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Module decompositions via Rickart modules

Abdullah Harmanci and Burcu Ungor

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Abstract. This work is devoted to the investigation of module decompositions which arise from Rickart modules, socle and radical of modules. In this regard, the structure and several illustrative examples of inverse split modules relative to the socle and radical are given. It is shown that a module M has decompositions $M = \mathrm{Soc}(M) \oplus N$ and $M = \mathrm{Rad}(M) \oplus K$ where N and K are Rickart if and only if M is $\mathrm{Soc}(M)$-inverse split and $\mathrm{Rad}(M)$-inverse split, respectively. Right $\mathrm{Soc}(\cdot)$-inverse split left perfect rings and semiprimitive right hereditary rings are determined exactly. Also, some characterizations for a ring R which has a decomposition $R = \mathrm{Soc}(R_R) \oplus I$ with I a hereditary Rickart module are obtained.

Introduction

Throughout this paper R denotes a ring with identity, modules are unital right R-modules, and S stands for the endomorphism ring $\mathrm{End}_R(M)$ of a module M which is under consideration. Right (left) Rickart rings (or principally projective rings) initially appeared in Maeda [11], and were further studied by Hattori [7], that is, a ring is called *right* (*left*) *Rickart* if every principal right (left) ideal is projective, equivalently, the right (left) annihilator of any single element is generated by an idempotent as a right (left) ideal. The concept of Rickart rings has been comprehensively studied in the literature and was also extended by Rizvi and Roman to

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the general module theoretic setting in [13], namely, a module M is said to be *Rickart* if for any $f \in S$, $\mathrm{Ker} f = eM$ for some $e^2 = e \in S$. It is obvious that a ring R is Rickart as an R-module if and only if it is a right Rickart ring. The notion of Rickart modules is further studied in [1] and [9]. Recently, in [14], the present authors consider conditions under which fully invariant submodules give rise to Rickart modules. In this direction, an F-inverse split module M is defined, and its properties are investigated where F is a fully invariant submodule of M. Recall that a submodule F of a module M is called *fully invariant* if $fF \subseteq F$ for every $f \in S$. For a module M, there are abundant fully invariant submodules of M. In [14], for any fully invariant submodule F of a module M, the module M is called *F-inverse split* if $f^{-1}(F)$ is a direct summand of M for every $f \in S$, and a basic decomposition of M is obtained, namely, M is F-inverse split if and only if $M = F \oplus N$ where N is a Rickart module.

Socles and radicals are important tools in studying the structure of modules and rings. For a module M, the socle $\mathrm{Soc}(M)$ and the radical $\mathrm{Rad}(M)$ are fully invariant submodules of M. Motivated by the works on Rickart modules and inverse split modules, it is of interest to investigate decompositions of inverse split modules which arise out of Rickart modules, socle and radical. In this direction, we perform some applications for the concept of an F-inverse split module M by specialising the fully invariant submodule F as the socle and the radical of M. Some relations between Rickart modules and $\mathrm{Soc}(\cdot)$-inverse split modules, $\mathrm{Rad}(\cdot)$-inverse split modules are obtained. It is proved that a finitely generated $\mathbb{Z}$-module is $\mathrm{Soc}(\cdot)$-inverse split if and only if its socle and torsion submodule are the same. A Loewy and $\mathrm{Soc}(\cdot)$-inverse split module is determined. It is also shown that every module which has a $\mathrm{Rad}(\cdot)$-inverse split projective cover inherits the same property. We present some applications for the $\mathrm{Soc}(\cdot)$-inverse split property to rings. Some relations between right $\mathrm{Soc}(\cdot)$-inverse split rings and certain classes of rings such as left perfect rings, right Kasch rings, semisimple rings, right nonsingular rings are provided. It is shown that the maximal right ring of quotients of a right $\mathrm{Soc}(\cdot)$-inverse split ring is right Rickart. We characterize a semiprimitive right hereditary ring in terms of $\mathrm{Rad}(\cdot)$-inverse split modules, and also a ring R which has a decomposition $R = \mathrm{Soc}(R_R) \oplus I$ with I a hereditary module in terms of $\mathrm{Soc}(\cdot)$-inverse split modules.

In what follows, by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{Z}_n$ we denote, respectively, the ring of integers, the ring of rational numbers and the ring of integers modulo n. For a module M, $E(M)$, $\mathrm{Soc}(M)$, $\mathrm{Rad}(M)$ and $Z(M)$ stand for the injective hull, socle, radical and singular submodule, respectively, and $J(R)$

denotes the Jacobson radical of a ring R. For the facts about the notions of socle, radical and inverse split modules which are pertinent to this paper see [2], [5], [14] and [15].

1. Soc$(\cdot)$-inverse split modules

For a module M, $\mathrm{Soc}(M)$ is the intersection of all essential submodules of M, equivalently, the sum of all simple submodules of M. If a ring R is semisimple, then every R-module M is semisimple, therefore $\mathrm{Soc}(M)$ is a direct summand of M. There are also some classes of rings over which every R-module M has an essential socle or $\mathrm{Soc}(M)$ need not be a direct summand of M. In this section, we deal with module decompositions caused by the socles of modules and Rickart modules.

Definition 1.1. A module M is called *Soc*(M)*-inverse split* if for every $f \in S$, $f^{-1}(\mathrm{Soc}(M))$ is a direct summand of M.

Obviously, every semisimple module M is $\mathrm{Soc}(M)$-inverse split. For a module M with zero socle, since $\mathrm{Ker} f = f^{-1}(0)$, M is Rickart if and only if it is $\mathrm{Soc}(M)$-inverse split. We now provide an effective characterization of $\mathrm{Soc}(\cdot)$-inverse split modules in terms of Rickart modules.

Theorem 1.2. The following are equivalent for a module M.
(1) M is $\mathrm{Soc}(M)$-inverse split.
(2) $M = \mathrm{Soc}(M) \oplus N$ where N is a Rickart module.
(3) M has a decomposition $M = K \oplus N$ where K is semisimple and N is Rickart with zero socle.

Proof. (1) $\Leftrightarrow$ (2) It is a direct consequence of [14, Theorem 2.3].
(2) $\Rightarrow$ (3) Clear by the fact that $\mathrm{Soc}(\mathrm{Soc}(M)) = \mathrm{Soc}(M)$.
(3) $\Rightarrow$ (2) Let $M = K \oplus N$ with K semisimple, N Rickart and $\mathrm{Soc}(N) = 0$. Then $\mathrm{Soc}(M) = \mathrm{Soc}(K) \oplus \mathrm{Soc}(N) = K$. This completes the proof. $\square$

The next examples show that there is no implication between Rickart and $\mathrm{Soc}(\cdot)$-inverse split properties of modules.

Examples 1.3. (1) Let M denote the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}$. Then $\mathrm{Soc}(M) = \mathbb{Z}_2$ and so $M = \mathrm{Soc}(M) \oplus \mathbb{Z}$. Since $\mathbb{Z}$ is Rickart, by Theorem 1.2, M is $\mathrm{Soc}(M)$-inverse split. On the other hand, M is not Rickart by [9, Example 2.5].

(2) Consider the ring $R = \prod_{i=1}^{\infty} R_i$ where $R_i = \mathbb{Z}_2$ for all i and let $M = R_R$. Then $\mathrm{Soc}(M) = \bigoplus_{i=1}^{\infty} R_i$ where $R_i = \mathbb{Z}_2$ for all i and it is an essential submodule of M. Thus $\mathrm{Soc}(M)$ is not a direct summand of M. Therefore M is not $\mathrm{Soc}(M)$-inverse split. On the other hand, $S \cong R$ and R is von Neumann regular. Hence M is Rickart by [9, Theorem 3.17].

We now investigate some conditions under which Rickart modules and $\mathrm{Soc}(\cdot)$-inverse split modules imply each other.

Proposition 1.4. Every indecomposable $\mathrm{Soc}(M)$-inverse split module M is Rickart, therefore M is Baer.

Proof. Let M be an indecomposable $\mathrm{Soc}(M)$-inverse split module and 1 denote the identity endomorphism of M. Since $1^{-1}(\mathrm{Soc}M) = \mathrm{Soc}(M)$, we have $\mathrm{Soc}(M) = 0$ or $\mathrm{Soc}(M) = M$. Being $\mathrm{Soc}(M) = 0$ implies that M is Rickart since for any $f \in S$, $f^{-1}(\mathrm{Soc}(M)) = f^{-1}(0)$ is a direct summand. If $\mathrm{Soc}(M) = M$, then M is semisimple, and so is Rickart. The last assertion follows from [9, Corollary 4.6]. □

Let R be a ring. In the literature, R is called a *right V-ring* if every simple R-module is injective. By [8, Theorem 3.75], R is a right V-ring if and only if for every R-module M, $\mathrm{Rad}(M) = 0$.

Proposition 1.5. Let R be a right Noetherian and right V-ring. Then every Rickart module M is $\mathrm{Soc}(M)$-inverse split.

Proof. Let M be a Rickart module. Since every simple module is injective and R is right Noetherian, $\mathrm{Soc}(M)$ is injective. Hence $\mathrm{Soc}(M)$ is a direct summand of M. Let $M = \mathrm{Soc}(M) \oplus N$ for some submodule N of M. By [9, Theorem 2.7], N is Rickart. Therefore M is $\mathrm{Soc}(M)$-inverse split. □

Proposition 1.6. The following are equivalent for a module M.

(1) M is a $\mathrm{Soc}(M)$-inverse split module and $\mathrm{Ker}f$ is a direct summand of $f^{-1}(\mathrm{Soc}(M))$ for all $f \in S$.

(2) M is Rickart and $\mathrm{Soc}(M)$ is a direct summand of M.

Proof. It follows from [14, Theorem 2.9]. □

Homomorphic images of $\mathrm{Soc}(\cdot)$-inverse split modules need not be $\mathrm{Soc}(\cdot)$-inverse split as the following example shows.

Example 1.7. Consider the ring $\mathbb{Z}$ as a $\mathbb{Z}$-module. Note that $\mathrm{Soc}(\mathbb{Z}) = 0$ and $\mathbb{Z}$ is Rickart. Thus $\mathbb{Z}$ is $\mathrm{Soc}(\mathbb{Z})$-inverse split. On the other hand, $\mathrm{Soc}(\mathbb{Z}/12\mathbb{Z}) = 2\mathbb{Z}/12\mathbb{Z}$ is an essential submodule of $\mathbb{Z}/12\mathbb{Z}$, and so it is not a direct summand of $\mathbb{Z}/12\mathbb{Z}$. Therefore $\mathbb{Z}/12\mathbb{Z}$ is not $\mathrm{Soc}(\mathbb{Z}/12\mathbb{Z})$-inverse split.

However, the $\mathrm{Soc}(\cdot)$-inverse split property is inherited by direct summands as shown below.

Proposition 1.8. Let M be a $\mathrm{Soc}(M)$-inverse split module. Then every direct summand N of M is $\mathrm{Soc}(N)$-inverse split.

Proof. Let N be a direct summand of M. According to [14, Proposition 2.12], N is $(N\cap \mathrm{Soc}(M))$-inverse split. Then the assertion holds by [2, Corollary 9.9]. □

In the next, we characterize finitely generated $\mathrm{Soc}(\cdot)$-inverse split abelian groups in terms of their torsion submodules.

Theorem 1.9. A finitely generated $\mathbb{Z}$-module M is $\mathrm{Soc}(M)$-inverse split if and only if $\mathrm{Soc}(M)$ is the torsion submodule of M.

Proof. Let M be a finitely generated $\mathbb{Z}$-module. We can decompose $M = t(M) \oplus N$ where $t(M)$ is the torsion submodule and N is the torsion-free submodule of M. Then $\mathrm{Soc}(N) = 0$, and $\mathrm{Soc}(t(M))$ is essential in $t(M)$. Hence $\mathrm{Soc}(M) = \mathrm{Soc}(t(M))$. Assume that M is $\mathrm{Soc}(M)$-inverse split. By Proposition 1.8, $t(M)$ is also $\mathrm{Soc}(M)$-inverse split. It follows $\mathrm{Soc}(M) = t(M)$. Conversely, assume that $\mathrm{Soc}(M) = t(M)$. Thus $M = \mathrm{Soc}(M) \oplus N$ where N is a free $\mathbb{Z}$-module. By [9, Theorem 2.26], N is Rickart. Therefore M is $\mathrm{Soc}(M)$-inverse split due to Theorem 1.2. □

Let M be a module with socle $S = \mathrm{Soc}(M)$. The (*lower*) *Loewy series* for M is defined transfinitely by: $S_0 = 0$, $S_1 = \mathrm{Soc}(M)$, $S_{\alpha+1}/S_\alpha = \mathrm{Soc}(M/S_\alpha)$ and, if α is a limit ordinal, $S_\alpha = \bigcup_{\beta<\alpha} \mathrm{S}_\beta$ (see [3, page 470]). If $M = S_\alpha$ for some ordinal α, then M is called a (*lower*) *Loewy module. The Loewy length* of such a module is $\mathrm{L}(M) = \gamma$, the least ordinal γ with $M = S_\gamma$. A ring R is said to be *right Loewy* in case the regular representation R_R is a Loewy module. A right Loewy ring is also called as *right semi-Artinian.*

Proposition 1.10. If a module M is semisimple, then it is $\mathrm{Soc}(M)$-inverse split. The converse holds if any nonzero submodule of M contains a minimal submodule. In particular, if R is right Loewy.

Proof. The first assertion is obvious. For the converse, let M be a Soc(M)-inverse split module. According to [2, 9.10 Corollary], Soc(M) is essential in M. By Theorem 1.2, Soc(M) is a direct summand of M. Therefore M is semisimple. □

Corollary 1.11. A module M is Loewy and Soc(M)-inverse split if and only if it is semisimple.

Proof. If M is Loewy, then every nonzero submodule of M contains a minimal submodule. By Proposition 1.10, we are done. □

Proposition 1.12. Let M be a module. Then the following hold.

(1) M is Soc(M)-inverse split and has the ascending chain condition on essential submodules if and only if M has a decomposition $M = \mathrm{Soc}(M) \oplus N$ where N is a Noetherian Rickart module.

(2) M is Soc(M)-inverse split and has the descending chain condition on essential submodules if and only if M has a decomposition $M = \mathrm{Soc}(M) \oplus N$ where N is an Artinian Rickart module.

Proof. (1) Let M be a Soc(M)-inverse split module. By Theorem 1.2, M has a decomposition $M = \mathrm{Soc}(M) \oplus N$ where N is a Rickart module. Since M has the ascending chain condition on essential submodules, $M/\mathrm{Soc}(M)$ is Noetherian by [5, 5.15(1)], and so N is Noetherian. Conversely, assume that $M = \mathrm{Soc}(M) \oplus N$ for some Noetherian Rickart module N. This implies that $M/\mathrm{Soc}(M)$ is Noetherian. Hence Theorem 1.2 and [5, 5.15(1)] complete the proof.

(2) It follows from Theorem 1.2 and [5, 5.15(2)] as in the proof of (1). □

Recall that a module M is called *hereditary* if every submodule of M is projective (see [15]). We now observe a characterization of a ring R which has a decomposition $R = \mathrm{Soc}(R_R) \oplus I$ with I a hereditary module in terms of Soc($\cdot$)-inverse split property.

Theorem 1.13. The following are equivalent for a ring R.

(1) Every free R-module M is Soc(M)-inverse split.

(2) Every projective R-module M is Soc(M)-inverse split.

(3) $R = \mathrm{Soc}(R_R) \oplus I$ where I is a hereditary Rickart R-module.

(4) $R = \mathrm{Soc}(R_R) \oplus I$ where I is a hereditary R-module.

Proof. (1) $\Leftrightarrow$ (2) By Proposition 1.8.

(3) $\Rightarrow$ (4) Obvious.

(2) $\Rightarrow$ (3) By (2), R is $\mathrm{Soc}(R_R)$-inverse split. Then $R = \mathrm{Soc}(R_R) \oplus I$ where I is a Rickart R-module due to Theorem 1.2. Let J be a submodule of I. There exists a short exact sequence $0 \to K \to P \xrightarrow{f} J \to 0$ with P a projective R-module. We can assume $f \in \mathrm{End}_R(P)$. Clearly, $\mathrm{Ker}f \subseteq f^{-1}(\mathrm{Soc}(P))$. Let $x \in f^{-1}(\mathrm{Soc}(P))$. Hence $fx \in \mathrm{Soc}(P) \cap J = \mathrm{Soc}(J)$. Note that being $R = \mathrm{Soc}(R_R) \oplus I$ implies $\mathrm{Soc}(I) = 0$, and so $\mathrm{Soc}(J) = 0$. Thus $fx = 0$. It follows that $x \in \mathrm{Ker}f$, so $\mathrm{Ker}f = f^{-1}(\mathrm{Soc}(P))$. By (2), P being $\mathrm{Soc}(P)$-inverse split implies that $\mathrm{Ker}f$ is a direct summand of P. Since $P/\mathrm{Ker}f \cong J$, J is projective. Therefore I is hereditary.

(4) $\Rightarrow$ (1) Let $M = \bigoplus_{\mathcal{J}} R$ be a free R-module with an index set $\mathcal{J}$ and $f \in S$. By (4), $M = \mathrm{Soc}(M) \oplus (\bigoplus_{\mathcal{J}} I)$. Then $\bigoplus_{\mathcal{J}} I = eM$ for some $e = e^2 \in S$ and $\mathrm{End}_R(\bigoplus_{\mathcal{J}} I) = eSe$. Since $\mathrm{Soc}(M) \subseteq f^{-1}(\mathrm{Soc}(M))$, by modularity condition, $f^{-1}(\mathrm{Soc}(M)) = \mathrm{Soc}(M) \oplus \big(\bigoplus_{\mathcal{J}} I \cap f^{-1}(\mathrm{Soc}(M))\big)$. We claim that $\bigoplus_{\mathcal{J}} I \cap f^{-1}(\mathrm{Soc}(M)) = \mathrm{Ker}(efe)$. Let $x \in \bigoplus_{\mathcal{J}} I \cap f^{-1}(\mathrm{Soc}(M))$. Since $eM = \bigoplus_{\mathcal{J}} I$, $efe(x) = ef(x) \in \bigoplus_{\mathcal{J}} I \cap \mathrm{Soc}(M) = 0$, and so $x \in \mathrm{Ker}(efe)$. Now let $y \in \mathrm{Ker}(efe)$. So $y \in \bigoplus_{\mathcal{J}} I$ and $efe(y) = ef(y) = 0$. Also, $f(y) = a + b$ for some $a \in \mathrm{Soc}(M)$ and $b \in \bigoplus_{\mathcal{J}} I$. Then $0 = ef(y) = ea + eb = ea + b$ implies $b \in \bigoplus_{\mathcal{J}} I \cap \mathrm{Soc}(M) = 0$. Hence $f(y) = a \in \mathrm{Soc}(M)$, and so $y \in \bigoplus_{\mathcal{J}} I \cap f^{-1}(\mathrm{Soc}(M))$. Thus $\bigoplus_{\mathcal{J}} I \cap f^{-1}(\mathrm{Soc}(M)) = \mathrm{Ker}(efe)$. On the other hand, by (4) and [15, 39.7], $\bigoplus_{\mathcal{J}} I$ is hereditary. It follows that $(\bigoplus_{\mathcal{J}} I)/\mathrm{Ker}(efe)$ is projective, so $\mathrm{Ker}(efe)$ is a direct summand of $\bigoplus_{\mathcal{J}} I$. Thus $f^{-1}(\mathrm{Soc}(M))$ is a direct summand of M. Therefore M is $\mathrm{Soc}(M)$-inverse split. $\square$

We conclude this section by performing an application for the notion of $\mathrm{Soc}(\cdot)$-inverse split modules by using semiperfectness relative to the socle. Recall that a module M is called *$Soc(M)$-semiperfect* if for any submodule N of M, there exists a projective direct summand A of M such that $A \subseteq N$ and $M = A \oplus B$ with $N \cap B \subseteq \mathrm{Soc}(M)$. Note that $\mathrm{Soc}(M)$ is an intersection of some submodules N of M with M/N singular. However, $M/\mathrm{Soc}(M)$ need be neither singular nor semisimple.

Proposition 1.14. Let M be a $\mathrm{Soc}(M)$-semiperfect module. Then the following hold.

(1) $M/\mathrm{Soc}(M)$ is semisimple.
(2) M is $\mathrm{Soc}(M)$-inverse split if and only if it is semisimple.
(3) Any submodule N of M with $N\cap \mathrm{Soc}(M) = 0$ is a Rickart module.
(4) If $\mathrm{Soc}(M)$ is a direct summand of M, then M is $\mathrm{Soc}(M)$-inverse split.

Proof. (1) Let $N/\mathrm{Soc}(M)$ be any submodule of $M/\mathrm{Soc}(M)$. By hypothesis, there exists a submodule U of M contained in N such that $M = U \oplus V$ and $V \cap N \subseteq \mathrm{Soc}(M)$. It is obvious that $[N/\mathrm{Soc}(M)] \oplus [\big(V+ \mathrm{Soc}(M)\big)/\mathrm{Soc}(M)] = M/\mathrm{Soc}(M)$. Therefore $M/\mathrm{Soc}(M)$ is semisimple.

(2) Assume that M is $\mathrm{Soc}(M)$-inverse split. There exists a submodule N of M such that $M = \mathrm{Soc}(M) \oplus N$. Let A be a submodule of N. By hypothesis there exists a projective submodule B such that $B \subseteq A$, $M = B \oplus C$ and $A \cap C \subseteq \mathrm{Soc}(M)$. Being $A \cap C \subseteq \mathrm{Soc}(M) \cap N$ implies $A \cap C = 0$ and $A = B$. Hence $M = A \oplus C$. Thus A is a direct summand of N and so N is semisimple. It follows that $N = 0$, therefore M is semisimple. The converse is clear.

(3) Let N be a submodule of M such that $N\cap \mathrm{Soc}(M) = 0$. By (1), $M/\mathrm{Soc}(M)$ is semisimple. Then N is semisimple as it is isomorphic to a submodule of $M/\mathrm{Soc}(M)$.

(4) It follows from (3). □

2. Applications to rings

This section is devoted to applications for the $\mathrm{Soc}(\cdot)$-inverse splitness to rings. Note that I is an ideal of a ring R if and only if it is a fully invariant submodule of R_R (resp. ${}_RR$). For an ideal I of R, in [14], R is called *right (left) I-inverse split* if it is I-inverse split as a right (left) R-module. In the light of I-inverse splitness where I is an ideal of a ring, we give the following definition.

Definition 2.1. A ring R is called *right (left) $\mathrm{Soc}(R_R)$-inverse split* if for every $f \in \mathrm{End}(R_R)$, $f^{-1}(\mathrm{Soc}(R_R))$ is a direct summand of R as a right (left) ideal.

Right (left) $\mathrm{Soc}({}_RR)$-inverse splitness is defined similarly. For a ring R, $\mathrm{Soc}(R_R)$-inverse split property is not left-right symmetric as shown below.

Example 2.2. Let R denote the ring $\begin{bmatrix} k\,k\,k \\ 0\,k\,0 \\ 0\,0\,k \end{bmatrix}$ where k is a semisimple ring. Then $\mathrm{Soc}(R_R) = \begin{bmatrix} 0\,k\,k \\ 0\,k\,0 \\ 0\,0\,k \end{bmatrix}$ (see [8, Example 7.14c]). Since $\mathrm{Soc}(R_R)$ is an essential right ideal of R, it is not a direct summand of R as a right ideal. Thus R is not right $\mathrm{Soc}(R_R)$-inverse split. On the other hand, $R = \mathrm{Soc}(R_R) \oplus I$ where I is the left ideal $\begin{bmatrix} k\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{bmatrix}$ of R. The ring k being semisimple implies that I is a semisimple R-module, and so I is Rickart. Therefore R is left $\mathrm{Soc}(R_R)$-inverse split by Theorem 1.2.

Proposition 2.3. Let R be a right $\mathrm{Soc}(R_R)$-inverse split ring. If the socle of $R/J(R)$ is finitely generated, then the following hold.

(1) The right socle $\mathrm{Soc}(R_R)$ is a direct summand of R as a left R-module.

(2) If R is left Rickart, then it is left $\mathrm{Soc}(R_R)$-inverse split.

Proof. (1) The ring R being right $\mathrm{Soc}(R_R)$-inverse split implies that $\mathrm{Soc}(R_R)$ is projective as a direct summand of the right R-module R. Since $\mathrm{Soc}(R/J(R))$ is finitely generated, then the right socle $\mathrm{Soc}(R_R)$ is also a direct summand as a left R-module by [6, Corollary to Theorem 1.2].

(2) It follows from (1) and Proposition 1.6. □

$\mathrm{Soc}(\cdot)$-inverse split rings occur in abundance in the context of the class of rings.

Example 2.4. Let R_1 be a semisimple ring and R_2 be a right Rickart ring with $\mathrm{Soc}({R_2}_{R_2}) = 0$. By Theorem 1.2, the ring $R = R_1 \times R_2$ of direct product of the rings R_1 and R_2 is a right $\mathrm{Soc}(R_R)$-inverse split ring. In particular, $\mathbb{Z}$ is a right Rickart ring. Let n be a positive integer and $\{p_i \mid i = 1, 2, \ldots, n\}$ a set of prime integers and consider the ring $R_1 = \prod_{i=1}^{n} \big(\mathbb{Z}/p_i\mathbb{Z}\big)$ and $R_2 = \mathbb{Z}$. Then $R = R_1 \times R_2$ is a right $\mathrm{Soc}(R_R)$-inverse split ring by Theorem 1.2, because $\mathrm{Soc}(R_R) = \mathrm{Soc}({R_1}_{R_1}) = \prod_{i=1}^{n} \big(\mathbb{Z}/p_i\mathbb{Z}\big)$ and $\mathbb{Z}$ is Rickart. Also $M = \prod_{i=1}^{n} \big(\mathbb{Z}/p_i\mathbb{Z}\big) \oplus \mathbb{Z}^{(I)}$ is $\mathrm{Soc}(M)$-inverse split as a $\mathbb{Z}$-module, because $\mathrm{Soc}(M) = \prod_{i=1}^{n} \big(\mathbb{Z}/p_i\mathbb{Z}\big)$ and $\mathbb{Z}^{(I)}$ is a Rickart $\mathbb{Z}$-module where I is any index set.

In the next, we give some relations between right $\mathrm{Soc}(\cdot)$-inverse split rings and certain classes of rings such as left perfect rings, right Kasch rings and semisimple rings. Recall that a ring R is called *right Kasch* if every simple R-module is embedded in R_R. By [8, Corollary 8.28], a ring R is right Kasch if and only if every maximal right ideal I of R is $I = r_R(t)$ for some $t \in R$. A ring R is said to be *right PS* if every simple right ideal is projective.

Theorem 2.5. Let R be a ring. Then the following are equivalent.

(1) R is right $\mathrm{Soc}(R_R)$-inverse split and left perfect.
(2) R is right $\mathrm{Soc}(R_R)$-inverse split and right Loewy.
(3) R is right $\mathrm{Soc}(R_R)$-inverse split and right Kasch.
(4) R is right $\mathrm{Soc}(R_R)$-inverse split and QF.
(5) R is right PS and QF.
(6) R is semisimple.

Proof. (1) $\Rightarrow$ (2) In [3], Bass shows in Theorem P that every left perfect ring is right Loewy (see also [2, Theorem 28.4]).

(2) $\Rightarrow$ (6) Clear by Corollary 1.11.

(3) $\Rightarrow$ (6) Let M be a simple R-module. Then $M = mR$ and $M \cong R/r_R(m)$ for some $0 \neq m \in M$. Since R is right Kasch, $M \cong I$ for some minimal right ideal I which is projective. Hence every simple R-module is projective. In particular, every maximal right ideal is a direct summand of R. Therefore R is semisimple.

(4) $\Rightarrow$ (5) By Theorem 1.2, $\mathrm{Soc}(R_R)$ is a direct summand of R, and so it is projective. Therefore R is right PS.

(5) $\Rightarrow$ (6) The ring R being right PS implies that $\mathrm{Soc}(R_R)$ is projective. Since R is QF, $\mathrm{Soc}(R_R)$ is also injective, hence it is a direct summand of R. On the other hand, the right socle of every QF-ring is an essential right ideal. Thus $\mathrm{Soc}(R_R) = R$. Therefore R is semisimple.

(6) $\Rightarrow$ (1), (6) $\Rightarrow$ (3) and (6) $\Rightarrow$ (4) are obvious. □

Lemma 2.6. Let R be a right $\mathrm{Soc}(R_R)$-inverse split ring. Then there exists a right ideal K of R such that $k^{-1}\mathrm{Soc}(R_R) = r_R(k)$ for every $k \in K$ where $k^{-1}\mathrm{Soc}(R_R) = \{a \in R \mid ka \in \mathrm{Soc}(R_R)\}$.

Proof. The ring R being right $\mathrm{Soc}(R_R)$-inverse split implies that there exists a right ideal K of R such that $R = \mathrm{Soc}(R_R) \oplus K$ by Theorem 1.2. Let $k \in K$. Then for any $x \in R$, $kx \in \mathrm{Soc}(R_R)$ implies $kx = 0$. The rest is clear. □

Theorem 2.7. Every right $\mathrm{Soc}(\cdot)$-inverse split ring is right nonsingular.

Proof. Let R be a right $\mathrm{Soc}(R_R)$-inverse split ring. Then $R = \mathrm{Soc}(R_R) \oplus K$ for some right ideal K of R due to Theorem 1.2. Hence $J(R) =$

$\mathrm{Rad}(\mathrm{Soc}(R_R)) \oplus \mathrm{Rad}(K)$. Since $\mathrm{Soc}(R_R)$ is semisimple, $\mathrm{Rad}(\mathrm{Soc}(R_R)) = 0$. Therefore by [12, Theorem 6.14], $\mathrm{Rad}(\mathrm{Soc}(R_R)) = \mathrm{Soc}(R_R) \cap J(R)$ and $J(R) = \mathrm{Rad}(K) \leqslant K$.

Let $a \in Z(K)$. Then $r_R(a)$ is essential in R. By Lemma 2.6, $a^{-1}\mathrm{Soc}(R_R) = r_R(a)$. The ring R being right $\mathrm{Soc}(R_R)$-inverse split implies that $a^{-1}\mathrm{Soc}(R_R)$ is a direct summand of R. Hence $r_R(a) = R$. It follows that $a = 0$, and so $Z(K) = 0$. Since $Z(R_R) = Z(\mathrm{Soc}(R_R)) \oplus Z(K)$, $Z(R_R) = Z(\mathrm{Soc}(R_R)) \subseteq \mathrm{Soc}(R_R)$.

Let $a \in Z(R_R)$. Then $r_R(a)$ is an essential right ideal of R and so $\mathrm{Soc}(R_R) \subseteq r_R(a)$. Hence $aZ(R_R) = 0$ or $a^2 = 0$. It follows that $Z(R_R)$ is a nil ideal. By [12, Corollary 6.8], $J(R)$ contains every nil right (left) ideal, so we have $Z(R_R) \subseteq J(R)$. Therefore $Z(R_R) = 0$. □

Theorem 2.8. Let R be a right $\mathrm{Soc}(R_R)$-inverse split ring. Then the maximal right ring of quotients $Q^r_{\max}(R)$ of R is a right Rickart ring.

Proof. By Theorem 2.7, R is right nonsingular. In the light of [4, Corollary 1.3.15] and [8, Johnson's Theorem 13.36], $Q^r_{\max}(R)$ is a von Neumann regular ring, therefore it is right Rickart. □

3. Rad(·)-inverse split modules

In this section we investigate inverse splitness of a module with respect to the dual notion to that of the socle. For a module M, $\mathrm{Rad}(M)$ is the intersection of all maximal submodules of M, equivalently, the sum of small submodules of M. It need not be small in M but, in ring case, $\mathrm{Rad}(R_R)$ is denoted by $J(R)$ and it is a small submodule of R_R. However, every module over a right perfect ring, or right cosemisimple ring, has a small radical. On the contrary, there is a class of modules M such that $\mathrm{Rad}(M)$ is a direct summand of M. In this section we study this class of modules M. The main aim is that whenever M has a decomposition such as $M = \mathrm{Rad}(M) \oplus N$ for some submodule N, to find out what a type of structure of N is. In this setting, we prove that N is a Rickart module for inverse split modules relative to the radical.

There is no close relation between Rad(·)-inverse split and Soc(·)-inverse split properties of modules since there exists a module M such that $\mathrm{Rad}(M) = M$ and $\mathrm{Soc}(M) \neq M$. There also exists a module M such that $\mathrm{Rad}(M) \neq M$ and $\mathrm{Soc}(M) = M$. Before stating the results we record some properties satisfied by the radical that we will use in the sequel without proof for an easy reference.

Lemma 3.1. The following hold for a module M.

(1) For any index set I, let $M = \bigoplus_{i \in I} M_i$. Then $\mathrm{Rad}(M) = \bigoplus_{i \in I} \mathrm{Rad}(M_i)$.

(2) Let $f\colon M \to M'$ be a module homomorphism. Then $f(\mathrm{Rad}(M)) \subseteq \mathrm{Rad}(M')$.

(3) $\mathrm{Rad}\big(M/\mathrm{Rad}(M)\big) = 0$.

(4) Let N be a submodule of M. Then $\mathrm{Rad}(N) \subseteq N \cap \mathrm{Rad}(M)$ where equality does not hold in general.

The properties stated in Lemma 3.1 are well known. We only note that there are modules such that $N \cap \mathrm{Rad}(M) = \mathrm{Rad}(N)$ does not hold in general.

Example 3.2. Let F be a field and consider the ring R, module M and its submodule N as

$$R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix},\ M = \begin{bmatrix} 0 & F \\ F & F \end{bmatrix},\ N = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}.$$

Then $\mathrm{Rad}(N) = 0$ and $\mathrm{Rad}(M) = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$. So $\mathrm{Rad}(M) = \mathrm{Rad}(M) \cap N \neq \mathrm{Rad}(N)$.

We now give the definition of the dual notion to that of the $\mathrm{Soc}(\cdot)$-inverse split modules.

Definition 3.3. A module M is called *Rad*(M)*-inverse split* if for every $f \in S$, $f^{-1}(\mathrm{Rad}(M))$ is a direct summand of M.

It is clear that every semisimple module M is $\mathrm{Rad}(M)$-inverse split. The module M defined in Example 3.2 is not $\mathrm{Rad}(M)$-inverse split. For a module M which is cogenerated by the class of simple modules, by [2, Proposition 9.16], M is Rickart if and only if it is $\mathrm{Rad}(M)$-inverse split.

In the next, a module decomposition is obtained by using a Rickart module and the radical of a module.

Theorem 3.4. The following are equivalent for a module M.

(1) M is $\mathrm{Rad}(M)$-inverse split.

(2) $M = \mathrm{Rad}(M) \oplus N$ where N is a Rickart module.

(3) M has a decomposition $M = K \oplus N$ where K has no maximal submodule and N is Rickart and cogenerated by the class of simple modules.

Proof. (1) $\Leftrightarrow$ (2) It follows from [14, Theorem 2.3] and Lemma 3.1(2).

(2) $\Rightarrow$ (3) Let K denote the submodule $\mathrm{Rad}(M)$. Then $M = K \oplus N$ where N is Rickart by (2). Since $\mathrm{Rad}(N) = 0$, N is cogenerated by the class of simple modules. On the other hand, $\mathrm{Rad}(M/N) = (\mathrm{Rad}(M) + N)/N = M/N$. This means that M/N has no maximal submodule. The submodule K being isomorphic to M/N implies that K has no maximal submodule.

(3) $\Rightarrow$ (2) Let $M = K \oplus N$ such that K has no maximal submodule and N is Rickart and cogenerated by the class of simple modules. Then by Lemma 3.1(1), $\mathrm{Rad}(M) = \mathrm{Rad}(K) \oplus \mathrm{Rad}(N) = \mathrm{Rad}(K) = K$. This completes the proof. □

The following examples illustrate that being a Rickart module or being a $\mathrm{Rad}(\cdot)$-inverse split module does not imply the other in general.

Examples 3.5. (1) Let p be a prime integer and consider the localization $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at $p\mathbb{Z}$ as a module over itself. Since $\mathbb{Z}_{(p)}$ is a domain, it is a Rickart module. On the other hand, the Jacobson radical of $\mathbb{Z}_{(p)}$ is $p\mathbb{Z}_{(p)}$ and it is not a direct summand of $\mathbb{Z}_{(p)}$. Therefore $\mathbb{Z}_{(p)}$ is not $J(\mathbb{Z}_{(p)})$-inverse split.

(2) Consider the Prüfer p-group $\mathbb{Z}_{p^\infty}$ for any prime integer p. Since $\mathbb{Z}_{p^\infty}$ has no maximal submodule, $\mathrm{Rad}(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty}$, and so $\mathbb{Z}_{p^\infty}$ is $\mathrm{Rad}(\mathbb{Z}_{p^\infty})$-inverse split, but it is not Rickart by [9, Example 2.17].

Proposition 3.6. Every indecomposable $\mathrm{Rad}(\cdot)$-inverse split module is either a radical module or a Rickart module for which every nonzero endomorphism is a monomorphism.

Proof. Let M be an indecomposable $\mathrm{Rad}(M)$-inverse split module. As $\mathrm{Rad}(M)$ is a direct summand of M, we have $\mathrm{Rad}(M) = M$ or $\mathrm{Rad}(M) = 0$. In the first case M is a radical module. Assume that $\mathrm{Rad}(M) = 0$. By Theorem 3.4, M is Rickart. Let $0 \neq f \in S$. Rickartness of M implies $\mathrm{Ker} f$ is a direct summand of M. Then $\mathrm{Ker} f = 0$. This completes the proof. □

Proposition 3.7. Every $\mathrm{Rad}(\cdot)$-inverse split module over a right V-ring is Rickart.

Proof. Let R be a right V-ring and M a $\mathrm{Rad}(M)$-inverse split module. Then $\mathrm{Rad}(M) = 0$. By Theorem 3.4, M is Rickart. □

Clearly, if a module M is Rickart with $\mathrm{Rad}(M) = 0$, then it is $\mathrm{Rad}(M)$-inverse split. In the next, we investigate under what conditions the converse of this statement is true. Recall that a module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M.

Semisimple modules, finitely generated modules and Noetherian modules are some examples of coatomic modules.

Proposition 3.8. Let M be a coatomic module. Then M is $\mathrm{Rad}(M)$-inverse split if and only if M is Rickart with $\mathrm{Rad}(M) = 0$.

Proof. Let M be $\mathrm{Rad}(M)$-inverse split and coatomic. Then $\mathrm{Rad}(M)$ is a small submodule of M. Also, it is a direct summand of M by Theorem 3.4. Hence $\mathrm{Rad}(M) = 0$, this implies that M is Rickart. □

By [2, Theorem 28.4] and Proposition 3.8, we obtain the next result immediately.

Corollary 3.9. Let R be a right perfect ring. Then for every R-module M, M is $\mathrm{Rad}(M)$-inverse split if and only if M is Rickart with $\mathrm{Rad}(M) = 0$.

Examples 3.10. (1) Let R be a right V-ring. It is known that for every module M, $\mathrm{Rad}(M) = 0$. One may suspect that such a module is Rickart but it is not the case, for example, consider the ring $R = \prod_{i=1}^{\infty} R_i$ where $R_i = \mathbb{Z}_2$ for all i. Since R is commutative von Neumann regular, it is a V-ring but not semisimple. By [9, Theorem 2.25], there exists an R-module which is not Rickart.

(2) Let M denote the $\mathbb{Z}$-module $\big(\bigoplus_{i\in I} Q_i\big) \oplus \big(\bigoplus_{j\in J} Z_j\big)$ where $Q_i = \mathbb{Q}$ and $Z_j = \mathbb{Z}$ for some index sets I and J. Note that $\mathrm{Rad}(\mathbb{Q}) = \mathbb{Q}$, $\mathrm{Rad}\big(\bigoplus_{j\in J} Z_j\big) = 0$. For any hereditary ring, every free right R-module is Rickart by [9, Theorem 2.26]. This fact and Theorem 3.4 imply that M is a $\mathrm{Rad}(M)$-inverse split $\mathbb{Z}$-module.

(3) Every divisible abelian group M is a $\mathrm{Rad}(M)$-inverse split $\mathbb{Z}$-module by the fact that $\mathrm{Rad}(M) = M$.

We now investigate when the homomorphic images of $\mathrm{Rad}(\cdot)$-inverse split modules have the same property.

Proposition 3.11. Let M and M' be R-modules and $M \xrightarrow{f} M'$ an epimorphism. If M is $\mathrm{Rad}(M)$-inverse split and $\mathrm{Ker} f \subseteq \mathrm{Rad}(M)$, then M' is $\mathrm{Rad}(M')$-inverse split.

Proof. Note that by [2, Proposition 9.15], $\mathrm{Ker} f \subseteq \mathrm{Rad}(M)$ implies that $f(\mathrm{Rad}(M)) = \mathrm{Rad}(M')$ and so $\mathrm{Rad}(M) = f^{-1}(\mathrm{Rad}(M'))$. Let M be $\mathrm{Rad}(M)$-inverse split. There exists a Rickart module N such that $M =$

$\text{Rad}(M) \oplus N$ by Theorem 3.4. Then $M' = f(\text{Rad}(M)) + f(N) = \text{Rad}(M') + f(N)$. Let $y \in f(\text{Rad}(M)) \cap f(N)$. There exist $a \in \text{Rad}(M)$, $b \in N$ such that $y = f(a) = f(b)$. Since $\text{Ker}f \subseteq \text{Rad}(M)$, $b \in \text{Rad}(M)$. Hence $b = 0$ and so $\text{Rad}(M') \cap f(N) = 0$. On the other hand, the restriction of f to N is an isomorphism between N and $f(N)$. Hence $f(N)$ is a Rickart module. Therefore M' is $\text{Rad}(M')$-inverse split. □

Let M and P be arbitrary modules. Recall that P is called a *small cover* of M if there exists an epimorphism $P \xrightarrow{f} M$ such that $\text{Ker}f$ is a small submodule of P. In case, P is a projective module and a small cover of M, P is called a *projective cover* of M. Every projective cover of a module is a small cover.

Corollary 3.12. The following hold.
(1) Let M be a module having a small cover P which is $\text{Rad}(P)$-inverse split. Then M is $\text{Rad}(M)$-inverse split.
(2) Let M be a module with a projective cover P. If P is $\text{Rad}(P)$-inverse split, then so is M.

The next result shows that direct summands of $\text{Rad}(\cdot)$-inverse split modules inherit the same property.

Proposition 3.13. Let M be a $\text{Rad}(M)$-inverse split module. Then every direct summand N of M is $\text{Rad}(N)$-inverse split.

Proof. Let N be a direct summand of M. Then $M = N \oplus K$ for some submodule K of M. In light of [14, Proposition 2.12], N is $(N \cap \text{Rad}(M))$-inverse split. On the other hand, $\text{Rad}(M)$ being fully invariant in M implies $\text{Rad}(M) = (N \cap \text{Rad}(M)) \oplus (K \cap \text{Rad}(M))$. Also, by Lemma 3.1(1), $\text{Rad}(M) = \text{Rad}(N) \oplus \text{Rad}(K)$. Hence $\text{Rad}(N) = N \cap \text{Rad}(M)$. Thus N is $\text{Rad}(N)$-inverse split. □

We now characterize semiprimitive right hereditary rings in terms of $\text{Rad}(\cdot)$-inverse split modules.

Theorem 3.14. The following are equivalent for a ring R.
(1) Every free R-module M is $\text{Rad}(M)$-inverse split.
(2) Every projective R-module M is $\text{Rad}(M)$-inverse split.
(3) R is semiprimitive right hereditary.

Proof. (1) $\Rightarrow$ (2) Let M be a projective R-module. Then M is a direct summand of a free R-module P. By (1), P is $\text{Rad}(P)$-inverse split, and so M is $\text{Rad}(M)$-inverse split by Proposition 3.13.

(2) $\Rightarrow$ (3) Since R is projective, it is $J(R)$-inverse split, and so $J(R)$ is a direct summand of R. Hence $J(R) = 0$, i.e., R is semiprimitive. Let I be a right ideal of R. There exist a free R-module P and a homomorphism $f\colon P \to I$ with $\mathrm{Im} f = I$. Being $J(R) = 0$ implies $\mathrm{Rad}(P) = 0$. By (2), P is $\mathrm{Rad}(P)$-inverse split, so it is Rickart. Thus $P/\mathrm{Ker} f \cong I$ is projective. Therefore R is right hereditary.

(3) $\Rightarrow$ (1) Let M be a free R-module. Since R is right hereditary, by [9, Theorem 2.26], M is Rickart. On the other hand, $\mathrm{Rad}(M) = 0$ because $J(R) = 0$. Therefore M is $\mathrm{Rad}(M)$-inverse split. □

We end this paper by presenting some applications of the notion of $\mathrm{Rad}(\cdot)$-inverse split modules by using semiperfect modules relative to the radical. Recall that a module M is called *Rad*(M)*-semiperfect* if for any submodule N of M, there exists a projective direct summand A of M such that $A \subseteq N$ and $M = A \oplus B$ with $N \cap B \subseteq \mathrm{Rad}(M)$.

Proposition 3.15. Let M be a $\mathrm{Rad}(M)$-semiperfect module. Then M is $\mathrm{Rad}(M)$-inverse split if and only if $M = \mathrm{Rad}(M) \oplus N$ where N is semisimple.

Proof. Let K be a submodule of M with $\mathrm{Rad}(M) \subseteq K$. There exists a direct summand A of M such that $M = A \oplus B$ and $K \cap B \subseteq \mathrm{Rad}(M)$. Then $M/\mathrm{Rad}(M) = \big(K/\mathrm{Rad}(M)\big) \oplus \big((B+ \mathrm{Rad}(M))/\mathrm{Rad}(M)\big)$. Hence $M/\mathrm{Rad}(M)$ is semisimple. Now suppose that M is a $\mathrm{Rad}(M)$-inverse split module. Then there exists a submodule N of M such that $M = \mathrm{Rad}(M) \oplus N$. Thus N is semisimple. The converse is clear due to Theorem 3.4. □

For any R-module M, by the fact that $MJ(R) \subseteq \mathrm{Rad}(M)$, the module $M/\mathrm{Rad}(M)$ has an $R/J(R)$-module structure.

Proposition 3.16. Let R be a right $J(R)$-semiperfect ring. Then the following hold.

(1) $R/J(R)$ is semisimple and R has a decomposition $R = A \oplus B$ with A semisimple and $J(R)$ is an essential right ideal in B and $B/J(R)$ is a semisimple right R-module.

(2) Every $R/J(R)$-module M has a decomposition $M = M_1 \oplus M_2$ with M_1 a semisimple $R/J(R)$-module and $\mathrm{Rad}(M)$ essential in M_2 as an $R/J(R)$-module.

Proof. (1) By a similar discussion in the proof of Proposition 3.15, $R/J(R)$ is semisimple. The second assertion is a direct consequence of

[10, Proposition 2.1]. (2) Let M be an $R/J(R)$-module. By (1), $M/\mathrm{Rad}(M)$ is a semisimple $R/J(R)$-module. Due to [10, Proposition 2.1], there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$ where M_1 is a semisimple $R/J(R)$-module and $\mathrm{Rad}(M)$ is essential in M_2 as an $R/J(R)$-module and $M_2/\mathrm{Rad}(M)$ is semisimple as an $R/J(R)$-module. □

Theorem 3.17. Let M be a $\mathrm{Rad}(M)$-semiperfect module with ascending chain condition on direct summands. Then M has a decomposition $M = \mathrm{Rad}(M) \oplus N$ where $\mathrm{Rad}(M)$ is projective and N is semisimple. In this case, M is $\mathrm{Rad}(M)$-inverse split.

Proof. Let M be a $\mathrm{Rad}(M)$-semiperfect module. There exists a projective direct summand $A_1 \subseteq \mathrm{Rad}(M)$ such that $M = A_1 \oplus B_1$. Then $\mathrm{Rad}(M) = A_1 \oplus \big(B_1 \cap \mathrm{Rad}(M)\big)$. Assume that $B_1 \cap \mathrm{Rad}(M) \neq 0$, by the same reasoning, there exists a projective direct summand $A_2 \subseteq B_1 \cap \mathrm{Rad}(M)$ such that $M = A_2 \oplus B_2$. Then $B_1 = A_2 \oplus \big(B_2 \cap B_1\big)$ and $\mathrm{Rad}(M) = A_2 \oplus A_1 \oplus \big(B_2 \cap B_1 \cap \mathrm{Rad}(M)\big)$ and $M = A_1 \oplus A_2 \oplus \big(B_2 \cap B_1\big)$. If $B_2 \cap B_1 \cap \mathrm{Rad}(M) \neq 0$, then there exists a projective direct summand $A_3 \subseteq B_2 \cap B_1 \cap \mathrm{Rad}(M)$ such that $M = A_3 \oplus B_3$. Then $B_2 \cap B_1 = A_3 \oplus (B_3 \cap B_2 \cap B_1)$, $M = A_3 \oplus A_2 \oplus A_1 \oplus \big(B_3 \cap B_2 \cap B_1\big)$ and $\mathrm{Rad}(M) = A_3 \oplus A_2 \oplus A_1 \oplus \big(B_3 \cap B_2 \cap B_1 \cap \mathrm{Rad}(M)\big)$. Continuing in this way, for each positive integer n, if $B_{n-1} \cap B_{n-2} \cap \cdots \cap B_1 \cap \mathrm{Rad}(M) \neq 0$, then we may find a projective direct summand A_n of M such that $M = A_n \oplus B_n$ and $A_n \subseteq B_{n-1} \cap B_{n-2} \cap \cdots \cap B_1 \cap \mathrm{Rad}(M) \neq 0$. By hypothesis, there must be a positive integer t such that the sequence $A_1 \subseteq A_1 \oplus A_2 \subseteq A_1 \oplus A_2 \oplus A_3 \subseteq \cdots$ of direct summands of M terminates at the step t and then $A_1 \oplus A_2 \oplus \cdots \oplus A_t = A_1 \oplus A_2 \oplus \cdots \oplus A_t \oplus A_{t+1} = \cdots$. At this step $B_1 \cap B_2 \cap \cdots \cap B_t \cap \mathrm{Rad}(M) = 0$ must be held. It follows that $M = A_t \oplus A_{t-1} \oplus \cdots \oplus A_1 \oplus (B_t \cap B_{t-1} \cap \cdots \cap B_1)$ and $\mathrm{Rad}(M) = A_1 \oplus A_2 \oplus \cdots \oplus A_t$ is projective. Let $N = B_t \cap B_{t-1} \cap \cdots \cap B_1$ and L be a submodule of N. By hypothesis, there exists a projective direct summand L_1 of M such that $M = L_1 \oplus L_2$ and $L_1 \subseteq L$ and $L_2 \cap L \subseteq \mathrm{Rad}(M)$. Then $L_2 \cap L = 0$. Hence $L = L_1$ is a direct summand of N. Thus N is semisimple. By Theorem 3.4, M is $\mathrm{Rad}(M)$-inverse split. □

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References

[1] N. Agayev, S. Halicioglu, A. Harmanci, *On Rickart modules*, Bull. Iran. Math. Soc., N.**38**(2), 2012, pp.433-445.

[2] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1992.

[3] H. Bass, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc., N.**95**, 1960, pp.466-488.

[4] G. F. Birkenmeier, J. K. Park, S. T. Rizvi, *Extensions of Rings and Modules*, Springer Science+Business Media, New York, 2013.

[5] N. V. Dung, D. V. Huynh, P. F. Smith, R. Wisbauer, *Extending Modules*, Pitman Research Notes in Math. Ser. 313, 1994.

[6] R. Gordon, *Rings in which minimal left ideals are projective*, Pacific J. Math., N.**31**, 1969, pp.679-692.

[7] A. Hattori, *A foundation of torsion theory for modules over general rings*, Nagoya Math. J., N.**17**, 1960, pp.147-158.

[8] T. Y. Lam, *Lectures on Modules and Rings*, Springer-Verlag, New York, 1999.

[9] G. Lee, S. T. Rizvi, C. S. Roman, *Rickart modules*, Comm. Algebra, N.**38**(11), 2010, pp.4005-4027.

[10] C. Lomp, *On semilocal modules and rings*, Comm. Algebra, N.**27**(4), 1999, pp.1921-1935.

[11] S. Maeda, *On a ring whose principal right ideals generated by idempotents form a lattice*, J. Sci. Hiroshima Univ. Ser. A, N.**24**, 1960, pp.509-525.

[12] N. H. McCoy, *The Theory of Rings*, Chelsea Publishing Co., Bronx, New York, 1973.

[13] S. T. Rizvi, C. S. Roman, *Baer property of modules and applications*, Advances in Ring Theory, 2005, pp.225-241.

[14] B. Ungor, S. Halicioglu, A. Harmanci, *Modules in which inverse images of some submodules are direct summands*, Comm. Algebra, N.**44**(4), 2016, pp.1496-1513.

[15] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach Science Publishers, Philadelphia, PA, 1991.

Contact information

A. Harmanci
Department of Mathematics,
Hacettepe University, Turkey
E-Mail(s): harmanci@hacettepe.edu.tr
Web-page(s): www.hacettepe.edu.tr

B. Ungor
Department of Mathematics, Ankara University, Turkey
E-Mail(s): bungor@science.ankara.edu.tr
Web-page(s): www.ankara.edu.tr

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