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# A formula for the number of weak endomorphisms on paths

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Abstract. A weak endomorphisms of a graph is a mapping on the vertex set of the graph which preserves or contracts edges. In this paper we provide a formula to determine the cardinalities of weak endomorphism monoids of finite undirected paths.

## Introduction and preliminaries

The motivation of this paper has come from [1], where Arworn gives an algorithm to determine the cardinalities of endomorphism monoids of finite undirected paths by using the square lattices. Furthermore, in [2], Arworn and Kim find the number of path homomorphisms by the lattices and the generalized catalan number, and in [5], Sirisathianwatthana and Pipattanajinda find the number of cycle weak homomorphisms.

Consider *finite simple graphs* $G$ with the *vertex set* $V(G)$ and the *edge set* $E(G)$. Let $G$ and $H$ be two graphs. A map $f : V(G) \to V(H)$ is a *homomorphism* if $f$ preserves the edges, i.e., if $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. Further, in [3], a map $f : V(G) \to V(H)$ is called a *weak homomorphism* (also called *egamorphism* in [4]) if $f$ preserves or contracts the edges, i.e., if $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$ whenever $\{x, y\} \in E(G)$. A (weak) homomorphism from $G$ to itself is called a *(weak) endomorphism* of $G$. Denote the set of (weak) endomorphisms of

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$G$ by $\mathrm{End}(G)$ ($\mathrm{WEnd}(G)$). Clearly $\mathrm{End}(G)$ and $\mathrm{WEnd}(G)$ form monoids by composition of mappings. Let $P_n = \{0, 1, 2, \ldots, n-1\}$ be an *undirected path of length* $n-1$, where $n \geqslant 1$. Denote the number of weak endomorphisms of the path $P_n$ by $|\,\mathrm{WEnd}(P_n)|$, and the number of weak endomorphisms of the path $P_n$ which maps 0 to $j$ by $|\,\mathrm{WEnd}^j(P_n)|$.

Let $n$ be a positive integer, the *multinomial coefficient* is

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \ldots r_k!} \tag{1}$$

where $n = r_1 + r_2 + \cdots + r_k$ and $k \in Z^+$. The next result is well-know and extends Formula (1)

$$\binom{n}{r_1, r_2, \ldots, r_k} = \binom{n-1}{r_1 - 1, r_2, \ldots, r_k} + \binom{n-1}{r_1, r_2 - 1, \ldots, r_k} + \cdots + \binom{n-1}{r_1, r_2, \ldots, r_k - 1}. \tag{2}$$

Next, we use the multinomial coefficient (1) and the extended formula (2) to find all shortest paths on three-dimensional square lattice.

Consider three-dimensional square lattices $M(i,j,k)$ in Figure 1 and $r$-ladder three-dimensional square lattices $M_r(i,j,k)$ in Figure 2 (here we choose $i = 6, j = 5, k = 4$ and $r = 2$),

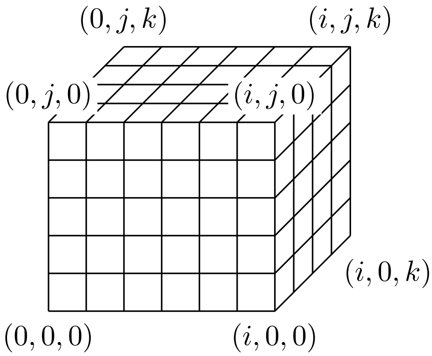


FIGURE 1

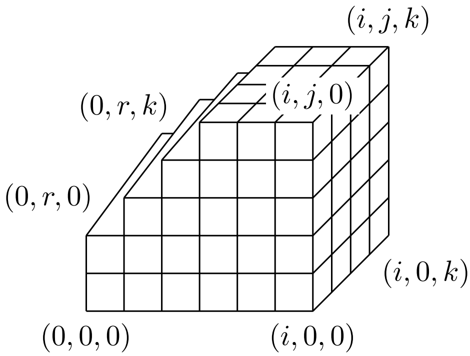


FIGURE 2

The shortest path on this three-dimensional square lattice from the point $(0,0,0)$ to any point $(i,j,k)$ can be obtained by going from the point $(0,0,0)$ to the point $(i,j,k)$ by $(1,0,0)$ or $(0,1,0)$ or $(0,0,1)$ and similarly for the next steps. And more generally from $(i_0, j_0, k_0)$ to $(i_0+1, j_0, k_0)$ or $(i_0, j_0+1, k_0)$ or $(i_0, j_0, k_0+1)$.

**Proposition 1.** *The numbers $M(i,j,k)$ and $M_r(i,j,k)$ $(r < j)$ of shortest paths from the point $(0,0,0)$ to any point $(i,j,k)$ in the three-dimensional square lattice and in the $r$-ladder three-dimensional square lattice are*

$$M(i,j,k) = \binom{i+j+k}{i,j,k},$$

*and*

$$\begin{aligned} M_r(i,j,k) &= M(i,j,k) - M(j-r-1, i+r+1, k) \\ &= \binom{i+j+k}{i,j,k} - \binom{i+j+k}{j-r-1, i+r+1, k}, \end{aligned}$$

*respectively.*

*Proof.* By using (1), (2) and induction. □

## 1. The number of weak endomorphisms on paths

In this section, we give an algorithm for the numbers of weak endomorphisms on paths by using the three-dimensional square lattice and the $r$-ladder three-dimensional square lattice.

In Figure 3, the possible weak endomorphisms of the path $P_4$ which map 0 to 0, i.e. the elements of $\mathrm{WEnd}^0(P_4)$ are indicated. There the numbers in the top line the elements of the domain and the numbers in the left column denote the elements of the image set.

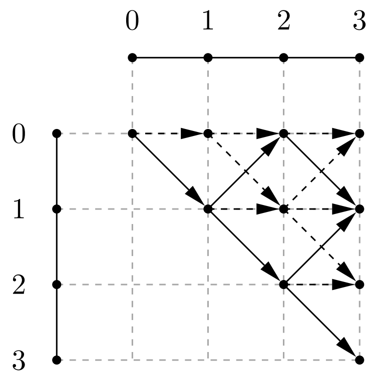


FIGURE 3

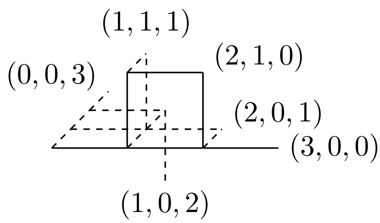


FIGURE 4

Take the mapping $f \in \mathrm{WEnd}^0(P_4)$ with $f(0) = f(1) = f(3) = 0$ and $f(2) = 1$, symbolized by the upper sequence of dashed arrows. Now we model this mapping by a shortest path in the 3-dimensional square lattice as follows: such that $f(0)$ and $f(x)$ is $(0,0,0)$ and $(i,j,k)$, respectively for some $x \in X = \{0,1,2,3\}$. Now we go from $(i,j,k)$ to $(i+1,j,k)$, $(i,j+1,k)$ or $(i,j,k+1)$, if $f(x+1) = f(x)+1$, $f(x+1) = f(x)-1$ or $f(x+1) = f(x)$, respectively. So $f$ is represented in the 3-dimensional square lattice by a shortest path from $(0,0,0)$ to $(1,1,1)$, compare Figure 4. Hence, the cardinality $|\operatorname{WEnd}^0(P_4)|$ is the summation of $M(i,j,k)$ and $M_r(i,j,k)$ where $i+j+k=3$.

So, by Figure 4 and Proposition 1, we get

$$\begin{aligned}
|\operatorname{WEnd}^0(P_4)| &= M(3,0,0) + M_0(2,1,0) + M(2,0,1) + M_0(1,1,1) \\
&\qquad + M(1,0,2) + M(0,0,3) \\
&= \binom{3}{3,0,0} + \binom{3}{2,1,0} - \binom{3}{0,3,0} \\
&\qquad + \binom{3}{2,0,1} + \binom{3}{1,1,1} - \binom{3}{0,2,1} \\
&\qquad + \binom{3}{1,0,2} + \binom{3}{0,0,3} \\
&= 13.
\end{aligned}$$

Similarly to Figure 3 and Figure 4, in Figure 5 the possible weak endomorphisms of the path $P_4$ which map 0 to 1, i.e. the elements of $\operatorname{WEnd}^1(P_4)$ are symbolized. This implies that the cardinality $|\operatorname{WEnd}^1(P_4)|$ is the summation of $M(i,j,k)$ and $M_r(i,j,k)$ where $i+j+k=3$, see Figure 6.

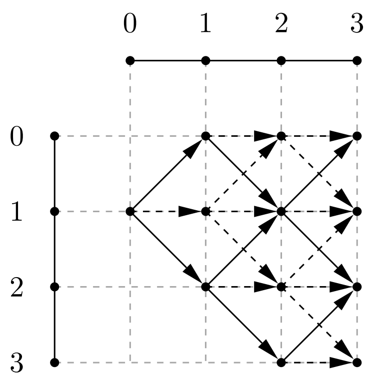


Figure 5

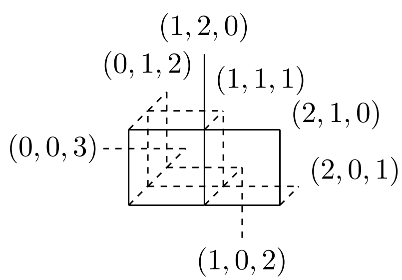


Figure 6

So, by Figure 6 and Proposition 1, we get

$$|\operatorname{WEnd}^1(P_4)| = M(2,1,0) + M_1(1,2,0) + M(2,0,1) + M(1,1,1)$$

$$
\begin{aligned}
&+ M(1,0,2) + M(0,1,2) + M(0,0,3)\\
= &\binom{3}{2,1,0} + \binom{3}{1,2,0} - \binom{3}{0,3,0}\\
&+ \binom{3}{2,0,1} + \binom{3}{1,1,1}\\
&+ \binom{3}{1,0,2} + \binom{3}{0,1,2} + \binom{3}{0,0,3}\\
= &\ 21.
\end{aligned}
$$

The next Proposition is as follows:

**Proposition 2.** *Let $n$ be positive integer and $j$ non-negative integer such that $j < n$. Then*
(1) $|\operatorname{WEnd}^j(P_n)| = |\operatorname{WEnd}^{n-j-1}(P_n)|$,
(2) $|\operatorname{WEnd}(P_{2n})| = 2\sum_{j=0}^{n-1} |\operatorname{WEnd}^j(P_{2n})|$,
(3) $|\operatorname{WEnd}(P_{2n+1})| = 2\sum_{j=0}^{n-1} |\operatorname{WEnd}^j(P_{2n+1})| + |\operatorname{WEnd}^n(P_{2n+1})|$.

Instead of a proof we look again at $P_4$. We use Proposition 2(1), Figures 4, 6 and Proposition 1, to get that $|\operatorname{WEnd}^0(P_4)| = |\operatorname{WEnd}^3(P_4)| = 13$ and $|\operatorname{WEnd}^1(P_4)| = |\operatorname{WEnd}^2(P_4)| = 21$. Thus, $|\operatorname{WEnd}(P_4)| = 2(13+21)=68$.

Next, we introduce the following notations:

$$eM(i,j) := \sum_{i_0=0}^{i}\sum_{j_0=0}^{j} M(i-i_0, j-j_0, i_0+j_0), \tag{3}$$

$$eM_r(i,j) := \sum_{i_0=0}^{i-j+r} M_r(i-i_0, j, i_0). \tag{4}$$

Further, the $eM_r(i,j) = 0$ if $i - j + r < 0$ call this (2.3). Thus, in the example of $P_4$, $|\operatorname{WEnd}^0(P_4)| = eM(3,0) + eM_0(2,1)$ and $|\operatorname{WEnd}^1(P_4)| = eM(2,1) + eM_1(1,2)$.

First we prove an auxiliary result:

**Proposition 3.** *Let $n$ be positive integer and $j$ non-negative integer such that $j < \frac{n}{2} - 1$. Then*

$$
\begin{aligned}
|\operatorname{WEnd}^j(P_n)| = &\sum_{i_0=0}^{i}\sum_{j_0=0}^{j}\binom{n-1}{i-i_0, j-j_0, i_0+j_0}\\
&+\sum_{s=1}^{n}\Big[\sum_{i_0=0}^{i-2s}\Big[\binom{n-1}{i-s-i_0, j+s, i_0} - \binom{n-1}{s-1, n-s-i_0, i_0}\Big]
\end{aligned}
$$

$$+\sum_{j_0=0}^{j-2s}\Big[\binom{n-1}{j-s-j_0, i+s, j_0} - \binom{n-1}{s-1, n-s-j_0, j_0}\Big]\Big],$$

*where* $n-1=i+j$.

*Proof.* Let $i=n-j-1$. To find $|\operatorname{WEnd}^j(P_n)|$, we compute according to the following Figure 7, drawn for $n=12, j=5$: where in particular $i=6, t=\lfloor\frac{i}{2}\rfloor=3, t'=\lfloor\frac{j}{2}\rfloor=2$, and $s=1,2,3$,

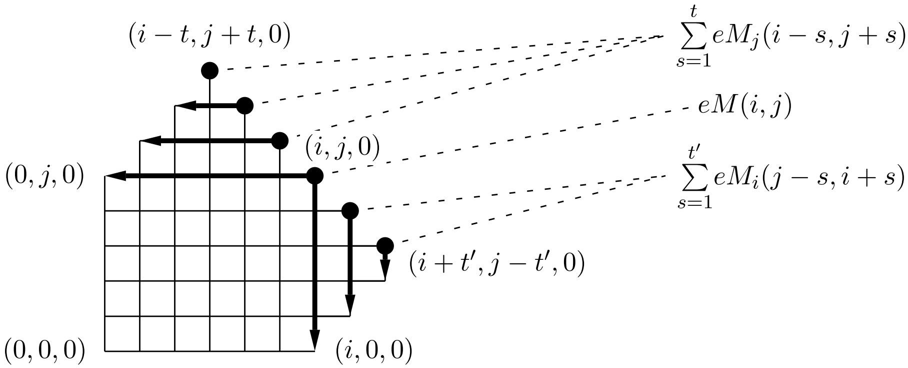


Figure 7

Thus,

$$|\operatorname{WEnd}^j(P_n)| = eM(i,j) + \sum_{s=1}^{n} eM_j(i-s, j+s) + \sum_{s=1}^{n} eM_i(j-s, i+s),$$

since $eM_j(i-s,j+s)=eM_i(j-s,i+s)=0$ if $s>n$ (we observe that $eM_j(i-s,j+s)=0$ and $eM_i(j-s,i+s)=0$ if $s>\lfloor\frac{i}{2}\rfloor$ and $s>\lfloor\frac{j}{2}\rfloor$, respectively). Equations (3), (4) and Proposition 1, imply that

$$\begin{aligned} eM(i,j) &= \sum_{i_0=0}^{i}\sum_{j_0=0}^{j} M(i-i_0, j-j_0, i_0+j_0),\\ &= \sum_{i_0=0}^{i}\sum_{j_0=0}^{j}\binom{n-1}{i-i_0, j-j_0, i_0+j_0}, \end{aligned}$$

$$\begin{aligned} \sum_{s=1}^{n} eM_j(i-s, j+s) &= \sum_{s=1}^{n}\Big[\sum_{i_0=0}^{i-2s} M_j(i-s-i_0, j+s, i_0)\Big]\\ &= \sum_{s=1}^{n}\Big[\sum_{i_0=0}^{i-2s}\Big[\binom{n-1}{i-s-i_0, j+s, i_0} - \binom{n-1}{s-1, n-s-i_0, i_0}\Big]\Big], \end{aligned}$$

$$\sum_{s=1}^{n} eM_i(j-s,i+s) = \sum_{s=1}^{n}\Big[\sum_{j_0=0}^{j-2s} M_i(j-s-j_0,i+s,j_0)\Big]$$
$$= \sum_{s=1}^{n}\Big[\sum_{j_0=0}^{j-2s}\Big[\binom{n-1}{j-s-j_0,i+s,j_0} - \binom{n-1}{s-1,n-s-j_0,j_0}\Big]\Big].$$

Therefore,

$$|\operatorname{WEnd}^j(P_n)| = \sum_{i_0=0}^{i}\sum_{j_0=0}^{j}\binom{n-1}{i-i_0,j-j_0,i_0+j_0}$$
$$+\sum_{s=1}^{n}\Big[\sum_{i_0=0}^{i-2s}\Big[\binom{n-1}{i-s-i_0,j+s,i_0} - \binom{n-1}{s-1,n-s-i_0,i_0}\Big]$$
$$+\sum_{j_0=0}^{j-2s}\Big[\binom{n-1}{j-s-j_0,i+s,j_0} - \binom{n-1}{s-1,n-s-j_0,j_0}\Big]\Big]. \qquad \square$$

Now we can prove the final result.

**Theorem 1.** *Let $n$ be positive integer and $j$ non-negative integer such that $j<n$. Then*

$$(1)\qquad |\operatorname{WEnd}(P_{2n})| = 2\sum_{j=0}^{n-1}\Big[eM(i,j) + \sum_{s=1}^{2n}\big[eM_j(i-s,j+s) + eM_i(j-s,i+s)\big]\Big], \text{ where } i=2n-j-1,$$

$$(2)\qquad |\operatorname{WEnd}(P_{2n+1})| = 2\sum_{j=0}^{n-1}\Big[eM(i,j) + \sum_{s=1}^{2n+1}\big[eM_j(i-s,j+s) + eM_i(j-s,i+s)\big]\Big] + eM(n,n) + 2\sum_{s=1}^{2n+1} eM_n(n-s,n+s), \text{ where } i=2n-j,$$

*where*

$$eM(i,j) = \sum_{i_0=0}^{i}\sum_{j_0=0}^{j} M(i-i_0,j-j_0,i_0+j_0)$$

*and*

$$eM_r(i,j) = \sum_{i_0=0}^{i-j+r} M_r(i-i_0,j,i_0).$$

*Proof.* (1) This is obvious by Proposition 2(2) and Proposition 3.

(2) From Proposition 2(3),

$$
\begin{aligned}
&|\operatorname{WEnd}(P_{2n+1})|\\
&\quad = 2\sum_{j=0}^{n-1}\Big[eM(i,j)+\sum_{s=1}^{2n+1}\big[eM_j(i-s,j+s)+eM_i(j-s,i+s)\big]\Big]\\
&\qquad + |\operatorname{WEnd}^n(P_{2n+1})|,
\end{aligned}
$$

where $i = 2n - j$.

Consider $j = n$. Then $i = n$. Thus

$$
\begin{aligned}
&|\operatorname{WEnd}^n(P_{2n+1})|\\
&\qquad = eM(n,n)+\sum_{s=1}^{2n+1}\big[eM_n(n-s,n+s)+eM_n(n-s,n+s)\big]\\
&\qquad = eM(n,n)+2\sum_{s=1}^{2n+1} eM_n(n-s,n+s). \qquad \square
\end{aligned}
$$

## 2. A sketched extension to homomorphisms from paths

Now we develop a method how to extend the obtained results to homomorphisms starting from a path to certain lexicographic products with this path.

We recall, the *lexicographic product* $G[H]$ of two graphs $G$ and $H$ has vertex set $V(G) \times V(H)$ and $\{(x_1, y_1), (x_2, y_2)\} \in E(G[H])$ whenever $\{x_1, x_2\} \in E(G)$, or $x_1 = x_2$ and $\{y_1, y_2\} \in E(H)$. Consider the behaviours of the homomorphism from $P_n$ to $P_n[K_m]$ where $K_m$ denotes the *complete graph* with the vertex set $V(K_m) = \{k_1, k_2, \ldots, k_m\}$, and of the homomorphism from $P_n$ to $P_n[C_m]$ where $C_m$ denotes the *cycle of length* $m-1$ with the vertex set $V(C_m) = \{0, 1, \ldots, m-1\}, m \geqslant 3$.

Take $f \in \operatorname{Hom}(P_n, P_n[K_2])$ such that $f(i) = (j, k_1)$ where $i, j \in V(P_n)$ and $k_1 \in V(K_2)$. Then $f(i+1) \in \{(j, k_2), (j+1, k_1), (j-1, k_1)\}$, if $j-1, j+1 \in V(P_n)$. So, for each $f : V(P_n) \to V(P_n[K_2])$, define $g : V(P_n) \to V(P_n)$ by $g(i) = j$ if $f(i) = (j, k_x)$; $k_x \in V(K_2)$. Then $g \in \operatorname{WEnd}^i(P_n)$ whenever $f \in \operatorname{Hom}^{(i,k_x)}(P_n, P_n[K_2])$, i.e. $f \in \operatorname{Hom}(P_n, P_n[K_2])$ which map 0 to $(i, k_x)$ for all $k_x \in V(K_2)$. Hence, $|\operatorname{Hom}(P_n, P_n[K_2])| = 2|\operatorname{WEnd}(P_n)|$. Thus by Theorem 1, we get the cardinality $|\operatorname{Hom}(P_n, P_n[K_2])|$.

This way, for each $f \in \operatorname{Hom}^{(j,k_x)}(P_n, P_n[K_2])$ is the shortest path on this 3-dimensional square lattice from the point $(0, 0, 0)$ to some point $(i, j_1, j_2)$ (and some $r$-ladder square lattice).

Consider an $(m+1)$-dimensional square lattice $M(i, j_1, j_2, \ldots, j_m)$ and an $r$-ladder $(m+1)$-dimensional square lattice $M_r(i, j_1, j_2, \ldots, j_m)$. The shortest path on the $(m+1)$-dimensional square lattice from the point $(0, 0, 0, \ldots, 0)$ to any point $(i, j_1, j_2, \ldots, j_m)$ can be obtained by going from the point $(0, 0, 0, \ldots, 0)$ to the point $(i, j_1, j_2, \ldots, j_m)$ by $(i_0, j_{01}, j_{02}, \ldots, j_{0m})$ together with $(i_0+1, j_{01}, j_{02}, \ldots, j_{0m})$, $(i_0, j_{01}+1, j_{02}, \ldots, j_{0m})$, $(i_0, j_{01}, j_{02}+1, \ldots, j_{0m})$, $\ldots$, $(i_0, j_{01}, j_{02}, \ldots, j_{0m}+1)$. Using (1), (2) and induction, we get the next proposition.

**Proposition 4.** *The numbers*

$$M(i, j_1, j_2, \ldots, j_m) \quad \textit{and} \quad M_r(i, r+m, j_2, \ldots, j_m) \quad (j_1 = r+m)$$

*of shortest paths from the point* $(0, 0, \ldots, 0)$ *to any point* $(i, j_1, j_2, \ldots, j_m)$ *in the* $(m+1)$*-dimensional square lattice and in the* $r$*-ladder* $(m+1)$*-dimensional square lattice is*

$$M(i, j_1, j_2, \ldots, j_m) = \binom{i + j_1 + j_2 + \ldots + j_m}{i, j_1, j_2, \ldots, j_m},$$

*and*

$$M_r(i, r+m, j_2, \ldots, j_m) = \binom{i + j_1 + j_2 + \ldots + j_m}{i, j_1, j_2, \ldots, j_m} - \binom{i + j_1 + j_2 + \ldots + j_m}{m-1, r+i+1, j_2, \ldots, j_m},$$

*respectively.*

Let $f \in \mathrm{Hom}(P_n, P_n[K_m])$ such that $f(i) = (j, k_x)$ where $i, j \in V(P_n)$ and $k_x \in V(K_m)$. Then $f(i+1) \in \{(j, k_1), (j, k_2), \ldots, (j, k_{x-1}), (j, k_{x+1}), (j, k_{x+2}), \ldots, (j, k_m)\} \cup \{(j+1, k_x), (j-1, k_x)\}$, if $j-1, j+1 \in V(P_n)$.

We can use the same technique for homomorphism from $P_n$ to $P_n[K_2]$ to get for each $f \in \mathrm{Hom}^{(j,k_x)}(P_n, P_n[K_m])$ the shortest path on this $(m+1)$-dimensional square lattice from the point $(0, 0, 0, \ldots, 0)$ to some point $(i, j_1, j_2, \ldots, j_m)$ (and the $r$-ladder square lattice). Further, take $f \in \mathrm{Hom}(P_n, P_n[C_m])$ such that $f(i) = (j, k)$ where $i, j \in V(P_n)$ and $k \in V(C_m)$. Then $f(i+1) \in \{(j, k-1), (j, k+1)\} \cup \{(j+1, k), (j-1, k)\}$, if $j-1, j+1 \in V(P_n)$ and $k-1 = m-1, k+1 = 0$ for $k = 0$ and $k = m-1$, respectively. Then for each $f \in \mathrm{Hom}^{(j,k)}(P_n, P_n[C_m])$ we get the shortest path on this 4-dimensional square lattice from the point $(0, 0, 0, 0)$ to some point $(i, j_1, j_2, j_4)$ (and the $r$-ladder square lattice). So, the algorithms for the numbers of homomorphisms from the path $P_n$ to the lexicographic products $P_n[K_m]$ and $P_n[C_m]$ can be found.

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