^{r'_1,r'_2}_{j,\beta}.d^{r'_1,r'_2}_{j,\beta}) < l(d^{r_1,r_2}_{i,\alpha}.d^{r_1,r_2}_{i,\alpha}).$$

Case 2: $2r'_1+r'_2 = 2r_1+r_2$ and $r'_1+r'_2 < r_1+r_2$ where $r_1(r'_1)$ is the number of pairs of $\{e\}$ horizontal edges and $r_2(r'_2)$ is the number of $\mathbb{Z}_2$-horizontal edges in $d^{r_1,r_2}_{i,\alpha}(d^{r'_1,r'_2}_{j,\beta})$ respectively, which implies that

Subcase 2.1: suppose that $r'_2 < r_2$, i.e., at least two $\mathbb{Z}_2$-horizontal edges of $d^{r'_1,r'_2}_{j,\beta}$ is connected to a $\mathbb{Z}_2$-horizontal edge of $d^{r_1,r_2}_{i,\alpha}$ to make a loop or one $\mathbb{Z}_2$-horizontal edge of $d^{r_1,r_2}_{i,\alpha}$ is connected to a $\mathbb{Z}_2$-through class of $d^{r'_1,r'_2}_{j,\beta}$ in the product $d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}$.

Subcase 2.2: suppose that $r'_1 < r_1$, i.e., at least two $\{e\}$ horizontal edges of $d^{r'_1,r'_2}_{j,\beta}$ is connected to a $\{e\}$ or $\mathbb{Z}_2$-horizontal edge of $d^{r_1,r_2}_{i,\alpha}$ to make a loop or one $\{e\}$-horizontal edge of $d^{r_1,r_2}_{i,\alpha}$ is connected to a $\{e\}$ or $\mathbb{Z}_2$-through class of $d^{r'_1,r'_2}_{j,\beta}$ in the product $d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}$.

Therefore the number of loops is strictly less than $2r'_1 + r'_2$, and thus

$$l\big(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}\big) \lneqq l\big(d^{r'_1,r'_2}_{j,\beta}d^{r'_1,r'_2}_{j,\beta}\big) = l\big(d^{r_1,r_2}_{i,\alpha}.d^{r_1,r_2}_{i,\alpha}\big)$$

Case 3: $2r'_1+r'_2 = 2r_1+r_2, r'_1+r'_2 = r_1+r_2$ and $\alpha < \beta$ where $r_1(r'_1)$ is the number of pairs of $\{e\}$ horizontal edges and $r_2(r'_2)$ is the number of $\mathbb{Z}_2$-horizontal edges in $d^{r_1,r_2}_{i,\alpha}\big(d^{r'_1,r'_2}_{j,\beta}\big)$ respectively and $\alpha(\beta)$ is the underlying partition of $d^{r_1,r_2}_{i,\alpha}\big(d^{r'_1,r'_2}_{j,\beta}\big)$, which implies that

$$l\big(d^{r_1,r_2}_{i,\alpha}.d^{r_1,r_2}_{i,\alpha}\big) = l\big(d^{r'_1,r'_2}_{j,\beta}.d^{r'_1,r'_2}_{j,\beta}\big) = 2r_1 + r_2 = 2r'_1 + r'_2$$

and $r_1 + r_2 = r'_1 + r'_2$.

Every $\{e\}$-through class of $U^{(d_i,P_i)}_{(d_i,P_i)}$ is uniquely connected to a $\{e\}$-through class of $d^{r'_1,r'_2}_{j,\beta}$ and vice versa and if $l\big(d^{r_1,r_2}_{i,\alpha}.d^{r'_1,r'_2}_{j,\beta}\big) = l\big(d^{r_1,r_2}_{i,\alpha}.d^{r_1,r_2}_{i,\alpha}\big) = l\big(d^{r'_1,r'_2}_{j,\beta}.d^{r'_1,r'_2}_{j,\beta}\big)$ then every $\{e\}(\mathbb{Z}_2)$-horizontal edge of $d^{r_1,r_2}_{i,\alpha}$ is connected uniquely to a $\{e\}(\mathbb{Z}_2)$-horizontal edge of $d^{r'_1,r'_2}_{j,\beta}$ and vice versa which implies that $d^{r_1,r_2}_{i,\alpha} = d^{r'_1,r'_2}_{j,\beta}$.

Thus, if $d_{i,\alpha}^{r_1,r_2} \neq d_{j,\beta}^{r'_1,r'_2}$ and $2r_1 + r_2 = 2r'_1 + r'_2$ and $r_1 + r_2 = r'_1 + r'_2$ then $l\big(d_{i,\alpha}^{r_1,r_2}.d_{j,\beta}^{r'_1,r'_2}\big) < l\big(d_{i,\alpha}^{r_1,r_2}.d_{i,\alpha}^{r_1,r_2}\big) = l\big(d_{j,\beta}^{r'_1,r'_2}.d_{j,\beta}^{r'_1,r'_2}\big)$.

(i)(b) and (i)(c) can be proved similarly to (i)(a).

(ii) It follows from (i) of Lemma 3.9, that the degree of the monomial $\big\{\prod a_{i\sigma(i)}\big\}_{\sigma\in\mathfrak{S}_{f_{2s_1+s_2}}}$, is strictly less than the degree of the monomial $\prod_{i=1}^{f_{2s_1+s_2}} a_{ii}$.

Thus, the determinant of the Gram matrix $G_{2s_1+s_2}^{k}$ of the algebra of $\mathbb{Z}_2$-relations is a non-zero monic polynomial with integer coefficients and the roots are all algebraic integers.

Similarly, we can prove for the determinant of the Gram matrices $\overrightarrow{G}_{2s_1+s_2}^{k}$ and G_s^k of signed partition algebras and partition algebras respectively. □

Lemma 3.10. *The Gram matrices* $G_{2s_1+s_2}^{k}$, $\overrightarrow{G}_{2s_1+s_2}^{k}$ *and* G_s^k *are symmetric.*

Proof. The proof follows from the Definition 3.8, since the top and bottom rows of the diagrams in $J_{2s_1+s_2}^{2k}$, $\overrightarrow{J}_{2s_1+s_2}^{2k}$, J_s^k have the same number of horizontal edges. □

Remark 3.11. Every partition diagram can be represented as a set partition and in set partition we can talk about subsets.

Thus a connected component of the diagram $d_{j,\beta}^{r'_1,r'_2}$ is contained in a connected component of $d_{i,\alpha}^{r_1,r_2}$ if the corresponding set partition of $d_{j,\beta}^{r'_1,r'_2}$ is contained in the set partition of $d_{i,\alpha}^{r_1,r_2}$.

We shall introduce a finer version of coarser diagrams.

Definition 3.12. (a) Let $d_{i,\alpha}^{r_1,r_2}, d_{j,\beta}^{r'_1,r'_2} \in J_{2s_1+s_2}^{2k}$. Define a relation on $J_{2s_1+s_2}^{2k}$ as follows: $d_{i,\alpha}^{r_1,r_2} < d_{j,\beta}^{r'_1,r'_2}$,

(i) if each $\{e\}$-through class of $d_{i,\alpha}^{r_1,r_2}$ is contained in a $\{e\}$-through class of $d_{j,\beta}^{r'_1,r'_2}$,

(ii) every $\mathbb{Z}_2$-through class of $d_{i,\alpha}^{r_1,r_2}$ is contained in a $\mathbb{Z}_2$-through class of $d_{j,\beta}^{r'_1,r'_2}$,

(iii) every $\{e\}$-horizontal edge of $d_{i,\alpha}^{r_1,r_2}$ is contained in a ($\{e\}$ or $\mathbb{Z}_2$) horizontal edge or ($\{e\}$ or $\mathbb{Z}_2$)-through class of $d_{j,\beta}^{r'_1,r'_2}$ and

(iv) every $\mathbb{Z}_2$-horizontal edge of $d_{i,\alpha}^{r_1,r_2}$ is contained in a $\mathbb{Z}_2$-horizontal edge or $\mathbb{Z}_2$-through class of $d_{j,\beta}^{r'_1,r'_2}$.

We say that $d_{j,\beta}^{r'_1,r'_2}$ is a coarser diagram of $d_{i,\alpha}^{r_1,r_2}$ and $(j,\beta,r'_1,r'_2) < (i,\alpha,r_1,r_2)$.

(b) Since $\overrightarrow{J}_{2s_1+s_2}^{2k} \subset J_{2s_1+s_2}^{2k}$ the relation defined on $J_{2s_1+s_2}^{2k}$ in (a) holds for the diagrams in $\overrightarrow{J}_{2s_1+s_2}^{2k}$.

(c) Define a relation on J_s^k as follows: $R^{d_{i,\alpha}^r} < R^{d_{j,\beta}^{r'}}$,

(i)′ if each through class of $R^{d_{i,\alpha}^r}$ is contained in a through class of $R^{d_{j,\beta}^{r'}}$,

(ii)′ if each horizontal edge of $R^{d_{i,\alpha}^r}$ is contained in a horizontal edge or through class of $R^{d_{j,\beta}^{r'}}$.

We say that $R^{d_{j,\beta}^{r'}}$ is a coarser diagram of $R^{d_{i,\alpha}^r}$ then $(j,\beta,r') < (i,\alpha,r)$. The relation $<$ holds for the diagrams in $\widetilde{J}_s^k$.

In our subsequent paper we establish the semisimplicity of our algebras.

Acknowledgement

The authors would like to express their gratitude and sincere thanks to the referee for all his(her) valuable comments and suggestions which in turn made the paper easy to read.

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Contact information

N. Karimilla Bi,
M. Parvathi

Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chepauk, Chennai -600 005, Tamilnadu, India.
E-Mail(s): sparvathi@hotmail.com

Received by the editors: 22.09.2015
and in final form 16.03.2018.