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Orthosymplectic Jordan superalgebras and the Wedderburn principal theorem*

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Abstract. An analogue of the Wedderburn Principal Theorem (WPT) is considered for finite-dimensional Jordan superalgebras $\mathcal{A}$ with solvable radical $\mathcal{N}$, $\mathcal{N}^2 = 0$, and such that $\mathcal{A}/\mathcal{N} \cong \mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$, where $\mathbb{F}$ is a field of characteristic zero.

We prove that the WPT is valid under some restrictions over the irreducible $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$-bimodules contained in $\mathcal{N}$, and show with counter-examples that these restrictions cannot be weakened.

Introduction

In recent works, see [10] and [11], the first author proved an analogue to the Wedderburn principal theorem for Jordan superalgebras when we have a finite dimensional Jordan superalgebra $\mathcal{A}$ with solvable radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$ and $\mathcal{A}/\mathcal{N}$ is a simple Jordan superalgebra of some of the following types: superalgebra of superform, Kac $\mathcal{K}_{10}$, $\mathcal{D}_t$, $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$. Some conditions were impossed over the solvable radical $\mathcal{N}$.

Similarly as [11], we consider a finite dimensional Jordan superalgebra $\mathcal{A}$ over an algebraically closed field of characteristic 0 $\mathbb{F}$, with solvable radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$ and $\mathcal{A}/\mathcal{N} \cong \mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$, to follow we show that if $\mathcal{N}$ considered as $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$-superbimodule does not contains any homomorphic image isomorphic to subbimodule $\mathcal{R}\mathrm{eg}\,(\mathfrak{J}\mathrm{osp}_{1|2}(\mathbb{F}))$ then,

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the Wedderburn principal theorem hold. Moreover, we shown that there is a counter-example to WPT for this case.

1. Preliminary results and notations

Recall that an algebra $\mathcal{A}$ is said to be a superalgebra if $\mathcal{A} = \mathcal{A}_0 \dotplus \mathcal{A}_1$ satisfies the relation $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j(\text{mod } 2)}$, i.e. $\mathcal{A}$ is a $\mathbb{Z}_2$ - graded algebra. Given an element $a \in \mathcal{A}_0 \cup \mathcal{A}_1$, $|a| = i$ denotes its parity, according to $a \in \mathcal{A}_i$.

Let $\Gamma = \mathbf{alg}\,\langle 1, e_i,\ i \in \mathbb{Z}^+ | e_ie_j + e_je_i = 0\rangle$ denote the Grassmann algebra. Then, $\Gamma = \Gamma_0 \dotplus \Gamma_1$, where Γ_0 and Γ_1, is spanned by all monomials of even and odd length respectively, and it is easy to see that Γ has a superalgebra structure.

Let $\mathcal{A} = \mathcal{A}_0 \dotplus \mathcal{A}_1$ be a superalgebra, let's the Grassmann envelope of $\mathcal{A}$ is the algebra $\Gamma(\mathcal{A}) = \Gamma_0 \otimes \mathcal{A}_0 + \Gamma_1 \otimes \mathcal{A}_1$. Assume that $\mathfrak{M}$ is a homogeneous variety of algebras (see, eg. [23]). The superalgebra $\mathcal{A}$ is said to be a $\mathfrak{M}$-superalgebra if the Grassmann envelope $\Gamma(\mathcal{A})$ belongs to $\mathfrak{M}$.

An associative superalgebra is just a $\mathbb{Z}_2$-graded associative algebra, but it is not so in general (see [21]). One can easily check that a superalgebra $\mathfrak{J} = \mathfrak{J}_0 \dotplus \mathfrak{J}_1$ is a Jordan superalgebra if and only if it satisfies the super identities

$$xy = (-1)^{|x||y|}yx, \tag{1}$$

$$\begin{aligned}((xy)z)t &+ (-1)^{|t|(|z|+|y|)+|z||y|}((xt)z)y + (-1)^{|x|(|y|+|z|+|t|)+|t||z|}((yt)z)x \\ &= (xy)(zt) + (-1)^{|t||z|+|t||y|}(xt)(yz) + (-1)^{|y||z|}(xz)(yt). \end{aligned}\tag{2}$$

In particular, the Jordan superalgebra $\mathfrak{J} = \mathfrak{J}_0 \dotplus \mathfrak{J}_1$ is a ($\mathbb{Z}_2$-graded) Jordan algebra if and only if $(\mathfrak{J}_1)^2 = 0$.

Let $\mathcal{A}$ be an associative superalgebra with multiplication ab, we consider a new multiplication $a \circ b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$, a, $b \in \mathcal{A}_0 \cup \mathcal{A}_1$. We can see, that with respect to this multiplication $\mathcal{A}$ has a structure of Jordan superalgebra, which we will denote as $\mathcal{A}^{(+)}$.

Wall proved in [22] that every associative simple finite-dimensional superalgebra over an algebraically closed field $\mathbb{F}$ is isomorphic to one of following associative superalgebras

(i) $\mathcal{A} = \mathcal{M}_{n|m}(\mathbb{F}), \quad \mathcal{A}_0 = \left\{\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right\}, \quad \mathcal{A}_1 = \left\{\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right\},$

(ii) $\mathcal{A} = \mathcal{Q}(n), \quad \mathcal{A}_0 = \left\{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right\}, \quad \mathcal{A}_1 = \left\{\begin{bmatrix} 0 & h \\ h & 0 \end{bmatrix}\right\}.$

where a, $h \in \mathcal{M}_n(\mathbb{F})$, $d \in \mathcal{M}_m(\mathbb{F})$, $b \in \mathcal{M}_{n\times m}(\mathbb{F})$, $c \in \mathcal{M}_{m\times n}(\mathbb{F})$.

Let $\mathcal{A}$ be an associative (super)algebra. A superinvolution $*$ in $\mathcal{A}$ is a graded linear mapping $* : \mathcal{A} \to \mathcal{A}$ such that $(a^*)^* = a$ and $(ab)^* = (-1)^{|a||b|}b^*a^*$. Let $\mathcal{H}(\mathcal{A}, *)$ be the set of symmetric elements of $\mathcal{A}$ relative to $*$, namely, $\mathcal{H}(\mathcal{A}, *) = \{a \in \mathcal{A}/a^* = a\}$. It is easy to see that $\mathcal{H}(\mathcal{A}, *) \subseteq \mathcal{A}^{(+)}$, and therefore $\mathcal{H}(\mathcal{A}, *)$ is a Jordan superalgebra.

Let I_n, I_m be identity matrices of order n and m respectively. We denote by t the usual transposition of matrices and let

$$U = -U^t = -U^{-1} = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}.$$

Consider the linear mapping osp : $\mathcal{M}_{n|2m}(\mathbb{F}) \to \mathcal{M}_{n|2m}(\mathbb{F})$, given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\mathrm{osp}} = \begin{bmatrix} I_n & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} a^t & -c^t \\ b^t & d^t \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & U^{-1} \end{bmatrix},$$

where $a \in \mathcal{M}_n$, $b, c^t \in \mathcal{M}_{n\times 2m}$ and $d \in \mathcal{M}_{2m}$.

We can see that osp is a superinvolution over superalgebra $\mathcal{M}_{n|2m}(\mathbb{F})$. So, the Jordan superalgebra $\mathcal{H}(\mathcal{M}_{n|2m}(\mathbb{F}), \mathrm{osp})$, denote by $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$, is determined by the following matrices set:

$$\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F}) = \left\{ \begin{bmatrix} a & b_1 & b_2 \\ -b_2^t & d_1 & d_2 \\ b_1^t & d_3 & d_1^t \end{bmatrix} \;\middle|\; a = a^t,\, d_2 = -d_2^t,\, d_3 = -d_3^t \right\}$$

where $a \in \mathcal{M}_n(\mathbb{F})$, b_1, $b_2 \in \mathcal{M}_{n\times m}(\mathbb{F})$, d_1, d_2, and $d_3 \in \mathcal{M}_m(\mathbb{F})$.

Simple finite-dimensional Jordan superalgebras over zero characteristic fields were classified by Kac [12] (see also [13]).

Now, we recall that a $\mathfrak{J}$-superbimodule $\mathcal{M} = \mathcal{M}_0 \dot{+} \mathcal{M}_1$ is a Jordan superbimodule if the corresponding split null extension $\mathcal{E} = \mathfrak{J} \oplus \mathcal{M}$ is a Jordan superalgebra. Besides, the split null extension is the vector space direct sum $\mathfrak{J} \oplus \mathcal{M}$ with multiplication that extends the multiplication of $\mathfrak{J}$, the action of $\mathfrak{J}$ on $\mathcal{M}$, and $\mathcal{M}^2 = 0$. Let $\mathcal{M}$ be a $\mathfrak{J}$-superbimodule, the opposite superbimodule $\mathcal{M}^{\mathrm{op}} = \mathcal{M}_0^{\mathrm{op}} \dot{+} \mathcal{M}_1^{\mathrm{op}}$ is defined by the conditions $\mathcal{M}_0^{\mathrm{op}} = \mathcal{M}_1$, $\mathcal{M}_1^{\mathrm{op}} = \mathcal{M}_0$, and the following action of $\mathfrak{J}$, $a \cdot m^{\mathrm{op}} = (-1)^{|a|}(am)^{\mathrm{op}}$, $m^{\mathrm{op}} \cdot a = (ma)^{\mathrm{op}}$, for any $a \in \mathfrak{J}_0 \cup \mathfrak{J}_1$, $m \in \mathcal{M}_0^{\mathrm{op}} \dot{\cup} \mathcal{M}_1^{\mathrm{op}}$. Whenever $\mathcal{M}$ is a Jordan $\mathfrak{J}$-superbimodule it is possible to see that $\mathcal{M}^{\mathrm{op}}$ is so as well. A *regular superbimodule* $\mathcal{R}\mathrm{eg}\,(\mathfrak{J})$ is defined on the vector superspace $\mathfrak{J}$ with the action of $\mathfrak{J}$ coinciding with the multiplication in $\mathfrak{J}$.

Irreducible bimodules over Jordan superalgebra $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$ were classified by Martinez and Zelmanov in [19], who proved that the only unital irreducible $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$-bimodules are the regular bimodule $\mathcal{R}\mathrm{eg}\,(\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F}))$ the bimodule $\mathfrak{S} = \mathcal{S}\mathrm{kew}(\mathcal{M}_{n|2m}(\mathbb{F}), \mathrm{osp})$, and their opposites, where

$$\mathfrak{S} = \left\{ \begin{bmatrix} a & b_1 & b_2 \\ b_2^t & d_1 & d_2 \\ -b_1^t & d_3 & -d_1^t \end{bmatrix} \;\middle|\; a = -a^t,\, d_2 = d_2^t,\, d_3 = d_3^t \right\} \tag{3}$$

$a \in \mathcal{M}_n(\mathbb{F})$, b_1, $b_2 \in \mathcal{M}_{n\times m}(\mathbb{F})$, d_1, d_2, and $d_3 \in \mathcal{M}_m(\mathbb{F})$.

1.1. The Peirce decomposition

Recall, that if $\mathfrak{J}$ is a Jordan (super)algebra with unity 1, and $\{e_1, \ldots, e_n\}$ is a set of pairwise orthogonal idempotents such that $1 = \sum_{i=1}^n e_i$, then $\mathfrak{J}$ admits Peirce decomposition [20], it is

$$\mathfrak{J} = \left(\bigoplus_{i=1}^{n} \mathfrak{J}_{ii}\right) \bigoplus \left(\bigoplus_{i<j} \mathfrak{J}_{ij}\right),$$

where

$$\mathfrak{J}_{ii} = \{\, x \in \mathfrak{J} : \quad e_i x = x, \}$$

and

$$\mathfrak{J}_{ij} = \{\, x \in \mathfrak{J} : \quad e_i x = \frac{1}{2}x, \quad e_j x = \frac{1}{2}x \,\}, \quad i \neq j$$

are the Peirce components of $\mathfrak{J}$ relative to the idempotents e_i, and e_j, moreover the following relations hold

$$\mathfrak{J}_{ij}^2 \subseteq \mathfrak{J}_{ii} + \mathfrak{J}_{jj}, \qquad \mathfrak{J}_{ij} \cdot \mathfrak{J}_{jk} \subseteq \mathfrak{J}_{ik}$$
$$\mathfrak{J}_{ij} \cdot \mathfrak{J}_{kl} = 0 \quad \text{when} \quad i \neq k, l \text{ and } j \neq k, l.$$

2. Main theorem

In this section we prove the central result of this paper. To start we introduce the following notation, by e_{ij}, $i, j = 1, \ldots, n+2m$, we denote the usual unit matrices.

For $i, j \in \{1, \ldots, n\}$ and $p, q \in \{1, \ldots, m\}$, denote $e_{ij}^n = e_{ij}$, $e_{ip}^{nm} = e_{i\,n+p}$, $e_{ip}^{n2m} = e_{i\,n+m+p}$, $e_{pq}^{m} = e_{n+p\,n+q}$, $e_{pi}^{mn} = e_{n+p\,i}$, $e_{pq}^{m2m} = e_{n+p\,n+m+q}$, $e_{pq}^{2m} = e_{n+m+p\,n+m+q}$, $e_{pq}^{2mm} = e_{n+m+p\,n+q}$ and $e_{pi}^{2mn} = e_{n+m+p\,i}$.

Consider $h_{ij} = e_{ij}^n + e_{ji}^n$ if $i \neq j$, $h_{ii} = e_{ii}^n$, $v_{pq} = e_{pq}^m + e_{qp}^{2m}$, $s_{pq} = e_{pq}^{m2m} - e_{qp}^{m2m}$, $\widetilde{s}_{pq} = e_{pq}^{2mm} - e_{qp}^{2mm}$, $u_{ip} = e_{ip}^{nm} + e_{pi}^{2mn}$, $k_{ip} = e_{ip}^{n2m} - e_{pi}^{mn}$.

With the previous notation, the Jordan superalgebra $\mathfrak{J} = \mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$ is spanned by $\{h_{ij}, v_{pq}, s_{pq}, \widetilde{s}_{pq}, u_{ip}, k_{ip}\}$ and its dimension is given by $\frac{(n+2m)^2+n-2m}{2}$.

From $a \circ b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$ we can see that the non-zero products of basis elements of $\mathfrak{J}$ are defined as follows:

$$
\begin{gathered}
h_{ii} \circ h_{ii} = h_{ii},\\
h_{ij} \circ h_{kl} = \frac{1}{2}(\delta_{jk}h_{il} + \delta_{li}h_{kj} + \delta_{jl}h_{ik} + \delta_{ik}h_{jl}) \quad \text{if } i \neq j, k \neq l,\\
s_{pq} \circ \widetilde{s}_{rt} = \frac{1}{2}(\delta_{qr}v_{pt} + \delta_{pt}v_{qr} - \delta_{qt}v_{pr} - \delta_{pr}v_{qt}),\\
v_{pq} \circ v_{rt} = \frac{1}{2}(\delta_{qr}v_{pt} + \delta_{pt}v_{rq}),\\
v_{pq} \circ s_{rt} = \frac{1}{2}(\delta_{qr}s_{pt} + \delta_{tq}s_{rp}), \quad v_{pq} \circ \widetilde{s}_{rt} = \frac{1}{2}(\delta_{pr}\widetilde{s}_{qt} + \delta_{pt}\widetilde{s}_{rq}),
\end{gathered} \tag{4}
$$

$$
\begin{gathered}
u_{kr} \circ h_{ij} = \frac{1}{2}(\delta_{jk}u_{ir} + \delta_{ik}u_{jr}),\\
k_{lp} \circ h_{ij} = \frac{1}{2}(\delta_{jl}k_{ip} + \delta_{il}k_{jp}) \quad \text{if } i \neq j,\\
u_{kr} \circ h_{ii} = \frac{1}{2}\delta_{ik}u_{ir}, \quad k_{lp} \circ h_{ii} = \frac{1}{2}\delta_{il}k_{ip},\\
u_{kr} \circ v_{pq} = \frac{1}{2}\delta_{rp}u_{kq}, \quad k_{lr} \circ v_{pq} = \frac{1}{2}\delta_{rq}k_{lp},\\
u_{ir} \circ s_{pq} = \frac{1}{2}(\delta_{rp}k_{iq} - \delta_{rq}k_{ip}), \quad k_{ir} \circ \widetilde{s}_{pq} = \frac{1}{2}(\delta_{rp}u_{iq} - \delta_{rq}u_{ip}),
\end{gathered} \tag{5}
$$

$$
\begin{gathered}
u_{ip} \circ u_{jq} = \frac{1}{2}\delta_{ij}\widetilde{s}_{pq}, \quad k_{ip} \circ k_{jq} = \frac{1}{2}\delta_{ij}s_{qp},\\
u_{ip} \circ k_{iq} = \frac{1}{2}v_{qp} - \delta_{pq}h_{ii},\\
u_{ip} \circ k_{jq} = \frac{1}{2}(\delta_{ij}v_{qp} - \delta_{pq}h_{ij}) \text{ if } i \neq j,
\end{gathered} \tag{6}
$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$. We note that the products in (4) and (5) are symmetric and the products in (6) is skew-symmetric.

Now we prove the following theorem.

Theorem. *Let $\mathcal{A}$ be a finite-dimensional Jordan superalgebra with a solvable radical $\mathcal{N}$, such that $\mathcal{N}^2 = 0$, and the quotient superalgebra $\mathfrak{J} = \mathcal{A}/\mathcal{N}$ is isomorphic ti $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$. If $n + m \geqslant 3$ or $n = m = 1$ and no homomorphic image of $\mathcal{N}$, considered as a $\mathfrak{J}$-bimodule, contains a subbimodule isomorphic to $\mathcal{R}\mathrm{eg}\,(\mathfrak{J}\mathrm{osp}_{1|2}(\mathbb{F}))$, then there exists a subsuperalgebra $\mathcal{S} \subset \mathcal{A}$ such that $\mathcal{S} \cong \mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$.*

Proof. Take $\mathfrak{J} = \mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$ and let $\mathfrak{J}$-mod denote the category of Jordan $\mathfrak{J}$-superbimodules. By Theorem 8.1 in [19], every $V \in \mathfrak{J}$-mod is completely reducible. Let $\mathfrak{M}(\mathfrak{J})$ by the set of V in $\mathfrak{J}$-mod such that V does not contain a bimodule isomorphic to $\mathcal{R}\mathrm{eg}\,(\mathfrak{J}\mathrm{osp}_{1|2}(\mathbb{F}))$, among its irreducible summands. Clearly, $\mathfrak{M}(\mathfrak{J})$ is closed with respect to subbimodules and homomorphic images, and by [10] (Theorem 3.3) we observe that it is suffices to prove the theorem when $\mathcal{A}$ is unital and $\mathcal{N}$ is irreducible. Following [19], (Theorem 6.3), there are four different types of unital irreducible $\mathfrak{J}$-bimodules $reg(\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F}))$, $\mathrm{Skew}(\mathcal{M}_{n|2m}, \mathrm{osp})$ and their opposites.

We have that

$$\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F}) = \mathrm{alg}\langle\, h_{ij},\, v_{pq},\, s_{pq},\, \widetilde{s}_{pq}\,\rangle \dotplus \mathrm{vect}\langle\, u_{ip},\, k_{ip}\,\rangle,$$

where $i, j = 1, \ldots, n$ and $p, q = 1, \ldots, m$.

Since WPT is valid for Jordan algebras, and using the fact that $\mathcal{A}/\mathcal{N} \cong \mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$ we can assume that there exists $\mathcal{S}_0 \subseteq \mathcal{A}_0$ such that $\mathcal{S}_0 \cong$ $\mathbf{alg}\,\langle\, h_{ij},\, v_{pq},\, s_{pq},\, \widetilde{s}_{pq}\,\rangle$. Therefore there exist H_{ij}, V_{pq}, S_{pq}, and $\widetilde{S}_{pq} \in \mathcal{A}_0$ for which the multiplications (4) are valid when we substitute h_{ij}, v_{pq}, s_{pq} and $\widetilde{s}_{pq}$ by H_{ij}, V_{pq}, S_{pq} and $\widetilde{S}_{pq}$, respectively.

We note that $\{H_{ii}, V_{pp}$ for $i = 1\ldots, n; p = 1\ldots, m\}$ is a set of pairwise orthogonal idempotents such that $H_{11} + \cdots + H_{nn} + V_{11} + \cdots + V_{mm} = 1$. Thus $\mathcal{A}$ has a Peirce decomposition with respect to its idempotents:

$$\mathcal{A} = \left(\bigoplus_{\substack{i\leqslant j\\ i=1}}^{n} (\mathcal{A})_{ij}\right) \bigoplus \left(\bigoplus_{\substack{i=1\ldots n\\ p=1\ldots,m}} (\mathcal{A})_{ip}\right) \bigoplus \left(\bigoplus_{\substack{p\leqslant q\\ p=1}}^{m} (\mathcal{A})_{pq}\right).$$

Now we need to find $\widetilde{U}_{ip}$ and $\widetilde{K}_{ip} \in \mathcal{A}_1$ such that the multiplications (5) and (6) hold when we change h_{ij}, v_{pq}, s_{pq}, $\widetilde{s}_{pq}$, u_{ip} and k_{ip} by H_{ij}, V_{pq}, S_{pq}, $\widetilde{S}_{pq}$, $\widetilde{U}_{ip}$ and $\widetilde{K}_{ip}$, respectively.

Since $\mathcal{A}_1/\mathcal{N}_1 \cong (\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F}))_1$, there exist $\bar{U}_{ip}$ and $\bar{K}_{ip} \in \mathcal{A}_1/\mathcal{N}_1$ such that (5) and (6) are valid in $\mathcal{A}/\mathcal{N}$ when we change h_{ij}, v_{pq}, s_{pq}, $\widetilde{s}_{pq}$, u_{ip} and k_{ip} by H_{ij}, V_{pq}, S_{pq}, $\widetilde{S}_{pq}$, $\bar{U}_{ip}$ and $\bar{K}_{ip}$, respectively.

Case 1. $\mathcal{N} \cong \mathcal{R}\mathrm{eg}\,(\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F}))$. Let g_{ij}, w_{pq}, z_{pq}, $\widetilde{z}_{pq}$, y_{ip} and $x_{ip} \in \mathcal{N}$ and assume that the isomorphism σ is determinated by

$$\sigma(\,g_{ij}) = h_{ij}, \quad \sigma(\,w_{pq}) = v_{pq}, \quad \sigma(\,z_{pq}) = s_{pq},$$
$$\sigma(\,\widetilde{z}_{pq}) = \widetilde{s}_{pq}, \quad \sigma(\,y_{ip}) = u_{ip}, \quad \sigma(\,x_{ip}) = k_{ip}.$$

So, we have that

$$\mathcal{N}_0 = \mathbf{span}\,\langle\, g_{ij},\ w_{pq},\ z_{pq},\ \widetilde{z}_{pq}\rangle \quad \text{and} \quad \mathcal{N}_1 = \mathbf{span}\,\langle\, y_{ip},\ x_{ip}\rangle.$$

The action of $\mathfrak{Josp}_{n|2m}(\mathbb{F})$ over $\mathcal{N}$ is determined by the equations (4),(5) and (6) when we replace g_{ij}, w_{pq}, z_{pq}, $\widetilde{z}_{pq}$, y_{ip}, and x_{ip} by h_{ij}, v_{pq}, s_{pq}, $\widetilde{s}_{pq}$, u_{ip}, and k_{ip}, respectively.

Lemma 1. *Let $\varphi : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ be the canonical homomorphism. For $i = 1, \dots, n$ and $p = 1, \dots, m$ let U_{ip} and $K_{ip} \in \mathcal{A}_1$ be preimages of $\bar{U}_{ip}$ and $\bar{K}_{ip}$ respectively, then*

$$\begin{aligned} U_{ip} \cdot H_{ij} &= \frac{1}{2} U_{jp}, \quad & U_{ip} \cdot V_{pq} &= \frac{1}{2} U_{iq}, \\ K_{ip} \cdot H_{ij} &= \frac{1}{2} K_{jp}, \quad & K_{ip} \cdot V_{qp} &= \frac{1}{2} K_{iq} \end{aligned} \tag{7}$$

$$U_{ip} \cdot S_{pq} = \frac{1}{2} K_{iq}, \quad K_{ip} \cdot \widetilde{S}_{pq} = \frac{1}{2} U_{iq} \tag{8}$$

Proof. To start we prove (7). From $\varphi(U_{ip} \cdot H_{ij}) = \frac{1}{2}\bar{U}_{jp}$, $\varphi(U_{ip} \cdot V_{pq}) = \varphi(U_{iq} \cdot V_{qq}) = \frac{1}{2}\bar{U}_{iq}$ and using the properties of Peirce decomposition for the Jordan superalgebra $\mathcal{A}$, we note that $U_{ip} \cdot H_{ij} \in (\mathcal{A}_0)_{j\,n+p}$. We can see that $\{\, y_{jp},\, x_{jp}\}$ is a generator set of $(\mathcal{N}_0)_{jp}$, and therefore we can assume that there exist $\eta^{uh,x}_{ipij}$, $\eta^{uh,y}_{ipij} \in \mathbb{F}$ such that

$$U_{ip} \cdot H_{ij} = \frac{1}{2} U_{jp} + \eta^{uh,y}_{ipij}\, y_{jp} + \eta^{uh,x}_{ipij}\, x_{jp}. \tag{9}$$

Similarly,

$$K_{ip} \cdot V_{qp} = \frac{1}{2} K_{iq} + \eta^{kv,y}_{ipqp}\, y_{iq} + \eta^{kv,x}_{ipqp}\, x_{iq}, \tag{10}$$

for some $\eta^{uv,x}_{ippq}$, $\eta^{uv,y}_{ippq} \in \mathbb{F}$.

Using (4) and replacing $x = U_{ip}$, $y = z = t = H_{ii}$ in (2) we have

$$2((U_{ip} \cdot H_{ii}) \cdot H_{ii}) \cdot H_{ii} + H_{ii} \cdot U_{ip} = 3(U_{ip} \cdot H_{ii}) \cdot H_{ii}. \tag{11}$$

If we replace (9) in (11), we obtain

$$\frac{3}{4} U_{ip} + \frac{5}{2} \eta^{uh,y}_{ipii}\, y_{ip} + \frac{5}{2} \eta^{uh,x}_{ipii}\, x_{ip} = \frac{3}{4} U_{ip} + 3\eta^{uh,y}_{ipii}\, y_{ip} + 3\eta^{uh,x}_{ipii}\, x_{ip}.$$

Hence, $\eta^{uh,x}_{ipii}\, x_{ip} + \eta^{uh,y}_{ipii}\, y_{ip} = 0$. Since y_{ip} and x_{ip} are linearly independent, we have $\eta^{uh,y}_{ipii} = \eta^{uh,x}_{ipii} = 0$. Similarly, we can prove that $\eta^{kh,y}_{ipii} = \eta^{kh,x}_{ipii} = 0$.

Using the fact that $\varphi(U_{ip} \cdot H_{ij}) = \varphi(U_{jp} \cdot H_{jj})$ we can conclude that $\eta^{uh,x}_{ipij} = \eta^{uh,x}_{jpjj} = \eta^{uh,y}_{ipij} = \eta^{uh,y}_{jpjj} = 0$.

Now, if we replace $x = K_{ip}$, $y = z = t = V_{pp}$ in (2) and using (4) we obtain

$$2((K_{ip} \cdot V_{pp}) \cdot V_{pp}) \cdot V_{pp} + V_{pp} \cdot K_{ip} = 3(K_{ip} \cdot V_{pp}) \cdot V_{pp}.$$

As in the case above, it is easy to see that $\eta^{kv,y}_{ippp} = \eta^{kv,x}_{ippp} = 0$. Again, using some properties of the canonical homomorphism we obtain $\eta^{kv,y}_{ipqp} = \eta^{kv,x}_{ipqp} = \eta^{kv,x}_{ippp} = 0$. The other equalities in (7) are proven similarly.

Now, we show the equality (8). Let $\eta^{us,y}_{ippq}$, $\eta^{us,x}_{ippq}$, $\eta^{k\widetilde{s},x}_{ippq}$ and $\eta^{k\widetilde{s},y}_{ippq}$ scalars such that

$$U_{ip} \cdot S_{pq} = \frac{1}{2}K_{iq} + \eta^{us,y}_{ippq} y_{iq} + \eta^{us,x}_{ippq} x_{iq}, \quad K_{ip} \cdot \widetilde{S}_{pq} = \frac{1}{2}U_{iq} + \eta^{k\widetilde{s},y}_{ippq} y_{iq} + \eta^{k\widetilde{s},x}_{ippq} x_{iq}. \tag{12}$$

Let $x = U_{ip}$, $y = S_{pq}$, $z = V_{pq}$ and $t = \widetilde{S}_{pq}$ in the equation (2). Using (4) and (7) we have,

$$((U_{ip} \cdot S_{pq}) \cdot V_{pq}) \cdot \widetilde{S}_{pq} - \frac{1}{4}U_{iq} = -\frac{1}{8}U_{iq} \tag{13}$$

Using (12) and computing the products we obtain $(\eta^{k\widetilde{s},y}_{iqpq} - \eta^{us,x}_{ippq})\, y_{iq} = 0$ and $\eta^{k\widetilde{s},x}_{ippq} x_{iq} = 0$. Hence $\eta^{k\widetilde{s},x}_{ippq} = 0$ (Similarly, we can prove $\eta^{us,y}_{ippq} = 0$), and

$$\eta^{k\widetilde{s},y}_{iqpq} - \eta^{us,x}_{ippq} = 0. \tag{14}$$

We note that $\varphi(U_{ir} \cdot S_{rq}) = \varphi(U_{ip} \cdot S_{pq}) = -\varphi(U_{ip} \cdot S_{qp}) = \frac{1}{2}\bar{K}_{iq}$. Thus we have $\eta^{us,x}_{irrq} = \eta^{us,x}_{ippq} = -\eta^{us,x}_{ipqp}$; but this relation only depend of i and q so we can write

$$U_{ip} \cdot S_{pq} = \frac{1}{2}K_{iq} + \eta^{us}_{iq} x_{iq}. \tag{15}$$

Analogously we have $K_{ip} \cdot \widetilde{S}_{pq} = \frac{1}{2}U_{iq} - \eta^{us}_{iq} y_{iq}$. The equation (14) can be rewritten as $\eta^{k\widetilde{s}}_{ip} = \eta^{us}_{iq}$, only depend of i and therefore $\eta^{k\widetilde{s}}_{i} = \eta^{us}_{i}$. Since $K_{ip} \cdot \widetilde{S}_{pq} = -K_{ip} \cdot \widetilde{S}_{qp}$, we obtain $\eta^{k\widetilde{s}}_{i} = -\eta^{k\widetilde{s}}_{i} = 0$ similarly, $\eta^{us}_{i} = 0$. Hence, $U_{ip} \cdot S_{pq} = \frac{1}{2}K_{iq}$, $K_{ip} \cdot \widetilde{S}_{pq} = \frac{1}{2}U_{iq}$. □

Lemma 2. *Let $\varphi : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ be the canonical homomorphism. For $i = 1, \cdots, n$ and $p = 1, \cdots, m$ let U_{ip} and $K_{ip} \in \mathcal{A}_1$ be preimages of $\bar{U}_{ip}$ and $\bar{K}_{ip}$ respectively, then*

$$U_{ip} \cdot U_{iq} = \frac{1}{2}\widetilde{S}_{pq}, \quad K_{jp} \cdot K_{jq} = \frac{1}{2}S_{qp}, \tag{16}$$

$$U_{ip}\cdot K_{iq}=\frac{1}{2}V_{qp},\quad U_{ip}\cdot K_{jp}=-\frac{1}{2}H_{ij},\quad U_{ip}\cdot K_{ip}=\frac{1}{2}V_{pp}-H_{ii}. \quad (17)$$

Now we prove (16). Let $\eta_{ipq}^{u,\widetilde{z}}$ and $\eta_{ipq}^{u,z}\in\mathbb{F}$, such that $U_{ip}\cdot U_{iq}=\frac{1}{2}\widetilde{S}_{pq}+\eta_{ipq}^{u,\widetilde{z}}\,\widetilde{z}_{pq}+\eta_{ipq}^{u,z}\,z_{pq}$. We note that $\varphi(U_{ip}\cdot U_{iq})=\varphi(U_{jp}\cdot U_{jq})$ then $\eta_{ipq}^{u,\widetilde{z}}$ only depend of p and q therefore $\eta_{jpq}^{u,\widetilde{z}}=\eta_{pq}^{u,\widetilde{z}}$ for all $j=1,\dots,n$. Now we can write $U_{ip}\cdot U_{iq}=\frac{1}{2}\widetilde{S}_{pq}+\eta_{pq}^{u,\widetilde{z}}\,\widetilde{z}_{pq}+\eta_{pq}^{u,z}\,z_{pq}$.

Substituting x, y, z and t in the equation (2) respectively by U_{ip}, U_{iq}, $\widetilde{S}_{qp}$ and V_{pq}, and using (4) we obtain $U_{iq}\cdot V_{pq}=U_{iq}\cdot\widetilde{S}_{qp}=V_{pq}\cdot\widetilde{S}_{rs}=0$, so because of (7) we obtain $((U_{ip}\cdot U_{iq})\cdot\widetilde{S}_{qp})\cdot V_{pq}=0$. Hence $\eta_{pq}^{u,z}\,w_{pq}=0$ and therefore $\eta_{pq}^{u,z}=0$. So, we have

$$U_{ip}\cdot U_{iq}=\frac{1}{2}\widetilde{S}_{pq}+\eta_{pq}^{u,\widetilde{z}}\,\widetilde{z}_{pq}. \quad (18)$$

By a similar process we can prove that

$$K_{ip}\cdot K_{iq}=\frac{1}{2}S_{qp}+\eta_{pq}^{k,z}\,z_{qp}. \quad (19)$$

Now we can consider the product $U_{ip}\cdot K_{jp}$. Let $\eta_{ijp}^{uk,g}\in\mathbb{F}$ such that $U_{ip}\cdot K_{jp}=-\frac{1}{2}H_{ij}+\eta_{ijp}^{uk,g}g_{ij}$. Knowing that $\varphi(U_{ip}\cdot K_{jp})=\varphi(U_{jp}\cdot K_{ip})=\varphi(U_{iq}\cdot K_{jq})$ we can affirm that $\eta_{ijp}^{uk,g}=\eta_{jip}^{uk,g}=\eta_{ijq}^{uk,g}$. Similarly, we have $\eta_{ipq}^{uk}=\eta_{jpq}^{uk}$. So we can write

$$U_{ip}\cdot K_{jp}=-\frac{1}{2}H_{ij}+\eta_{g}^{uk}g_{ij}\quad\text{and}\quad U_{ip}\cdot K_{iq}=\frac{1}{2}V_{qp}+\eta_{pq}^{uk}w_{qp}. \quad (20)$$

If we replace x, y, z, and t, respectively, by U_{ip}, U_{iq}, S_{qp} and V_{pq} in the equation (2), and using (4) and (7) we obtain

$$((U_{ip}\cdot U_{iq})\cdot S_{qp})\cdot V_{pq}+\frac{1}{2}(U_{iq}\cdot S_{qp})\cdot U_{iq}=\frac{1}{2}U_{iq}\cdot(U_{iq}\cdot S_{qp})$$

Replacing (15) and (18) in the equality above we obtain $\eta_{pq}^{u}-\eta_{qp}^{uk}=0$. Similarly we can show that $0=\eta_{pq}^{k}-\eta_{pq}^{uk}$.

$$\eta_{pq}^{u}=\eta_{qp}^{uk}\quad\text{and}\quad\eta_{pq}^{k}=\eta_{pq}^{uk}. \quad (21)$$

Using (21) we obtain $\eta_{pq}^{u}+\eta_{pq}^{k}=\eta_{qp}^{uk}+\eta_{pq}^{uk}$. We note that $\eta_{pq}^{u}=-\eta_{qp}^{u}$ and $\eta_{pq}^{k}=-\eta_{qp}^{k}$. Thus $\eta_{qp}^{u}+\eta_{qp}^{k}=\eta_{pq}^{uk}+\eta_{qp}^{uk}=\eta_{pq}^{u}+\eta_{pq}^{k}$, hence $\eta_{pq}^{u}+\eta_{pq}^{k}=0$ and therefore $\eta_{pq}^{uk}+\eta_{qp}^{uk}=0$.

Finally, we show the equality (17). Let $x = U_{ip}$, $y = K_{ip}$ and $z = t = H_{ij}$ in the equality (2). As we did before, we have

$$((U_{ip} \cdot K_{ip}) \cdot H_{ij}) \cdot H_{ij} + \frac{1}{2} U_{ip} \cdot K_{ip} = (U_{ip} \cdot K_{ip}) \cdot {H_{ij}}^2 + \frac{1}{2} U_{jp} \cdot K_{jp}.$$

Thus $(\eta_{jp}^{uk,w} - \eta_{ip}^{uk,w}) w_{pp} + (\eta_{jp}^{uk,g} - \eta_{ip}^{uk,g}) g_{jj} = 0$. Since w_{pp} and g_{ii} are linearly independent we have $\eta_{jp}^{uk,w} = \eta_{ip}^{uk,w} = \eta_p^{uk,w}$ and $\eta_{jp}^{uk,g} = \eta_{ip}^{uk,g} = \eta_p^{uk,g}$.

If we take $x = U_{ip}$, $y = K_{ip}$, $z = V_{pq}$ and $t = V_{qp}$ in (2) we can show that $\eta_p^{uk,w} = \eta_q^{uk,w} = \eta_w^{uk}$ and $\eta_p^{uk,g} = \eta_q^{uk,g} = \eta_g^{uk}$ and therefore, we obtain

$$U_{ip} \cdot K_{ip} = \frac{1}{2} V_{pp} - H_{ii} + \eta_w^{uk} w_{pp} + \eta_g^{uk} g_{ii}. \tag{22}$$

Let $x = U_{ip}$, $y = U_{iq}$, $z = K_{ip}$ and $t = K_{iq}$ in the equation (2). Thus we obtain

$$\begin{aligned} &((U_{ip} \cdot U_{iq}) \cdot K_{ip}) \cdot K_{iq} + U_{iq} \cdot ((U_{ip} \cdot K_{iq}) \cdot K_{ip}) + ((U_{iq} \cdot K_{iq}) \cdot K_{ip}) \cdot U_{ip} \\ &\quad = (U_{ip} \cdot U_{iq}) \cdot (K_{ip} \cdot K_{iq}) + (U_{iq} \cdot K_{ip}) \cdot (U_{ip} \cdot K_{iq}) \\ &\qquad + (U_{iq} \cdot K_{iq}) \cdot (K_{ip} \cdot U_{ip}). \end{aligned} \tag{23}$$

If we replace (18), (19) and (22) in (23) and using the fact $\eta_p^{us} = \eta_p^{k\widetilde{s}} = 0$ and $\eta_{pq}^{u} = \eta_{qp}^{uk} = -\eta_{pq}^{uk}$ and $\eta_{pq}^{k} = \eta_{pq}^{uk}$ we obtain

$$(2\eta_w^{uk} - 2\eta_{pq}^{u}) w_{qq} + (2\eta_w^{uk} - \eta_g^{uk}) w_{pp} - 2\eta_g^{uk} g_{ii} = 0.$$

Due to the fact that w_{pp}, w_{qq} and g_{ii} are linearly independent, $\eta_{pq}^{u} = \eta_w^{uk} = \eta_g^{uk} = 0$. Hence $\eta_{pq}^{uk} = \eta_{pq}^{k} = 0$. □

We note that the relations (7)–(17) are valid when $n = 1$ and $m \geqslant 2$ or $n \geqslant 2$ and $m = 1$ and therefore, the WPT is valid.

Case 2. $\mathcal{N} \cong \mathcal{S}\mathrm{kew}(\mathcal{M}_{n|2m}(\mathbb{F}), \mathrm{osp})$. We denote

$$a_{ij} = e_{ij}^{n} - e_{ji}^{n}, \quad \widetilde{a}_{pq} = e_{pq}^{m} - e_{qp}^{2m}, \quad f_{pq} = e_{pq}^{m2m} + e_{qp}^{m2m},$$
$$\widetilde{f}_{pq} = e_{pq}^{2mm} + e_{qp}^{2mm}, \quad b_{ip} = e_{ip}^{nm} - e_{pi}^{2mn}, \quad c_{ip} = e_{ip}^{n2m} + e_{pi}^{mn}.$$

Since (3) we can see that $\mathcal{N}$ is generated by a_{ij}, $\widetilde{a}_{pq}$, f_{pq}, $\widetilde{f}_{pq}$, b_{ip} and c_{ip}, moreover,

$$\mathcal{N}_0 = \mathbf{span}\, \langle a_{ij}, \widetilde{a}_{pq}, f_{pq}, \widetilde{f}_{pq} \rangle \quad \text{and} \quad \mathcal{N}_1 = \mathbf{span}\, \langle b_{ip},\, c_{ip} \rangle$$

It is easy to see that the action of $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$ over $\mathcal{S}\mathrm{kew}(\mathcal{M}_{n|2m}(\mathbb{F}),$ osp) is determined by the following multiplication table

$$
\begin{gathered}
h_{ii} \circ a_{ii} = \frac{1}{2} a_{ii}, \\
h_{kl} \circ a_{ij} = \frac{1}{2}(\delta_{jk} a_{il} + \delta_{li} a_{kj} + \delta_{jl} a_{ik} + \delta_{ik} a_{lj}) \quad \text{if } i \neq j, k \neq l, \\
\widetilde{s}_{pq} \circ f_{rt} = \frac{1}{2}(\delta_{pr} \widetilde{a}_{tq} + \delta_{ps} \widetilde{a}_{rq} - \delta_{qr} \widetilde{a}_{tp} - \delta_{qt} \widetilde{a}_{pr}), \\
\widetilde{s}_{pq} \circ \widetilde{a}_{rt} = \frac{1}{2}(\delta_{qr} \widetilde{f}_{pt} - \delta_{pr} \widetilde{f}_{qt}), \\
s_{pq} \circ \widetilde{f}_{rt} = \frac{1}{2}(\delta_{qr} \widetilde{a}_{pt} + \delta_{qt} \widetilde{a}_{pr} - \delta_{pr} \widetilde{a}_{qt} - \delta_{pt} \widetilde{a}_{qr}), \\
s_{pq} \circ \widetilde{a}_{rt} = \frac{1}{2}(\delta_{pt} f_{qr} - \delta_{qt} f_{pr}), \\
v_{pq} \circ \widetilde{a}_{rt} = \frac{1}{2}(\delta_{qr} \widetilde{a}_{pt} + \delta_{pt} \widetilde{a}_{rq}), \quad v_{pq} \circ f_{rt} = \frac{1}{2}(\delta_{qr} f_{pt} + \delta_{tq} f_{pr}), \\
v_{pq} \circ \widetilde{f}_{rt} = \frac{1}{2}(\delta_{pr} \widetilde{f}_{qt} + \delta_{pt} \widetilde{f}_{qr}),
\end{gathered}
\tag{24}
$$

$$
\begin{gathered}
h_{ij} \circ b_{kr} = \frac{1}{2}(\delta_{jk} b_{ir} + \delta_{ik} b_{jr}), \quad h_{ii} \circ b_{kr} = \frac{1}{2} \delta_{ik} b_{ir}, \\
v_{pq} \circ b_{kr} = \frac{1}{2} \delta_{rp} b_{kq}, \quad h_{ij} \circ c_{kr} = \frac{1}{2}(\delta_{jk} c_{ir} + \delta_{ik} k_{jr}), \\
h_{ii} \circ c_{kr} = \frac{1}{2} \delta_{ik} c_{ir}, \quad v_{pq} \circ c_{kr} = \frac{1}{2} \delta_{rq} c_{kp}, \\
s_{pq} \circ b_{ir} = \frac{1}{2}(\delta_{pr} c_{iq} - \delta_{qr} c_{ip}), \quad \widetilde{s}_{pq} \circ c_{ir} = \frac{1}{2}(\delta_{rp} b_{iq} - \delta_{qr} b_{ip}) \\
u_{ip} \circ a_{kj} = \frac{1}{2}(\delta_{ij} b_{kp} - \delta_{ik} b_{jp}), \quad u_{ip} \circ \widetilde{a}_{qr} = \frac{1}{2} \delta_{pq} b_{ir}, \\
k_{ip} \circ a_{jk} = \frac{1}{2}(\delta_{ik} c_{jp} - \delta_{ij} c_{kp}), \quad k_{ip} \circ \widetilde{a}_{qr} = -\frac{1}{2} \delta_{pr} c_{iq}, \\
u_{ip} \circ f_{qr} = \frac{1}{2}(\delta_{pq} c_{ir} + \delta_{pr} c_{iq}), \quad k_{ip} \circ \widetilde{f}_{qr} = \frac{1}{2}(\delta_{pq} b_{ir} + \delta_{pr} b_{iq}), \\
u_{ir} \circ s_{pq} = \frac{1}{2}(\delta_{rp} k_{iq} - \delta_{rq} k_{ip}), \quad k_{ir} \circ \widetilde{s}_{pq} = \frac{1}{2}(\delta_{rp} u_{iq} - \delta_{rq} u_{ip}),
\end{gathered}
\tag{25}
$$

$$
\begin{gathered}
u_{ip} \circ b_{jq} = \frac{1}{2} \delta_{ij} \widetilde{f}_{pq}, \quad u_{ip} \circ c_{jq} = \frac{1}{2}(\delta_{pq} a_{ij} - \delta_{ij} \widetilde{a}_{qp}) \\
k_{ip} \circ c_{jq} = -\frac{1}{2} \delta_{ij} f_{pq}, \quad k_{ip} \circ b_{jq} = \frac{1}{2}(\delta_{pq} a_{ji} - \delta_{ij} \widetilde{a}_{pq}).
\end{gathered}
\tag{26}
$$

The products in (24), (25) are symmetric and the products in (26) are skew-symmetric.

Similarly as was prove to case 1, we have that if $n+m \geqslant 3$ then there exist an analogous to Lemma 1 and 2. Therefore there exist U_{ip} and $K_{ip} \in \mathcal{A}_1$ such that $\mathrm{vect}\langle \mathrm{U_{ip}, K_{ip}, i = 1, \ldots, n,\ p = 1, \ldots, m} \rangle \cong (\mathfrak{J}\mathrm{osp}_{\mathrm{n|2m}}(\mathbb{F}))_1$, thus the WPT is valid.

Now we prove the theorem for $m=n=1$ and $\mathcal{N} \cong \mathcal{S}\mathrm{kew}(\mathcal{M}_{1|2}(\mathbb{F}), \mathrm{osp})$. Let h, v, $\bar{u}$ and $\bar{k}$ such that $(\mathbb{F}\cdot h + \mathbb{F}\cdot v) \dotplus (\mathbb{F}\cdot \bar{u} + \mathbb{F}\cdot \bar{k}) \cong \mathfrak{J}\mathrm{osp}_{1|2}(\mathbb{F})$. We need to find $\widetilde{u}$, $\widetilde{k} \in \mathcal{A}_1$ such that $\varphi(\widetilde{u}) = \bar{u}$ and $\varphi(\widetilde{k}) = \bar{k}$. Moroever, $\widetilde{u}h = \widetilde{u}v = \frac{1}{2}u$, $\widetilde{k}h = \widetilde{k}v = \frac{1}{2}k$, and $\widetilde{u}\widetilde{k} = \frac{1}{2}v - h$.

Let $\mathcal{N} = (\mathbb{F}\cdot\widetilde{a} + \mathbb{F}\cdot f + \mathbb{F}\cdot\widetilde{f}) \dotplus (\mathbb{F}\cdot b + \mathbb{F}\cdot c)$ and consider the following action of $\mathfrak{J}\mathrm{osp}_{1|2}(\mathbb{F})$ over $\mathcal{N}$

$$\begin{gathered} v\widetilde{a} = \widetilde{a}, \quad vf = f, \quad v\widetilde{f} = \widetilde{f}, \quad bh = bv = \frac{1}{2}b, \quad ch = cv = \frac{1}{2}c, \\ \widetilde{u}\widetilde{a} = \widetilde{k}\widetilde{f} = \frac{1}{2}b, \quad \widetilde{k}\widetilde{a} = \widetilde{u}f = \frac{1}{2}c \end{gathered} \tag{27}$$

$$\widetilde{u}b = \widetilde{f}, \quad \widetilde{u}c = -\frac{1}{2}\widetilde{a} = \widetilde{k}b, \quad \widetilde{k}c = -f, \tag{28}$$

where (27) and (28) are commutative and anticommutative, respectively. It is easy to see that $\mathcal{N} \cong \mathcal{S}\mathrm{kew}(\mathcal{M}_{1|2}(\mathbb{F}), \mathrm{osp})$.

Let $\xi_{\widetilde{a}}$, ξ_f and $\xi_{\widetilde{f}} \in \mathbb{F}$ such that $uk = \frac{1}{2}v - h + \xi_{\widetilde{a}}\widetilde{a} + \xi_f f + \xi_{\widetilde{f}}\widetilde{f}$. Let's prove that there exist α_b, α_c, β_b and $\beta_c \in \mathbb{F}$ such that

$$\widetilde{u} = u + \alpha_b b + \alpha_c c, \quad \widetilde{k} = k + \beta_b b + \beta_c c \quad \text{and} \quad \widetilde{u}\cdot\widetilde{k} = \frac{1}{2}v - h.$$

We note that $\varphi(\widetilde{u}) = \bar{u}$ and $\varphi(\widetilde{k}) = \bar{k}$. Using (27) we have

$$\widetilde{u}h = \widetilde{u}v = \frac{1}{2}\widetilde{u} \quad \text{and} \quad \widetilde{k}h = \widetilde{k}v = \frac{1}{2}\widetilde{k}.$$

Now $\widetilde{u}\widetilde{k} = \frac{1}{2}v - h$ if and only if

$$(\xi_{\widetilde{a}} + \frac{1}{2}\alpha_b - \frac{1}{2}\beta_c)\widetilde{a} + (\xi_f + \alpha_c)f + (\xi_{\widetilde{f}} + \beta_b)\widetilde{f} = 0.$$

Since $\widetilde{a}$, f, and $\widetilde{f}$ are linearly independent we have

$$\xi_{\widetilde{a}} + \frac{1}{2}\alpha_b - \frac{1}{2}\beta_c = \xi_f + \alpha_c = \xi_{\widetilde{f}} + \beta_b = 0,$$

and therefore $2\xi_{\widetilde{a}} + \alpha_b = \beta_c$, $\xi_f = -\alpha_c$, $\xi_{\widetilde{f}} = -\beta_b$ is a solution, hence the WPT is valid.

We note that if $\mathcal{N}$ is isomorphic to anyone of superbimodules opposites, then by the Pierce properties, we have that the radical part in any product is zero and therefore the equalities (7), (8), (16) and (17) hold when we change H_{ij}, V_{pq}, S_{pq}, $\widetilde{S}_{pq}$, U_{ij} and K_{ip} respectively by h_{ij}, v_{pq}, s_{pq}, $\widetilde{s}_{pq}$, u_{ij} and k_{ip}, and therefore, WPT is true.

3. A counter-example

Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ where $\mathcal{A}_0 = \mathbb{F}\cdot h + \mathbb{F}\cdot v + \mathbb{F}\cdot g + \mathbb{F}\cdot w$ and $\mathcal{A}_1 = \mathbb{F}\cdot u + \mathbb{F}\cdot k + \mathbb{F}\cdot y + \mathbb{F}\cdot x$, and $\mathcal{N} = (\mathbb{F}\cdot g + \mathbb{F}\cdot w) \dot{+} (\mathbb{F}\cdot y + \mathbb{F}\cdot x)$. The non-zero multiplications in $\mathcal{A}$ are given by

$$\begin{gathered} h^2 = h, \quad v^2 = v, \quad hg = g, \quad vw = w, \quad uh = uv = \frac{1}{2}u, \\ kh = kv = \frac{1}{2}k, \quad yh = yv = \frac{1}{2}y, \quad ug = uw = \frac{1}{2}y, \\ xh = xv = \frac{1}{2}x, \quad kg = kw = \frac{1}{2}x, \end{gathered} \tag{29}$$

$$ux = \frac{1}{2}w - g, \quad yk = \frac{1}{2}w - g, \quad uk = \frac{1}{2}v - h + g, \tag{30}$$

where the products (29) are symmetric and (30) are skew-symmetric.

Using (2), and the table of multiplications above it is easy to show that $\mathcal{A}$ is a Jordan superalgebra. Moreover, if $\mathfrak{Josp}_{1|2}(\mathbb{F})_0 = \mathbb{F}\cdot h_{11} + \mathbb{F}\cdot v_{11}$ while $\mathfrak{Josp}_{1|2}(\mathbb{F})_1 = \mathbb{F}\cdot u_{11} + \mathbb{F}\cdot k_{11}$. Consider the mapping $\varphi : \mathcal{A}/\mathcal{N} \to \mathfrak{Josp}_{1|2}(\mathbb{F})$ and $\psi : \mathcal{N} \to \mathfrak{Josp}_{1|2}(\mathbb{F})$ given by $\varphi(h) = \psi(g) = h_{11}$, $\varphi(v) = \psi(w) = v_{11}$, $\varphi(u) = \psi(y) = u_{11}$ and $\varphi(k) = \psi(x) = k_{11}$.

We can see that φ is an isomorphism between $\mathcal{A}/\mathcal{N}$ and $\mathfrak{Josp}_{1|2}(\mathbb{F})$, while ψ is an isomorphism between $\mathcal{N}$ and $\mathcal{R}\mathrm{eg}\,(\mathfrak{Josp}(1|2))$.

If we assume that the WPT is valid for $\mathcal{A}$, then there exist h, $v \in \mathcal{A}_0$ and $\widetilde{u}$, $\widetilde{k} \in \mathcal{A}_1$ such that, the following products are commutative $h^2 = h$, $v^2 = v$, $h\widetilde{u} = v\widetilde{u} = \frac{1}{2}\widetilde{u}$, $h\widetilde{k} = v\widetilde{k} = \frac{1}{2}\widetilde{k}$, and anticommutative product $\widetilde{u}\widetilde{k} = \frac{1}{2}\widetilde{v} - \widetilde{h}$ hold, and $\widetilde{u} \equiv u (\mathrm{mod}\ \mathcal{N})$ and $\widetilde{k} \equiv k (\mathrm{mod}\ \mathcal{N})$.

Consider α_x, α_y, β_x and $\beta_y \in \mathbb{F}$ such that $\widetilde{u} = u + \alpha_y y + \alpha_x x$ and $\widetilde{k} = k + \beta_y y + \beta_x x$. We note that

$$\begin{aligned} \widetilde{u}\widetilde{k} &= (u + \alpha_y y + \alpha_x x)(k + \beta_x x + \beta_y y) = uk + \alpha_y yk + \beta_x ux \\ &= \frac{1}{2}v - h + g + \frac{1}{2}\alpha_y w - \alpha_y g + \frac{1}{2}\beta_x w - \beta_x g \\ &= \frac{1}{2}v - h + (1 - \alpha_y - \beta_x)g + \frac{1}{2}(\alpha_y + \beta_x)w. \end{aligned}$$

So $\widetilde{u}\widetilde{k} = \frac{1}{2}v - h$ if and only if $2(1-\alpha_y-\beta_x)g + (\alpha_y+\beta_x)w = 0$. Due to g and w are linearly independent, we have $1 = \alpha_y + \beta_x = 0$ and so we have a contradiction.

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