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RESEARCH ARTICLE

Dickson's theorem for Bol loops*

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Abstract. Dickson characterized groups in terms of one-sided invertibility. In this note, we give comparable characterizations for Bol and Moufang loops.

Introduction

Dickson [3] characterized groups as follows: a group is a semigroup Q which possesses a left (right) unit, under which every element of Q is left (right) invertible. In this note we prove the similar result for the left (right) Bol loops ([1],[2],[5]–[12]). As a consequence we formulate a similar result for the Moufang loops.

Definition 1. A groupoid $Q(\cdot)$ is called:

1) a right division groupoid (right quasigroup) if the equation $a \cdot x = b$ has a solution (unique solution) $x \in Q$ for every $a, b \in Q$;
2) a left division groupoid (left quasigroup) if the equation $y \cdot a = b$ has a solution (unique solution) $y \in Q$ for every $a, b \in Q$;
3) a quasigroup (division groupoid) if $Q(\cdot)$ is a right and left quasigroup (right and left division groupoid);

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4) left cancellative if the equation $a \cdot x = b$ has no more than one solution $x \in Q$ for every $a, b \in Q$, i.e. for arbitrary $a, x_1, x_2 \in Q$:

$$a \cdot x_1 = a \cdot x_2 \longrightarrow x_1 = x_2;$$

5) right cancellative if the equation $y \cdot a = b$ has no more than one solution $x \in Q$ for every $a, b \in Q$, i.e. for arbitrary $a, y_1, y_2 \in Q$:

$$y_1 \cdot a = y_2 \cdot a \longrightarrow y_1 = y_2;$$

6) a loop if $Q(\cdot)$ is a quasigroup with a unit;
7) a left Bol loop if $Q(\cdot)$ is a loop satisfying the left Bol identity:

$$(x \cdot yx)z = x(y \cdot xz); \tag{1}$$

8) a right Bol loop if $Q(\cdot)$ is a loop satisfying the right Bol identity:

$$z(xy \cdot x) = (zx \cdot y)x; \tag{2}$$

9) a Moufang loop if $Q(\cdot)$ is a loop satisfying one of the following left and right Moufang identities:

$$x(y \cdot xz) = (xy \cdot x)z, \tag{3}$$

$$(zx \cdot y)x = z(x \cdot yx); \tag{4}$$

10) right (left) alternative if $Q(\cdot)$ satisfies the right (left) alternative identity:

$$y(x \cdot x) = (y \cdot x)x, (x \cdot x)y = x(x \cdot y);$$

11) alternative if $Q(\cdot)$ is a right and left alternative.

It is well known that the Moufang identities (3) and (4) are equivalent in the class of loops ([1], [6], [9]). For applications of right (left) quasigroup operations in geometry and topology (knot theory) see [4, 8, 10]. For right (left) loops see [13].

1. The main result

Let us start at the following auxiliary result.

Lemma 1. *1) Every idempotent element of a left alternative and right division groupoid $Q(\cdot)$ is a left unit of $Q(\cdot)$;*

2) Every idempotent element of a right alternative and left division groupoid $Q(\cdot)$ is a right unit of $Q(\cdot)$;

3) Every idempotent element of an alternative and division groupoid is a unit of $Q(\cdot)$.

Proof. 1) Let $e \in Q$ be an idempotent element of $Q(\cdot)$, i.e. $e \cdot e = e$. For any $b \in Q$ there exist $x \in Q$ such that $e \cdot x = b$, then by left alternativity:

$$e \cdot b = e(e \cdot x) = (e \cdot e)x = e \cdot x = b. \qquad \square$$

Theorem 1. *1) Let $Q(\cdot)$ be a left cancellative groupoid with a right unit, satisfying the left Bol identity* (1)*.*

If every element of Q is a right invertible under one of the right units, then $Q(\cdot)$ is a loop (left Bol loop).

2) Let $Q(\cdot)$ be a right cancellative groupoid with a left unit, satisfying the right Bol identity (2)*.*

If every element of Q is a left invertible under one of the left units, then $Q(\cdot)$ is a loop (right Bol loop).

Proof. 1) Let e be a right unit of $Q(\cdot)$ and for every $a \in Q$ there exist an element $a' \in Q$ that $a \cdot a' = e$. Let us denote: $a'' = (a')'$.

First, we prove that $a' \cdot a = e$. Namely, by the Bol left identity (1) we have:

$$a' \cdot (a \cdot a') = a' \cdot e = a',$$
$$a' \cdot a = a' \cdot (a \cdot e) = a' \cdot (a \cdot (a' \cdot a'')) = (a' \cdot (a \cdot a')) \cdot a'' =$$
$$= (a' \cdot e) \cdot a'' = a' \cdot a'' = e.$$

It is evident the uniqueness of the inverse element $a' \in Q$ by left cacellativity of $Q(\cdot)$. Hence, $a'' = a$. Moreover, by the left Bol identity (1) we obtain:

$$(x \cdot x'x)z = x(x' \cdot xz) \quad \text{and} \quad (x \cdot e)z = x(x' \cdot xz),$$

i.e.

$$x \cdot z = x(x' \cdot xz).$$

According to the left cancellative property, we obtain: $z = x' \cdot xz$ for arbitrary $x, z \in Q$.

For any $a, b \in Q$ the unique solution of $a \cdot x = b$ is the $x = a' \cdot b$. Indeed, if $x = a' \cdot b$, then $a \cdot x = b$, since $a''(a' \cdot b) = b$ and $a'' = a$.

Let us consider the equation: $y \cdot a = b$, where $a, b \in Q$. If $y \cdot a = b$, then $(a \cdot ya)a' = ab \cdot a'$, i.e. $a(y \cdot aa') = ab \cdot a'$ by the left Bol identity (1), or $ay = ab \cdot a'$, $y = a'(ab \cdot a')$. And from $y = a'(ab \cdot a')$ it follows that $y \cdot a = b$ by the left Bol identity (1):

$$y \cdot a = (a' \cdot (ab \cdot a')) \cdot a = a'(ab \cdot a'a) = a'(ab \cdot e) = a'(ab) = b.$$

Hence, $Q(\cdot)$ is a quasigroup.

In order to prove that the right unit $e \in Q$ is a unit of $Q(\cdot)$, first we note that by setting $y = e$ in the left Bol identity (1), we immediately obtain the left alternative law: $(x \cdot x)z = x(x \cdot z)$. Now, according to the previous Lemma 1, every idempotent element of $Q(\cdot)$ is a left unit of $Q(\cdot)$. In particular, the right unit $e \in Q$ is a unit of $Q(\cdot)$.

2) The proof is dual to the proof of 1). □

Corollary 1. *1) Let $Q(\cdot)$ be a left cancellative groupoid with a right unit, satisfying the left Moufang identity* (3).

If every element of Q is a right invertible under one of the right units, then $Q(\cdot)$ is a loop (Moufang loop).

2) Let $Q(\cdot)$ be a right cancellative groupoid with a left unit, satisfying the right Moufang identity (4).

If every element of Q is a left invertible under one of the left units, then $Q(\cdot)$ is a loop (Moufang loop).

Proof. 1) Note, that by setting $z = e$ (which is a right unit of $Q(\cdot)$) in the left Moufang identity (3) we obtain the flexible law: $(xy)x = x(yx)$. Hence, the identity (3) is converted to the left Bol identity (1). So we can use the result 1) of the previous Theorem 1.

2) The proof is similar to the proof of 1). □

Corollary 2. *1) Every left cancellative, alternative and division groupoid with the left Bol identity* (1) *is a loop (Moufang loop).*

2) Every right cancellative, alternative and division groupoid with the right Bol identity (2) *is a loop (Moufang loop).*

Proof. 1) For any element $a \in Q$ there exists an element $e_a \in Q$ such that $a \cdot e_a = a$, since $Q(\cdot)$ is a division groupoid. Applying this equality in the identity of right alternativity, we find:

$$a \cdot e_a^2 = (a \cdot e_a) \cdot e_a = a \cdot e_a,$$

i.e. e_a is an idempotent element of $Q(\cdot)$: $e_a^2 = e_a$ by left cancellativity of $Q(\cdot)$.

Hence, $Q(\cdot)$ has a unit by the previous Lemma 1. Consequently, we can use the result 1) of the previous theorem. Thus, $Q(\cdot)$ is a left Bol loop. Then the right alternative low together with the left Bol identity (1) implies that:

$$(x \cdot yx)x = x(y \cdot xx) = x(yx \cdot x).$$

Setting $yx = z$, we get the flexible law: $(xz)x = x(zx)$, which implies that the left Bol loop $Q(\cdot)$ is a Moufang loop.

2) The proof is similar to the proof of 1). □

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