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Finite groups admitting a dihedral group of automorphisms*

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Abstract. Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β and let $F = \langle \alpha\beta \rangle$. Suppose that D acts on a finite group G by automorphisms in such a way that $C_G(F) = 1$. In the present paper we prove that the nilpotent length of the group G is equal to the maximum of the nilpotent lengths of the subgroups $C_G(\alpha)$ and $C_G(\beta)$.

1. Introduction

Throughout the paper all groups are finite. Let F be a nilpotent group acted on by a group H via automorphisms and let the group G admit the semidirect product FH as a group of automorphisms so that $C_G(F) = 1$. By a well known result [1] due to Belyaev and Hartley, the solvability of G is a drastic consequence of the fixed point free action of the nilpotent group F. A lot of research, [7, 10, 11, 13–15], investigating the structure of G has been conducted in case where FH is a Frobenius group with kernel F and complement H. So the immediate question one could ask was whether the condition of being Frobenius for FH could be weakened or not. In this direction we introduced the concept of a Frobenius-like group in [8] as a generalization of Frobenius group and investigated the structure of G when the group FH is Frobenius-like [3],[4],[5],[6]. In particular,

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we obtained in [3] the same conclusion as in [10]; namely the nilpotent lengths of G and $C_G(H)$ are the same, when the Frobenius group FH is replaced by a Frobenius-like group under some additional assumptions. In a similar attempt in [16] Shumyatsky considered the case where FH is a dihedral group and proved the following.

Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β and let $F = \langle \alpha\beta \rangle$. (Here, $D = FH$ where $H = \langle \alpha \rangle$) Suppose that D acts on the group G by automorphisms in such a way that $C_G(F) = 1$. If $C_G(\alpha)$ and $C_G(\beta)$ are both nilpotent then G is nilpotent.

In the present paper we extend his result as follows.

Theorem. *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β and let $F = \langle \alpha\beta \rangle$. Suppose that D acts on the group G by automorphisms in such a way that $C_G(F) = 1$. Then the nilpotent length of G is equal to the maximum of the nilpotent lengths of the subgroups $C_G(\alpha)$ and $C_G(\beta)$.*

After completing the proof we realized that it follows as a corollary of the main theorem of a recent paper [2] by de Melo. The proof we give relies on the investigation of D-towers in G in the sense of [17] and the following proposition which, we think, can be effectively used in similar situations.

Proposition. *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β. Suppose that D acts on a q-group Q for some prime q and let V be a kQD-module for a field k of characteristic different from q such that the group $F = \langle \alpha\beta \rangle$ acts fixed point freely on the semidirect product VQ. If $C_Q(\alpha)$ acts nontrivially on V then we have $C_V(\alpha) \neq 0$ and $\mathrm{Ker}(C_Q(\alpha)$ on $C_V(\alpha)) = \mathrm{Ker}(C_Q(\alpha)$ on $V)$.*

Notation and terminology are standard unless otherwise stated.

2. Proof of the proposition

We first present a lemma to which we appeal frequently in our proofs.

Lemma. *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β and let $F = \langle \alpha\beta \rangle$. Suppose that D acts on the group S by automorphisms in such a way that $C_S(F) = 1$. Then the following hold.*

(i) *For each prime p dividing its order, the group S contains a unique D-invariant Sylow p-subgroup.*

(ii) *Let N be a normal D-invariant subgroup of S. Then $C_{S/N}(F) = 1$, $C_{S/N}(\alpha) = C_S(\alpha)N/N$ and $C_{S/N}(\beta) = C_S(\beta)N/N$.*
(iii) *$S = C_S(\alpha)C_S(\beta)$.*

Proof. See the proofs of Lemma 2.6, Lemma 2.7 and Lemma 2.8 in [16]. □

We are now ready to prove the proposition.

Notice that $V = C_V(\alpha)C_V(\beta)$ by Lemma (iii) applied to the action of D on V. Suppose first that $C_V(\alpha) = 0$. Then $[V, \beta] = 0$ whence $[Q, \beta] \leqslant \mathrm{Ker}(Q \text{ on } V)$ by the Three Subgroup Lemma. Set $\overline{Q} = Q/\mathrm{Ker}(Q \text{ on } V)$. We observe that $C_Q(F) = 1$ implies $C_{\overline{Q}}(F) = 1$ by Lemma (ii). This forces $C_{\overline{Q}}(\alpha) = 1$. As the equality $C_{\overline{Q}}(\alpha) = \overline{C_Q(\alpha)}$ holds by Lemma (ii), we get $C_Q(\alpha)$ acts trivially on V. This contradiction shows that $C_V(\alpha) \neq 0$ establishing the first claim.

To ease the notation we set $H = \langle \alpha \rangle$ and $K = \mathrm{Ker}(C_Q(H) \text{ on } C_V(H))$. Here $D = FH$. To prove the second claim we use induction on $\dim_k V + |QD|$. We choose a counterexample with minimum $\dim_k V + |QD|$ and proceed over several steps.

1) *We may assume that k is a splitting field for all subgroups of QFH.*

We consider the QD-module $\bar{V} = V \otimes_k \bar{k}$ where $\bar{k}$ is the algebraic closure of k. Notice that $\dim_k V = \dim_{\bar{k}} \bar{V}$ and $C_{\bar{V}}(H) = C_V(H) \otimes_k \bar{k}$. Therefore once the proposition has been proven for the group QD on $\bar{V}$, it becomes true for QD on V also.

2) *V is an indecomposable QD-module on which Q acts faithfully.*

Notice that V is a direct sum of indecomposable QD-submodules. Let W be one of these indecomposable QD-submodules on which K acts nontrivially. If $W \neq V$, then the proposition is true for the group QD on W by induction. That is,

$$\mathrm{Ker}(C_Q(H) \text{ on } C_W(H)) = \mathrm{Ker}(C_Q(H) \text{ on } W)$$

and hence

$$K = \mathrm{Ker}(K \text{ on } C_W(H)) = \mathrm{Ker}(K \text{ on } W)$$

which is a contradiction with the assumption that K acts nontrivially on W. Hence $V = W$.

Recall that $\overline{Q} = Q/\mathrm{Ker}(Q \text{ on } V)$ and consider the action of the group $\overline{Q}D$ on V assuming $\mathrm{Ker}(Q \text{ on } V) \neq 1$. An induction argument gives $\mathrm{Ker}(C_{\overline{Q}}(H) \text{ on } C_V(H)) = \mathrm{Ker}(C_{\overline{Q}}(H) \text{ on } V)$. This leads to a contradiction as $C_{\overline{Q}}(H) = \overline{C_Q(H)}$ by Lemma(ii). Thus we may assume that Q acts faithfully on V.

3) *Let Ω denote the set of Q-homogeneous components of V. K acts trivially on every element W in Ω such that $Stab_H(W) = 1$ and so H fixes an element of Ω.*

Let W be in Ω such that $Stab_H(W) = 1$. Then the sum $X = W + W^\alpha$ is direct. It is straightforward to verify that $C_X(H) = \{v + v^\alpha : v \in W\}$. By definition, K acts trivially on $C_X(H)$. Note also that K normalizes both W and W^α as $K \leqslant Q$. It follows now that K is trivial on X and hence on W. This shows that H fixes at least one element of Ω because otherwise $K = 1$, a contradiction.

4) *F acts transitively on Ω.*

Let $\Omega_i, i = 1, \ldots, s$ be all distinct D-orbits of Ω. Then $V = \bigoplus_{i=1}^{s} \bigoplus_{W \in \Omega_i} W$. Since $\bigoplus_{W \in \Omega_i} W$ is QD-invariant for each i we have $s = 1$ by (2), that is, D acts transitively on Ω. Let W be an H-invariant element of Ω whose existence is guaranteed by (3). Then the F-orbit containing W in Ω is the whole of Ω.

From now on W denotes an H-invariant element of Ω. It should be noted that the group $Z(Q/\operatorname{Ker}(Q \text{ on } W))$ acts by scalars on the homogeneous Q-module W, and so $[Z(Q), H] \leqslant \operatorname{Ker}(Q \text{ on } W)$. Set $F_1 = Stab_F(W)$ and let T be a transversal containing 1 for F_1 in F . Then $F = \bigcup_{t \in T} F_1 t$ and so $V = \bigoplus_{t \in T} W^t$. Note that an H-orbit on $\Omega = \{W^t : t \in T\}$ is of length at most 2.

5) *The number of H-invariant elements in Ω is at most 2, and is equal to 2 if and only if $|F/F_1|$ is even. Furthermore $V = U \oplus X$ where X is a Q-submodule centralized by K and U is the direct sum of all H-invariant elements in Ω.*

If W^t is H-invariant then $W^{t\alpha} = W^t$ implies $t^\alpha t^{-1} \in F_1$. On the other hand $t^\alpha t^{-1} = t^{-2}$ since α inverts F. That is, tF_1 is an element of F/F_1 of order at most 2. If $tF_1 = F_1$ then $t = 1$. Otherwise tF_1 is the unique element of order 2 in F/F_1. Thus the number of H-invariant elements in Ω is at most 2 and if it is equal to 2 then $|F/F_1|$ is even. If conversely F/F_1 is of even order, let yF_1 be the unique element of order 2 in F/F_1. Then $y^\alpha F_1 = yF_1$ and so $(W^y)^\alpha = W^{y^\alpha} = W^y \neq W$. This shows that there exist exactly two H-invariant elements in Ω if and only if F/F_1 is of even order.

6) *Since $1 \neq K \trianglelefteq C_Q(H)$, we can choose a nonidentity element $z \in K \cap Z(C_Q(H))$. Set $L = \langle z \rangle$. Then $Q = L^{F_2} C_Q(U)$ where $F_2 = Stab_F(U)$.*

It follows from an induction argument applied to the action of $L^F D$ on V that $Q = L^F$. Let $F_2 = Stab_F(U)$ and observe that for any $f \in$

$F - F_2$, $U^f \leqslant X$ and hence is centralized by L by (5). Thus we get $Q = L^{F_2}C_Q(U) = L^{F_2}C_Q(W)$.

7) *Set* $Y = F_{q'}$. *Then* $Y \cap F_1 \neq Y \cap F_2$.

Suppose that $Y \cap F_1 = Y \cap F_2$. Pick a simple commutator $c = [z^{f_1}, \ldots, z^{f_m}]$ of maximal weight in the elements $z^f, f \in F_1$ such that $c \notin C_Q(W)$. Since $Q = L^{F_2}C_Q(W)$, the weight of this commutator is equal to the nilpotency class of $Q/C_Q(W)$. It should be noted that the nilpotency classes of $Q/C_Q(W)$ and Q are the same, since Q can be embedded into the direct product of $Q/C_Q(W^f)$ as f runs through F. Hence $c \in Z(Q)$. Clearly, $C_Q(F) = 1$ implies $C_Q(Y) = 1$ and hence $\prod_{x\in Y} c^x = 1$, as $\prod_{x\in Y} c^x$ is contained in $Z(Q)$ and is fixed by Y. In fact we have

$$1 = \prod_{x\in Y} c^x = \prod_{x\in Y-F_1} c^x \prod_{x\in Y\cap F_1} c^x.$$

Recall that $[Z(Q), F_1] \leqslant C_Q(W)$ and hence $[Z(Q), F_1] \leqslant \bigcap_{f\in F} C_Q(W^f) = C_Q(V) = 1$. This gives $\prod_{x\in Y\cap F_1} c^x = c^{|Y\cap F_1|}$. On the other hand, for any $f \in F_1$ and any $x \in Y - F_1$, $fx \notin F_2$ and so z centralizes $W^{(fx)^{-1}}$, that is, $z^{fx} \in C_Q(W)$. Therefore c^x lies in $C_Q(W)$ for any x in $Y - F_1$. It follows that $\prod_{x\in Y-F_1} c^x \in C_Q(W)$. This forces that $c^{|Y\cap F_1|} \in C_Q(W)$ which is impossible as $c \notin C_Q(W)$.

8) *Final contradiction.*

By (5) and (7), $|F_2 : F_1| = 2$ and q is odd. Now $Z_2(Q) = [Z_2(Q), H]C_{Z_2(Q)}(H)$ as $(|Q|, |H|) = 1$. Notice that $U = W \oplus W^t$ for some $t \in T$ which may be assumed to lie in $F_2 = Stab_F(U)$. We have $[Z_2(Q), L, H] \leqslant [Z(Q), H] \leqslant C_Q(W) \cap C_Q(W^t) = C_Q(U)$. We also have $[L, H, Z_2(Q)] = 1$ as $[L, H] = 1$. It follows now by the Three Subgroup Lemma that $[H, Z_2(Q), L] \leqslant C_Q(U)$. On the other hand $[C_{Z_2(Q)}(H), L] = 1$ by the definition of L. Thus $[L, Z_2(Q)] \leqslant C_Q(U)$. Then we have $[L^{F_2}, Z_2(Q)] \leqslant C_Q(U)$, as U is F_2- invariant, which yields that $[Q, Z_2(Q)] \leqslant C_Q(U)$. Thus $[Q, Z_2(Q)] \leqslant \bigcap_{f\in F} C_Q(U)^f = C_Q(V) = 1$ and hence Q is abelian.

Now $[Q, F_1H] \leqslant C_Q(W)$ due to the scalar action of $Q/C_Q(W)$ on W. Notice that $C_W(H) = 0$ because otherwise L is trivial on W due to its action by scalars. So H inverts every element of W. Since $Stab_F(W^t) = Stab_F(W)^t = {F_1}^t = F_1$, we can replace W by W^t and conclude that H inverts every element in U. That is, H acts by scalars and hence lies in the center of $QF_2H/C_{QF_2}(U)$. On the other hand H inverts $F_2/C_{F_2}(U)$. It follows that $|F_2/C_{F_2}(U)| = 1$ or 2. Since $|F_2 : F_1| = 2$, we have $F_1 \leqslant C_{F_2}(U)$. This contradicts the fact that $C_W(F_1) = 0$ as $C_V(F) = 0$.

3. Proof of the theorem

Suppose that $n = f(G) \geqslant f(C_G(\alpha)) \geqslant f(C_G(\beta))$ and set $H = \langle \alpha \rangle$. We may assume by Proposition 5 in [9] that $C_G(F) = 1$ implies $[G, F] = G$. In view of Lemma (i) for each prime p dividing the order of G there is a unique D-invariant Sylow p-subgroup of G. This yields the existence of an irreducible D-tower $\widehat{P}_1, \ldots, \widehat{P}_n$ in the sense of [17] where

(a) $\widehat{P}_i$ is a D-invariant p_i-subgroup, p_i is a prime, $p_i \neq p_{i+1}$, for $i = 1, \ldots, n-1$;
(b) $\widehat{P}_i \leqslant N_G(\widehat{P}_j)$ whenever $i \leqslant j$;
(c) $P_n = \widehat{P}_n$ and $P_i = \widehat{P}_i / C_{\widehat{P}_i}(P_{i+1})$ for $i = 1, \ldots, n-1$ and $P_i \neq 1$ for $i = 1, \ldots, n$;
(d) $\Phi(\Phi(P_i)) = 1$, $\Phi(P_i) \leqslant Z(P_i)$, and $\exp(P_i) = p_i$ when p_i is odd for $i = 1, \ldots, n;$
(e) $[\Phi(P_{i+1}), P_i] = 1$ and $[P_{i+1}, P_i] = P_{i+1}$ for $i = 1, \ldots, n-1$;
(f) $(\Pi_{j<i}\widehat{P_j})FH$ acts irreducibly on $P_i/\Phi(P_i)$ for $i = 1, \ldots, n$;
(g) $P_1 = [P_1, F]$.

Set now $X = \prod_{i=1}^{n} \widehat{P}_i$. As $P_1 = [P_1, D]$ by *(g)*, we observe that $X = [X, D]$. If X is proper in G, by induction we have $n = f(X) = f(C_X(H))$ and so the theorem follows. Hence $X = G$. Notice that G is nonabelian and hence $C_G(H) \neq 1$, that is $f(C_G(H) \geqslant 1$. Therefore the theorem is true if $G = F(G)$. We set next $\overline{G} = G/F(G)$. As $\overline{G}$ is a nontrivial group such that $\overline{G} = \left[\overline{G}, F\right]$, it follows by induction that $f(\overline{G}) = n - 1 = f(C_{\overline{G}}(H))$. This yields that $[C_{\overline{\widehat{P}_{n-1}}}(H), \ldots, C_{\overline{\widehat{P}_1}}(H)]$ is nontrivial. Since $C_{\overline{\widehat{P}_i}}(H) = \overline{C_{\widehat{P}_i}(H)}$ for each i by Lemma (ii), we have $Y = [C_{\widehat{P}_{n-1}}(H), \ldots, C_{\widehat{P}_1}(H)] \nleq F(G) \cap \widehat{P}_{n-1} = C_{\widehat{P}_{n-1}}(\widehat{P}_n)$.

By the Proposition applied to the action of the group $\widehat{P}_{n-1}FH$ on the module $\widehat{P}_n/\Phi(\widehat{P}_n)$ we get

$$\mathrm{Ker}(C_{\widehat{P}_{n-1}}(H) \text{ on } C_{\widehat{P}_n/\Phi(\widehat{P}_n)}(H)) = \mathrm{Ker}(C_{\widehat{P}_{n-1}}(H) \text{ on } \widehat{P}_n/\Phi(\widehat{P}_n)).$$

It follows now that Y does not centralize $C_{\widehat{P}_n}(H)$ and hence $f(C_G(H) = n = f(G)$. This completes the proof.

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