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# On divergence and sums of derivations

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Abstract. Let $K$ be an algebraically closed field of characteristic zero and $A$ a field of algebraic functions in $n$ variables over $\mathbb{K}$. (i.e. $A$ is a finite dimensional algebraic extension of the field $\mathbb{K}(x_1, \ldots, x_n)$ ). If $D$ is a $\mathbb{K}$-derivation of $A$, then its divergence $divD$ is an important geometric characteristic of $D$ ($D$ can be considered as a vector field with coefficients in $A$). A relation between expressions of $divD$ in different transcendence bases of $A$ is pointed out. It is also proved that every divergence-free derivation $D$ on the polynomial ring $\mathbb{K}[x, y, z]$ is a sum of at most two jacobian derivation.

## Introduction

Let $\mathbb{K}$ be a field of characteristic zero and $A$ a field of algebraic functions in $n$ variables over $\mathbb{K}$, i.e. $A$ is a finite dimensional algebraic extension of the field $\mathbb{K}(x_1, \ldots, x_n)$. If $D$ is a $\mathbb{K}$-derivation of $A$, then the divergence divD of the derivation $D$ is a very important geometric characteristic of $D$ ($D$ can be considered as a vector field with coefficients in $A$). In the first part of the paper, a relation between expressions of divD in different transcendence bases is pointed out (Theorem 1). This theorem generalizes a result of the paper [4]. Naturally, the divergence of a derivation does not change if we pass from one coordinate system in a polynomial ring to another coordinate system. In particular, the set

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of all divergence-free derivations of the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ is invariant under action of automorphisms of this ring. Such derivations form a very important subalgebra $L_0$ of the Lie algebra $Der_{\mathbb{K}}(\mathbb{K}[x_1, \ldots, x_n])$ of all $\mathbb{K}$-derivations on $\mathbb{K}[x_1, \ldots, x_n]$. The Lie algebra $L_0$ was studied by many authors (see, for example, [1], [5], [4]). Note that the algebra $L_0$ contains all the jacobian derivations of $\mathbb{K}[x_1, \ldots, x_n]$ which are the simplest divergence-free derivations. So, it is interesting to know relations between divergence-free derivation and jacobian derivations. It is proved that every divergence-free derivation of the polynomial ring $\mathbb{K}[x, y, z]$ is a sum of at most two jacobian derivations (Theorem 2). Note that every divergence-free derivation of the ring $\mathbb{K}[x, y]$ is a jacobian derivation. A divergence-free derivation of the polynomial ring $\mathbb{K}[x, y, z]$ is pointed out that is not a jacobian one (Proposition 1).

We use standard notation. The ground field $\mathbb{K}$ is of characteristic zero, $\mathbb{K}(x_1, \ldots, x_n)$ is the field of rational functions. We denote by $A$ a field of algebraic functions in $n$ variables over the field $\mathbb{K}$. Let $\{y_1, \ldots, y_n\}$ be a transcendence basis of $A$ (over $\mathbb{K}$). Then every derivation $\frac{\partial}{\partial y_i}$ of the subfield $\mathbb{K}(y_1, \ldots, y_n) \subseteq A$ can be uniquely extended to a derivation of the field $A$. We denote this extension for convenience by the same notation $\frac{\partial}{\partial y_i}$. Denote by $Y$ the set $\{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\}$ of $\mathbb{K}$-derivations of $A$. If $a_1, \ldots, a_{n-1}$ are elements of $A$, then the jacobian derivation $D_{a_1,\ldots,a_{n-1}}$ is defined by the rule: $D_{(a_1,\ldots,a_{n-1})}(h) = \det J(a_1, \ldots, a_{n-1}, h)$, where $J(a_1, \ldots, a_{n-1}, h)$ is the Jacobi matrix of the functions $a_1, \ldots, a_{n-1}, h \in A$. The divergence $\mathrm{divD}$ of a derivation $D \in \mathrm{Der}_{\mathbb{K}}(A)$, $D = \sum p_i \frac{\partial}{\partial x_i}$ is defined by the rule: $\mathrm{divD} = \sum_{\mathrm{i=1}}^{\mathrm{n}} \frac{\partial \mathrm{p_i}}{\partial \mathrm{x_i}}$.

## 1. On behavior of divergence under change of a transcendence basis

Let $A \supseteq \mathbb{K}(x_1, \ldots, x_n)$ be a field of algebraic functions. It is known that the Lie algebra $\mathrm{Der}_{\mathbb{K}}(A)$ of all $\mathbb{K}$-derivations of $A$ is vector space over $A$ of dimension $n$ (but not a Lie algebra over $A$). The set $X = \{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$ of partial differentiations is a basis of $\mathrm{Der}_{\mathbb{K}}(A)$ over $A$. Thus every element $D \in \mathrm{Der}_{\mathbb{K}} R$ can be uniquely written in the form

$$D = \sum_{i=1}^{n} p_i \frac{\partial}{\partial x_i}, \; p_1, \ldots, p_n \in A. \tag{1}$$

Let $y_1, \ldots, y_n \in A$ be a transcendence basis of the field $A$ over the field $\mathbb{K}$. Then $Y = \{\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\}$ is also a basis of the linear space

$\mathrm{Der}_{\mathbb{K}}(A)$ over the field $A$. Therefore for a given derivation $D \in \mathrm{Der}_{\mathbb{K}}(A)$ there exist elements $q_1, \ldots, q_n \in A$ such that

$$D = \sum_{i=1}^{n} q_i \frac{\partial}{\partial y_i}, \; q_1, \ldots, q_n \in A. \tag{2}$$

Denote the divergence of the derivation $D$ in the transcendence bases $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively by

$$\mathrm{div}_X D = \sum_{i=1}^{n} \frac{\partial p_i}{\partial x_i}, \qquad \mathrm{div}_Y D = \sum_{i=1}^{n} \frac{\partial q_i}{\partial y_i}.$$

**Theorem 1.** *Let $D \in \mathrm{Der}_{\mathbb{K}}(A)$. Then*

$$\mathrm{div}_X D = \mathrm{div}_Y D + \frac{D(\Delta)}{\Delta}, \quad \textit{where } \Delta = \det\Big(\frac{\partial y_i}{\partial x_j}\Big)_{i,j=1}^{n}.$$

*Proof.* Since $\frac{\partial y_i}{\partial y_j} = \delta_{ij}$ we have by (2) the following equalities

$$q_i = D(y_i) = \sum_{j=1}^{n} p_j \frac{\partial y_i}{\partial x_j}, \quad i = 1, \ldots, n. \tag{3}$$

The derivations $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}$ form a basis of the vector space $\mathrm{Der}_{\mathbb{K}}(A)$, so we can write

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} r_i^j \frac{\partial}{\partial y_i}, \quad j = 1, \ldots, n. \tag{4}$$

for some $r_i^j \in A$, $i, j = 1, \ldots, n$. These elements can be found from (4):

$$r_i^j = \frac{\partial y_i}{\partial x_j}, \quad i, j = 1, \ldots, n.$$

Thus we have

$$\frac{\partial}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial y_i}{\partial x_j} \frac{\partial}{\partial y_i}, \quad j = 1, \ldots, n.$$

Analogously we get

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}, \qquad i = 1, \ldots, n.$$

Using the relation (3) we obtain

$$\begin{aligned}\operatorname{div}_Y D &= \sum_{i=1}^n \frac{\partial q_i}{\partial y_i} = \sum_{i=1}^n \frac{\partial}{\partial y_i}\Big(\sum_{j=1}^n p_j \frac{\partial y_i}{\partial x_j}\Big)\\ &= \sum_{i,j=1}^n \frac{\partial p_j}{\partial y_i}\,\frac{\partial y_i}{\partial x_j} + \sum_{i,j=1}^n p_j \frac{\partial}{\partial y_i}\Big(\frac{\partial y_i}{\partial x_j}\Big). \qquad (5)\end{aligned}$$

The first summand in the right side of (5) can be written (using (3)) in the form

$$\sum_{i,j=1}^n \frac{\partial p_j}{\partial y_i}\,\frac{\partial y_i}{\partial x_j} = \sum_{j=1}^n\Big(\sum_{i=1}^n \frac{\partial y_i}{\partial x_j}\,\frac{\partial}{\partial y_i}\Big)(p_j) = \sum_{j=1}^n \frac{\partial}{\partial x_j}(p_j) = \operatorname{div}_X D.$$

Write the second summand in the right side of (5) (using (4) and the equality $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$) in the form

$$\begin{aligned}\sum_{i,j=1}^n p_j \frac{\partial}{\partial y_i}\Big(\frac{\partial y_i}{\partial x_j}\Big) &= \sum_{i,j,k=1}^n p_j \frac{\partial x_k}{\partial y_i}\,\frac{\partial}{\partial x_j}\Big(\frac{\partial y_i}{\partial x_k}\Big)\\ &= \sum_{i,k=1}^n \frac{\partial x_k}{\partial y_i}\Big(\sum_{j=1}^n p_j \frac{\partial}{\partial x_j}\Big)\Big(\frac{\partial y_i}{\partial x_k}\Big) = \sum_{i,k=1}^n \frac{\partial x_k}{\partial y_i} D\Big(\frac{\partial y_i}{\partial x_k}\Big).\end{aligned}$$

The matrix $\big(\frac{\partial x_k}{\partial y_i}\big)_{k,i=1}^n$ is inverse to the matrix $\big(\frac{\partial y_k}{\partial x_i}\big)_{k,i=1}^n$, so we have

$$\frac{\partial x_k}{\partial y_i} = \frac{A_k^i}{\Delta}, \qquad i, j = 1, \dots, n,$$

where $A_i^k$ is the cofactor of the element $\frac{\partial y_k}{\partial x_i}$ in the determinant $\Delta = \det\Big(\frac{\partial y_k}{\partial x_i}\Big)_{k,i=1}^n$. Thus

$$\begin{aligned}\sum_{i,k=1}^n \frac{\partial x_k}{\partial y_i} D\Big(\frac{\partial y_i}{\partial x_k}\Big) &= \frac{1}{\Delta}\sum_{i,k=1}^n A_k^i D\Big(\frac{\partial y_i}{\partial x_k}\Big)\\ &= \frac{1}{\Delta}\sum_{k=1}^n \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & D\big(\frac{\partial y_1}{\partial x_k}\big) & \cdots & \frac{\partial y_1}{\partial x_n}\\ \vdots & \cdots & \vdots & \cdots & \vdots\\ \frac{\partial y_n}{\partial x_1} & \cdots & D\big(\frac{\partial y_n}{\partial x_k}\big) & \cdots & \frac{\partial y_n}{\partial x_n} = \end{vmatrix} = \frac{D(\Delta)}{\Delta}.\end{aligned}$$

The proof is complete. □

## 2. Divergence-free and jacobian derivations

Some known results about divergence-free derivations are collected in the next Lemma (see, for example, [2]) or [3]):

**Lemma 1.** 1) *If $D$ is a jacobian derivation of the ring $\mathbb{K}[x_1, \dots, x_n]$, then $divD = 0$;*

2) *Every divergence-free derivation of the polynomial ring $\mathbb{K}[x, y]$ is a jacobian derivation;*

3) *If $D_1, D_2$ are divergence-free derivations of the ring $\mathbb{K}[x_1, \dots, x_n]$, then so are $D_1 + D_2$ and $[D_1, D_2]$.*

**Theorem 2.** *Let $D$ be a $\mathbb{K}$-derivation of the polynomial ring $\mathbb{K}[x, y, z]$ with* $\mathrm{divD} = 0$. *Then there exist jacobian derivations $D_1$ and $D_2$ of the ring $\mathbb{K}[x, y, z]$ such that $D = D_1 + D_2$.*

*Proof.* Write $D$ in the form

$$D = p(x, y, z)\frac{\partial}{\partial x} + q(x, y, z)\frac{\partial}{\partial y} + r(x, y, z)\frac{\partial}{\partial z},$$

where $p, q, r \in \mathbb{K}[x, y, z]$. Then by the conditions of the theorem, $p'_x + q'_y + r'_z = \mathrm{divD} = 0$. First find a jacobian derivation $D_1$ of the ring $\mathbb{K}[x, y, z]$ of the form

$$D_1 = p(x, y, z)\frac{\partial}{\partial x} + q_1(x, y, z)\frac{\partial}{\partial y}$$

for some $q_1 \in \mathbb{K}[x, y, z]$. Denote by $s = s(x, y, z)$ a polynomial in $\mathbb{K}[x, y, z]$ such that $s'_y = p$, i.e. $s = \int p(x, y, z)dy$ (it is obvious that such a polynomial does exist). Denote by $D_1 = D_{(s,z)}$ the jacobian derivation determined by the polynomials $s, z \in \mathbb{K}[x, y, z]$. It is easy to see that

$$D_1 = p(x, y, z)\frac{\partial}{\partial x} - s'_x(x, y, z)\frac{\partial}{\partial y}.$$

Set $D_2 = D - D_1$. Let us show that $D_2$ is a jacobian derivation of the ring $\mathbb{K}[x, y, z]$. It is obvious that

$$D_2 = q_2(x, y, z)\frac{\partial}{\partial y} + r(x, y, z)\frac{\partial}{\partial z}, \text{ where } q_2 = q - s'_x.$$

Consider $q_2(x, y, z)$ and $r(x, y, z)$ as polynomials of variables $y, z$ with coefficients in the ring $\mathbb{K}[x]$. Since $\mathrm{div}D_2 = \mathrm{div}D - \mathrm{div}D_1 = 0$ we have $\frac{\partial q_2}{\partial y} + \frac{\partial r}{\partial z} = 0$. Denote for convenience $\varphi = -r$, $\psi = q_2$. Then the vector field

$\varphi\frac{\partial}{\partial y}+\psi\frac{\partial}{\partial z}$ is potential since the equality $\varphi'_z=\psi'_y$ holds. Therefore there exists a polynomial $t(x,y,z)$ such that $t'_y=\varphi$, $t'_z=\psi$ (the polynomial $t$ can be obtained by formal integrating the polynomials $\psi$ and $\varphi$ on variables $y$ and $z$ respectively).

$$t(x,y,z)=\int_{0,0,0}^{M(x,y,z)}\varphi(x,y,z)dy+\psi(x,y,z)dz.$$

Thus $t'_y=-r(x,y,z)$, $t'_z=q_2(x,y,z)$. Let us consider the jacobian derivation $D_{(t,x)}$. It is obvious that

$$D_{(t,x)}=t'_z\frac{\partial}{\partial y}-t'_y\frac{\partial}{\partial z}=q_2(x,y,z)\frac{\partial}{\partial y}+r(x,y,z)\frac{\partial}{\partial z}=D_2.$$

So we have $D=D_1+D_2$, where $D_1$, $D_2$ are jacobian derivations. $\square$

**Proposition 1.** *The derivation $D=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-2z\frac{\partial}{\partial z}$ of the polynomial ring $\mathbb{K}[x,y,z]$ is divergence-free but not a jacobian derivation.*

*Proof.* By Theorem 10.1.1 from [3], it holds $\mathrm{Ker}D\neq\mathbb{K}$. Take any polynomial $f\in\mathrm{Ker}D$ and write it as a sum $f=f_0+f_1+\ldots+f_n$ of homogeneous components. Since the derivation $D$ is homogeneous, all the polynomials $f_i$ are also in $\mathrm{Ker}D$. Therefore we can assume without loss of generality that $f$ is a homogenous polynomial of degree $k$. The equality $D(f)=0$ means that $xf'_x+yf'_y-2zf'_z=0$ and therefore it holds

$$kf-3zf'_z=0 \tag{6}$$

(here $xf'_x+yf'_y+zf'_z=kf$ since $\deg f=k$). Consider the polynomial $f$ as a polynomial of $z$ with coefficients in $\mathbb{K}[y,z]$ and write

$$f=\varphi_0+\varphi_1z+\ldots+\varphi_mz^m,\ \varphi_i\in\mathbb{K}[x,y],m=\deg_z f.$$

Then $3zf'_z=3z\varphi_1+6z^2\varphi_2+\ldots+3m\varphi_mz^m$ and using (6) we get

$$\varphi_0=0,\ z\varphi_1(k-3)=0,\ z^2\varphi_2(k-6)=0,\ldots,z^m\varphi_m(k-3m)=0. \tag{7}$$

Since $\varphi_m\neq0$ (because of equality $m=\deg_z f$) we have $k=3m$ and

$$\varphi_1=0,\ldots,\varphi_{m-1}=0.$$

The latter means that $f=z^m\varphi_m$, where $k=3m$. The equality $\deg f=k$ implies $\deg\varphi_m=2m$ and therefore $\varphi_m=\varphi_m(x,y)$ is a homogeneous polynomial of degree $2m$. Denote for convenience $\psi_{2m} = \varphi_m$. Then $f = z^m \cdot \psi_{2m}$ is a homogeneous polynomial of degree $3m$.

Conversely, if $f = z^m \cdot \psi_{2m}$ is a homogeneous polynomial of degree $3m$, where $\psi_{2m} = \psi_{2m}(x, y)$, then $3zf'_z = 3mf'$. Setting $k = 3m$ we get $zh'_z = 3kf$, i.e. the polynomial $f$ satisfies the equality (6). Then

$$xf'_x + yf'_y - 2zf'_z = 0$$

and therefore $f \in \mathrm{Ker}D$. Thus $\mathrm{Ker}D$ is a linear combination of homogeneous polynomials of the form $z^m\psi_{2m}$, where $\psi_{2m}$ is a homogeneous polynomial of degree $2m$ in variables $x, y$.

Now suppose $D = D_{(a,b)}$ for some $a, b \in \mathbb{K}[x, y, z]$. Then $a, b \in \mathrm{Ker}D$ and omitting the constant terms in the polynomials $a$ and $b$ we get

$$a = \alpha_1 z\varphi_2 + \alpha_2 z^2\varphi_4 + \ldots + \alpha_m z^m\varphi_{2m},\ b = \beta_1 z\psi_2 + \beta_2 z^2\psi_4 + \ldots + \beta_s z^s\psi_{2s}$$

for some $\varphi_i, \psi_j \in \mathbb{K}[x, y]$, $\alpha_i, \beta_j \in \mathbb{K}$, $\alpha_m \neq 0$, $\beta_s \neq 0$. Then

$$D(x) = D_{(a,b)}(x) = a'_y b'_y - a'_z b'_y.$$

On the other hand, $D(x) = x$ and therefore $a'_y b'_y - a'_z b'_y = x$. The polynomials $a'_y$ and $b'_y$ are divided by $z$, so the polynomial $x$ is also divided by $z$. The obtained contradiction shows that $D$ is not a jacobian derivation. The proof is complete. □

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