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RESEARCH ARTICLE

Cross-cap singularities counted with sign

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Abstract. A method for computing the algebraic number of cross-cap singularities for mapping from m-dimensional compact manifold with boundary $M \subset \mathbb{R}^m$ into $\mathbb{R}^{2m-1}$, m is odd, is presented. As an application, the intersection number of an immersion $g\colon S^{m-1}(r) \to \mathbb{R}^{2m-2}$ is described as the algebraic number of cross-caps of a mapping naturally associated with g.

Introduction

Mappings from the m-dimensional, smooth, orientable manifold M into $\mathbb{R}^{2m-1}$ are natural object of study. In [9],Whitney described typical mappings from M into $\mathbb{R}^{2m-1}$. Those mappings have only isolated critical points, called cross-caps (or Whintey umbrellas).

According to [1, Theorem 4.6], [11, Lemma 2], a mapping $M \to \mathbb{R}^{2m-1}$ has a cross-cap at $p \in M$, if and only if in the local coordinate system near p this mapping has the form

$$(x_1, \dots, x_m) \mapsto (x_1^2, x_2, \dots, x_m, x_1x_2, \dots, x_1x_m).$$

In [11], for m odd, Whitney presented a method to associate a sign with a cross-cap. Put $\zeta(f)$ to be an algebraic sum of cross-caps of $f : M \to \mathbb{R}^{2m-1}$, where M is m-dimensional compact orientable manifold. Then according to Whitney, [11, Theorem 3], $\zeta(f) = 0$, if M is closed. If M

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has a boundary, then following Whitney, [11, Theorem 4], for a homotopy $f_t : M \to \mathbb{R}^{2m-1}$ regular in some open neighbourhood of ∂M, if the only singular points of f_0 and f_1 are cross-caps, then $\zeta(f_0) = \zeta(f_1)$. Moreover arbitrarily close to any mapping $h : M \to \mathbb{R}^{2m-1}$, there is mapping regular near boundary, with only cross-caps as singular points (see [11]). In the case where m even, it is impossible to associate sign with cross-cap in the same way as in the odd case, but if m is even, it is enough to consider number of cross-caps mod 2, to get similar results (see [11]).

In [6], the authors studied a mapping α from a compact and oriented $(n-k)$-manifold M into the Stiefel manifold $\widetilde{V}_k(\mathbb{R}^n)$, for $n-k$ even. They constructed a mapping $\widetilde{\alpha} : S^{k-1} \times M \to \mathbb{R}^n \setminus \{0\}$ associated with α, and defined $\Lambda(\alpha)$ as half of topological degree of $\widetilde{\alpha}$. In case $M = S^{n-k}$, they showed that $\Lambda(\alpha)$ corresponds with the class of α in $\pi_{n-k}\widetilde{V}_k(\mathbb{R}^n) \simeq \mathbb{Z}$. According to [6], in the case where $M \subset \mathbb{R}^{n-k+1}$ is an algebraic hypersurface and α is polynomial, with some additional assumptions concerning M and α, $\Lambda(\alpha)$ can be presented as a sum of signatures of two quadratic forms defined on $\mathbb{R}[x_1, \dots, x_{n-k+1}]$. And so, easily computed.

In this paper we prove that in the case where m is odd, for $f : (M, \partial M) \to \mathbb{R}^{2m-1}$, where $M \subset \mathbb{R}^m$, $\zeta(f)$ can be expressed as $\Lambda(\alpha)$, for some α associated with f. And so, with some additional assumptions concerning M and f, $\zeta(f)$ can be easily computed for polynomial mapping f. Moreover we present a method that can be used to check effectively that f has only cross-caps as singular points. In case when m is even, the effective method to compute number of cross-caps modulo 2 is presented in [5].

Take a smooth map $g : \mathbb{R}^m \to \mathbb{R}^{2m-2}$, let us assume that $g|_{S^{m-1}}$ is an immersion. In [10], Whitney introduced the intersection number $I(g|_{S^{m-1}})$ of immersion $g|_{S^{m-1}}$. In this paper we show that $I(g|_{S^{m-1}})$, can be presented as an algebraic sum of cross-caps of the mapping $(\omega, g)|\bar{B}^m$, where ω is sum of squares of coordinates.

Take $f : (\mathbb{R}^m, 0) \to \mathbb{R}^{2m-1}$ with cross-cap at 0. In [3], Ikegami and Saeki defined the sign of a cross-cap singularity for mapping f as the intersection number of immersion $f|_S : S = f^{-1}(S^{2m-2}(\epsilon)) \to S^{2m-2}(\epsilon)$, for ϵ small enough. It is easy to see that this definition complies with Whitney definition from [11]. In [3], the authors showed that for generic map (in sense of [3]) $g : (\mathbb{R}^m, 0) \to \mathbb{R}^{2m-1}$, the number of cross-caps appearing in a C^∞ stable perturbation of g, counted with signs, is an invariant of the topological $\mathcal{A}_+$-equivalence class of g, and is equal to the intersection number of $g|_S : S = g^{-1}(S^{2m-2}(\epsilon)) \to S^{2m-2}(\epsilon)$. Using our methods, this number can be easily computed for polynomial mappings.

We use notation $S^n(r)$, $B^n(r)$, $\bar{B}^n(r)$ for sphere, open ball, closed ball (resp.) centred at the origin of radios r and dimension n. If we omit symbol r, we assume that $r = 1$.

1. Cross-cap singularities

Let M, N be smooth manifolds. Take a smooth mapping $f : M \to N$.

Lemma 1. *Let W be a submanifold of N. Take $p \in M$ such that $f(p) \in W$. Let us assume that there is a neighbourhood U of $f(p)$ in N and a smooth mapping $\phi\colon U \to \mathbb{R}^s$ such that* $\operatorname{rank} D\phi(f(p)) = k = \operatorname{codim} W$ *and* $W \cap U = \phi^{-1}(0)$. *Then $f \pitchfork W$ at p if and only if* $\operatorname{rank} D(\phi \circ f)(p) = k$.

Proof. Of course $\operatorname{Ker} D\phi(f(p)) = T_{f(p)}W$, and so we get $\dim T_{f(p)}N = \dim \operatorname{Ker} D\phi(f(p)) + k$. Then:

$$\begin{aligned} f \pitchfork W \text{ at } p &\iff T_{f(p)}N = T_{f(p)}W + Df(p)T_pM \\ &\iff T_{f(p)}N = \operatorname{Ker} D\phi(f(p)) + Df(p)T_pM. \end{aligned}$$

The above equality holds if and only if there exist vectors $v_1, \ldots, v_k$ in $Df(p)T_pM$, such that any nontrivial combination of $v_1, \ldots, v_k$ is outside the $\operatorname{Ker} D\phi(f(p))$ and so $\operatorname{rank} D\phi(f(p))\,[v_1 \ldots v_k] = k$. We get that $f \pitchfork W$ at p if and only if $\operatorname{rank} D(\phi \circ f)(p) = k$. □

By $j^1 f$ we mean the canonical mapping associated with f, from M into the spaces of 1-jets $J^1(M, N)$. We say that $f : M \to N$ is 1-*generic*, if $j^1 f \pitchfork S_r$, for $r \geqslant 0$, where $S_r = \{\sigma \in J^1(M, N) \mid \operatorname{corank} \sigma = r\}$. Put $S_r(f) = \{x \in M \mid \operatorname{corank} Df(p) = r\} = (j^1 f)^{-1}(S_r)$.

Let us assume that M and N are manifolds of dimension m and $2m-1$ respectively. In this case (see [1]) $\operatorname{codim} S_r = r^2 + r(m-1)$, and so $\operatorname{codim} S_1 = m$ and $\operatorname{codim} S_r > m$, for $r \geqslant 2$. So f is 1-generic if and only if $f \pitchfork S_1$ and $S_r(f) = \varnothing$ for $r \geqslant 2$. The typical singularity for mapping $f : M \to N$ is a cross-cap singularity. Following [9], [11], [1] we present equivalent definitions of a cross-cap.

Definition 1. A point p is a cross-cap of a mapping $f : M \to N$ if the following equivalent conditions are fulfilled:

1) $p \in S_1(f)$ and $j^1 f \pitchfork S_1$ at p;
2) there are coordinate systems near p and $f(p)$, such that

$$\frac{\partial f}{\partial x_1}(p) = 0 \tag{1}$$

and vectors

$$\frac{\partial^2 f}{\partial x_1^2}(p), \frac{\partial f}{\partial x_2}(p), \ldots, \frac{\partial f}{\partial x_m}(p), \frac{\partial^2 f}{\partial x_1 \partial x_2}(p), \ldots, \frac{\partial^2 f}{\partial x_1 \partial x_m}(p) \quad (2)$$

are linearly independent;

3) there are coordinate systems near p and $f(p)$ such that the mapping f has the form

$$(x_1, \ldots, x_m) \mapsto (x_1^2, x_2, \ldots, x_m, x_1 x_2, \ldots, x_1 x_m).$$

According to [9, Section 2], if p is a cross-cap singularity and (1) holds, then vectors (2) are linearly independent.

Take $f = (f_1, \ldots, f_{2m-1}) : \mathbb{R}^m \to \mathbb{R}^{2m-1}$. Put $\mu : \mathbb{R}^m \to \mathbb{R}^s$ such that $\mu(x)$ is given by all the m-minors of $Df(x)$. Of course $s = \binom{2m-1}{m}$.

Lemma 2. *A point $p \in \mathbb{R}^m$ is a cross-cap singularity of f if and only if* $\operatorname{rank} Df(p) = m - 1$ *and* $\operatorname{rank} D\mu(p) = m$.

Proof. A point p is a cross-cap singularity if and only if $p \in S_1(f)$ and $j^1 f \pitchfork S_1$ at p. Note that $p \in S_1(f)$ if and only if $\operatorname{rank} Df(p) = m - 1$.

Of course $J^1(\mathbb{R}^m, \mathbb{R}^{2m-1}) \cong \mathbb{R}^m \times \mathbb{R}^{2m-1} \times M(2m-1, m)$, where $M(2m-1, m)$ is a space of real matrices of dimension $(2m-1) \times m$. Take an open neighbourhood U of $j^1 f(p)$ in $J^1(\mathbb{R}^m, \mathbb{R}^{2m-1})$, and a mapping

$$\phi : U \to \mathbb{R}^s,$$

where $\phi(x, y, [a_{ij}])$ is given by all m-minors of $[a_{ij}]$. We may assume that

$$\det \frac{\partial(f_1, \ldots, f_{m-1})}{\partial(x_1, \ldots, x_{m-1})}(p) \neq 0.$$

Put $A = [a_{ij}]_{1 \leqslant i,j \leqslant m-1}$ the submatrix of $[a_{ij}]$, then for U small enough, $\det A \neq 0$. Let M_i be the determinant of submatrix of $[a_{ij}]$ composed of first $m-1$ rows and row number $(m+i-1)$, for $i = 1, \ldots, m$. Then

$$M_i = (-1)^{2m-1+i} \det A \cdot a_{m+i-1,m} + b_i,$$

for $i = 1, \ldots, m$ and b_i does not depend on $a_{mm}, \ldots, a_{2m-1,m}$, and so

$$\operatorname{rank} \frac{\partial(M_1, \ldots, M_m)}{\partial(a_{m,m}, \ldots, a_{2m-1,m})} = m.$$

We get that

$$\operatorname{rank} D\phi(j^1 f(p)) \geqslant m.$$

Let us recall that $\operatorname{codim} S_1 = m$. We can choose U small enough such that

$$\phi^{-1}(0) = U \cap S_1.$$

So we get that $\operatorname{rank} D\phi(j^1 f(p)) = \operatorname{codim} S_1 = m$. Of course $\phi \circ j^1 f = \mu$ in the small neighbourhood of p. According to Lemma 1, $j^1 f \pitchfork S_1$ at p if and only if $\operatorname{rank} D\mu(p) = m$.

□

2. Algebraic sum of cross-cap singularities

First we want to recall some well-known facts concerning the topological degree. Let $(N, \partial N)$ be n-dimensional compact oriented manifold with boundary. For smooth mapping $f : N \to \mathbb{R}^n$ such that $f|_{\partial N} : \partial N \to \mathbb{R}^n \setminus \{0\}$, by $\deg f|_{\partial N}$ or $\deg(f, N, 0)$ we denote the topological degree of mapping $f/|f| : \partial N \to S^{n-1}$. Note that if $f^{-1}(0)$ is a finite set then

$$\deg f|_{\partial N} = \sum_{p \in f^{-1}(0)} \deg_p f,$$

where $\deg_p f$ stands for the local topological degree of f at p (see [8]).

Let M be a m-dimensional manifold and m be odd. Take a smooth mapping $f : M \to \mathbb{R}^{2m-1}$ and let $p \in M$ be a cross-cap of f. According to [11], p is called positive (negative) if the vectors (2) determine the negative (positive) orientation of $\mathbb{R}^{2m-1}$. According to [11, Lemma 3], this definition does not depend on choosing the coordinate system on M.

Let us assume, that $f : \mathbb{R}^m \to \mathbb{R}^{2m-1}$ is a smooth mapping such that 0 is a cross-cap of f. Of course it is an isolated critical point of f. Denote by v_i the ith column of Df, for $i = 1, \ldots, m$. There exists $r > 0$ such that $v_1(x), \ldots, v_m(x)$ are linearly independent for $x \in \bar{B}^m(r) \setminus \{0\}$. Following [6] we can define

$$\begin{aligned}\widetilde{\alpha}(\beta, x) &= \beta_1 v_1(x) + \ldots + \beta_m v_m(x) \\ &= Df(x)(\beta) : S^{m-1} \times \bar{B}^m(r) \to \mathbb{R}^{2m-1}.\end{aligned}$$

Then the topological degree of the mapping

$$\widetilde{\alpha}|_{S^{m-1} \times S^{m-1}(r)} : S^{m-1} \times S^{m-1}(r) \to \mathbb{R}^{2m-1} \setminus \{0\}$$

is well defined. By [6, Proposition 2.4], $\deg(\widetilde{\alpha}|_{S^{m-1} \times S^{m-1}(r)})$ is even.

Theorem 1. *Let m be odd. If 0 is a cross-cap of a mapping $f : \mathbb{R}^m \to \mathbb{R}^{2m-1}$, then it is positive if and only if $\frac{1}{2}\deg(\widetilde{\alpha}|_{S^{m-1}\times S^{m-1}(r)}) = -1$, and so it is negative if and only if $\frac{1}{2}\deg(\widetilde{\alpha}|_{S^{m-1}\times S^{m-1}(r)}) = +1$.*

Proof. We can find linear coordinate system $\phi : \mathbb{R}^m \to \mathbb{R}^m$, such that $\phi(0) = 0$ and $f \circ \phi$ fulfills condition (1) at 0. Denote by A the matrix of ϕ. Let $w_1, \ldots, w_m$ denote columns of $D(f \circ \phi)$. Then $w_1(0) = 0$ and since 0 is a cross-cap then vectors

$$\frac{\partial w_1}{\partial x_1}(0), \quad w_2(0), \quad \ldots, \quad w_m(0), \quad \frac{\partial w_1}{\partial x_2}(0), \quad \ldots, \quad \frac{\partial w_1}{\partial x_m}(0) \qquad (3)$$

are linearly independent. Put $\widetilde{\gamma}(\beta, x) = (\beta_1 w_1(x) + \ldots + \beta_m w_m(x)) : S^{m-1} \times \bar{B}^m(r) \to \mathbb{R}^{2m-1}$. We can assume that r is such that $\widetilde{\gamma} \neq 0$ on $S^{m-1} \times \bar{B}^m(r) \setminus \{0\}$. Let us see that

$$\widetilde{\gamma}(\beta, x) = D(f \circ \phi)(x) \cdot \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = Df(\phi(x)) \cdot A \cdot \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$$
$$= Df(\phi(x)) \cdot \begin{bmatrix} \phi_1(\beta) \\ \vdots \\ \phi_m(\beta) \end{bmatrix}.$$

So $\widetilde{\gamma} = \widetilde{\alpha} \circ (\phi \times \phi)$. It is easy to see that $\phi \times \phi$ preserve the orientation of $S^{m-1} \times S^{m-1}(r)$. We can assume that $r > 0$ is so small, that $\deg(\widetilde{\alpha}|_{S^{m-1}\times S^{m-1}(r)}) = \deg(\widetilde{\alpha}|_{\phi(S^{m-1})\times\phi(S^{m-1}(r))})$. So we get that

$$\deg(\widetilde{\gamma}|_{S^{m-1}\times S^{m-1}(r)}) = \deg(\widetilde{\alpha}|_{\phi(S^{m-1})\times\phi(S^{m-1}(r))}) \deg(\phi \times \phi) =$$
$$= \deg(\widetilde{\alpha}|_{S^{m-1}\times S^{m-1}(r)}).$$

Since $f \circ \phi$ fulfils (1), vectors $w_2, \ldots, w_m$ are independent on $\bar{B}^m(r)$. Let us see that $\widetilde{\gamma}(\beta, x) = 0$ on $S^{m-1} \times \bar{B}^m(r)$ if and only if $x = 0$ and $\beta = (\pm 1, 0, \ldots, 0)$. So $\deg(\widetilde{\gamma}|_{S^{m-1}\times S^{m-1}(r)})$ is a sum of local topological degrees of $\widetilde{\gamma}$ at $(1, 0, \ldots, 0; 0, \ldots, 0)$ and at $(-1, 0, \ldots, 0; 0, \ldots, 0)$.

Near the point $(1, 0, \ldots, 0; 0, \ldots, 0)$ the well-oriented parametrisation of $S^{m-1} \times \bar{B}^m(r)$ is given by

$$(\beta_2, \ldots, \beta_m; x) = (\sqrt{1 - \beta_2^2 - \ldots - \beta_m^2}, \beta_2, \ldots, \beta_m; x).$$

And then the derivative matrix of $\widetilde{\gamma}$ at $(1, 0, \ldots, 0; 0, \ldots, 0)$ has a form

$$A_1 = \begin{bmatrix} w_2(0) & \ldots & w_m(0) & \frac{\partial w_1}{\partial x_1}(0) & \ldots & \frac{\partial w_1}{\partial x_m}(0) \end{bmatrix}.$$

Near $(-1,0,\ldots,0;0,\ldots,0)$ the well-oriented parametrisation of $S^{m-1}\times\bar{B}^m(r)$ is given by

$$(\beta_2,\ldots,\beta_m;x)=(-\sqrt{1-\beta_2^2-\ldots-\beta_m^2},-\beta_2,\ldots,\beta_m;x).$$

And then the derivative matrix of $\widetilde{\gamma}$ at $(-1,0,\ldots,0;0,\ldots,0)$ has a form

$$A_2=\begin{bmatrix}-w_2(0) & \ldots & w_m(0) & -\frac{\partial w_1}{\partial x_1}(0) & \ldots & -\frac{\partial w_1}{\partial x_m}(0)\end{bmatrix}.$$

Let us recall that m is odd. System of vectors (3) is independent, so 0 is a regular value of $\widetilde{\gamma}$, and

$$\frac{1}{2}\deg(\widetilde{\gamma}|_{S^{m-1}\times S^{m-1}(r)})=\frac{1}{2}(\operatorname{sgn}\det A_1+\operatorname{sgn}\det A_2)=\operatorname{sgn}\det A_1.$$

Moreover 0 is a positive cross-cap if and only if vectors (3) determine negative orientation of a $\mathbb{R}^{2m-1}$, i. e. if and only if $\frac{1}{2}\deg(\widetilde{\alpha}|_{S^{m-1}\times S^{m-1}(r)})=-1$. □

Let $U\subset\mathbb{R}^m$ be an open bounded set and $f:\overline{U}\to\mathbb{R}^{2m-1}$ be smooth. We say that f is *generic* if only critical points of f are cross-caps and f is regular in the neighborhood of ∂U. Let us denote by $\zeta(f)$ the algebraic sum of cross-caps of f. Then using Theorem 1 we get the following.

Proposition 1. *Let $U\subset\mathbb{R}^m$, (m is odd), be a bounded m-dimensional manifold such that $\overline{U}$ is an m-dimensional manifold with a boundary. For $f:\overline{U}\subset\mathbb{R}^m\to\mathbb{R}^{2m-1}$ generic, $\zeta(f)=-\frac{1}{2}\deg(\widetilde{\alpha})$, where $\widetilde{\alpha}(\beta,x)=Df(x)(\beta):S^{m-1}\times\partial U\to\mathbb{R}^{2m-1}\setminus\{0\}$.*

Proposition 2. *Let $U\subset\mathbb{R}^m$, (m is odd), be a bounded m-dimensional manifold such that $\overline{U}$ is an m-dimensional manifold with a boundary. Take $h:\overline{U}\subset\mathbb{R}^m\to\mathbb{R}^{2m-1}$ a smooth mapping such that h is regular in a neighborhood of ∂U. Then for every generic $f:\overline{U}\subset\mathbb{R}^m\to\mathbb{R}^{2m-1}$ close enough to h in C^1-topology we have, $\zeta(f)=-\frac{1}{2}\deg(\widetilde{\alpha})$, where $\widetilde{\alpha}(\beta,x)=Dh(x)(\beta):S^{m-1}\times\partial U\to\mathbb{R}^{2m-1}\setminus\{0\}$.*

3. Examples

To compute some examples we want first to recall the theory presented in [6].

Take $\alpha=(\alpha_1,\ldots,\alpha_k):\mathbb{R}^{n-k+1}\to M(n,k)$ a polynomial mapping, $n-k$ even, where $M(n,k)$ is a space of real matrices of dimension $n\times k$.

By $[a_{ij}(x)]$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k$, we denote the matrix given by $\alpha(x)$ (i.e. $\alpha_j(x)$ stands in the jth column). Then one can define $\widetilde{\alpha} : \mathbb{R}^k \times \mathbb{R}^{n-k+1} \to \mathbb{R}^n$ as

$$\widetilde{\alpha}(\beta, x) = \beta_1\alpha_1(x) + \ldots + \beta_k\alpha_k(x) = [a_{ij}(x)] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}.$$

Let I be the ideal in $\mathbb{R}[x_1, \ldots, x_{n-k+1}]$ generated by all $k \times k$ minors of $[a_{ij}(x)]$, and $V(I) = \{x \in \mathbb{R}^{n-k+1} \mid h(x) = 0 \text{ for all } h \in I\}$.

Take

$$m(x) = \det \begin{bmatrix} a_{12}(x) & \ldots & a_{1k}(x) \\ \\ a_{k-1,2}(x) & \ldots & a_{k-1,k}(x) \end{bmatrix}.$$

For $k \leqslant i \leqslant n$, we define

$$\Delta_i(x) = \det \begin{bmatrix} a_{11}(x) & \ldots & a_{1k}(x) \\ & \ldots & \\ a_{k-1,1}(x) & \ldots & a_{k-1,k}(x) \\ a_{i1}(x) & \ldots & a_{ik}(x) \end{bmatrix}.$$

Put $\mathcal{A} = \mathbb{R}[x_1, \ldots, x_{n-k+1}]/I$. Let us assume that $\dim \mathcal{A} < \infty$, so that $V(I)$ is finite. For $h \in \mathcal{A}$, we denote by $T(h)$ the trace of the linear endomorphism $\mathcal{A} \ni a \mapsto h \cdot a \in \mathcal{A}$. Then $T : \mathcal{A} \to \mathbb{R}$ is a linear functional.

Let $u \in \mathbb{R}[x_1, \ldots, x_{n-k+1}]$. Assume that $\overline{U} = \{x \mid u(x) \geqslant 0\}$ is bounded and $\nabla u(x) \neq 0$ at each $x \in u^{-1}(0) = \partial U$. Then $\overline{U}$ is a compact manifold with boundary, and $\dim \overline{U} = n - k + 1$.

Put $\delta = \partial(\Delta_k, \ldots, \Delta_n)/\partial(x_1, \ldots, x_{n-k+1})$. With u and δ we associate quadratic forms Θ_δ, $\Theta_{u\cdot\delta} : \mathcal{A} \to \mathbb{R}$ given by $\Theta_\delta(a) = T(\delta \cdot a^2)$ and $\Theta_{u\cdot\delta}(a) = T(u \cdot \delta \cdot a^2)$.

Theorem 2. *[6, Theorem 3.3] If $n - k$ is even, $\alpha = (\alpha_1, \ldots, \alpha_k) : \mathbb{R}^{n-k+1} \to M(n, k)$ is a polynomial mapping such that* $\dim \mathcal{A} < \infty$, *$I + \langle m \rangle = \mathbb{R}[x_1, \ldots, x_{n-k+1}]$ and quadratic forms $\Theta_\delta, \Theta_{u\cdot\delta} : \mathcal{A} \to \mathbb{R}$ are non-degenerate, then the restricted mapping $\alpha|_{\partial U}$ goes into $\widetilde{V}_k(\mathbb{R}^n)$ and*

$$\Lambda(\alpha|_{\partial U}) = \frac{1}{2} \deg(\widetilde{\alpha}|_{S^{k-1} \times \partial U}) = \frac{1}{2}(\text{signature}\, \Theta_\delta + \text{signature}\, \Theta_{u\cdot\delta}),$$

where $\widetilde{\alpha}(\beta, x) = \beta_1\alpha_1(x) + \ldots + \beta_k\alpha_k(x)$.

Using the theory presented in [6], particularly [6, Theorem 3.3], and computer system SINGULAR ([2]), one can apply the results from Sections 1 and 2 to compute algebraic sum of cross-caps for polynomial mappings.

Example 1. Let us take $f : \mathbb{R}^3 \to \mathbb{R}^5$ given by

$$f(x, y, z) = (12y^2 + z, 6x^2 + y^2 + 6y, 18xy + 13y^2 + 9x,$$
$$8x^2z + 10xz^2 + 5x^2 + 3xz, x^2y + 4xyz + yz + 4z^2).$$

Applying Lemma 2 and using Singular one can check that f is 1-generic. Moreover, according to Proposition 1 and [6], one can check that

$$\zeta(f|_{\bar{B}^3(\sqrt{3})}) = 2, \quad \zeta(f|_{\bar{B}^3(10)}) = 1.$$

We can also check that f has 11 cross-caps in $\mathbb{R}^3$, 6 of them are positive, 5 negative.

Example 2. Take $f : \mathbb{R}^5 \to \mathbb{R}^9$ given by

$$f(s, t, x, y, z) = (y, z, t, 20x^2 + 17sz + x, 13sy + 13sz + 5t, 25st + 4x^2 + 28z,$$
$$3x^2 + 19yz + 22s, 11ts^2 + 8t^2z + xz, 27txy + 9sxz + 20st).$$

One may check that f is 1-generic, has 3 cross-caps in $\mathbb{R}^5$ and

$$\zeta(f|_{\bar{B}^3(1/10)}) = 0, \quad \zeta(f|_{\bar{B}^3(2)}) = -1, \quad \zeta(f|_{\bar{B}^3(1000)}) = 1.$$

4. Intersection number of immersions

Take n-dimensional, compact, oriented manifold N and immersion $g : N \to \mathbb{R}^{2n}$. As in [10] we say that an immersion $g\colon N \to \mathbb{R}^{2n}$ has a *regular self-intersection* at the point $g(p) = g(q)$ if

$$Dg(p)T_pN + Dg(q)T_qN = \mathbb{R}^{2n}.$$

An immersion $g\colon N \to \mathbb{R}^{2n}$ is called *completely regular* if it has only regular self-intersections and no triple points.

Assume that n is **even**. Let $g\colon N \to \mathbb{R}^{2n}$ be a completely regular immersion having a regular self-intersection at the point $g(p) = g(q)$. Let $u_1, \dots, u_n \in T_pN$, $v_1, \dots, v_n \in T_qN$ be sets of well-oriented, independent vectors in respective tangent spaces of N. Then the vectors $Dg(p)u_1, \dots, Dg(p)u_n$, $Dg(q)v_1, \dots, Dg(q)v_n$ form a basis in $\mathbb{R}^{2n}$. As in [10] we will say that the self-intersection at the point $g(p) = g(q)$ is *positive* or *negative* according to whether this basis determines the positive or negative orientation of $\mathbb{R}^{2n}$.

Following [10], *the intersection number* $I(g)$ of a completely regular immersion g is the algebraic sum of its self-intersections. For any immersion

$g: N \to \mathbb{R}^{2n}$ *the intersection number* $I(g)$ is defined as the intersection number of a completely regular immersion $\tilde{g}$, regularly homotopic to g (homotopy by immersions). For other equivalent description of $I(g)$ see [7], [4].

As in previous Sections we assume that m is odd. Take a smooth map $g = (g_1, \ldots, g_{2m-2}) : \mathbb{R}^m \to \mathbb{R}^{2m-2}$. Denote by $\omega = x_1^2 + \ldots + x_m^2$. Then $S^{m-1}(r) = \{x|\ \omega(x) = r^2\}$. According to [4, Lemma 18], $g|_{S^{m-1}(r)}$ is an immersion if and only if

$$\operatorname{rank}\begin{bmatrix} 2x_1 & \ldots & 2x_m \\ \frac{\partial g_1}{\partial x_1}(x) & \ldots & \frac{\partial g_1}{\partial x_m}(x) \\ & \ldots & \\ \frac{\partial g_{2m-2}}{\partial x_1}(x) & \ldots & \frac{\partial g_{2m-2}}{\partial x_m}(x) \end{bmatrix} = m,$$

for $x \in S^{m-1}(r)$.

Take $0 < r_1 < r_2$, such that $g|_{S^{m-1}(r_1)}$ and $g|_{S^{m-1}(r_2)}$ are immersions. Denote by $P = \{x|\ r_1^2 \leqslant w(x) \leqslant r_2^2\}$. Then P is an m-dimensional oriented manifold with boundary. Then $(\omega, g) : \mathbb{R}^m \to \mathbb{R}^{2m-1}$ is a regular map in the neighbourhood of ∂P. Let us define $\widetilde{\alpha} : S^{m-1} \times P \to \mathbb{R}^{2m-1}$ as

$$\widetilde{\alpha}(\beta, x) = \begin{bmatrix} 2x_1 & \ldots & 2x_m \\ \frac{\partial g_1}{\partial x_1}(x) & \ldots & \frac{\partial g_1}{\partial x_m}(x) \\ & \ldots & \\ \frac{\partial g_{2m-2}}{\partial x_1}(x) & \ldots & \frac{\partial g_{2m-2}}{\partial x_m}(x) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$$

Proposition 3. *Let us assume that* $g|_{S^{m-1}(r_1)}$ *and* $g|_{S^{m-1}(r_2)}$ *are immersions, then*

$$I(g|_{S^{m-1}(r_2)}) - I(g|_{S^{m-1}(r_1)}) = \zeta((\omega, g)|_P).$$

Proof. Let us recall that m is odd. Then

$$\deg(\widetilde{\alpha}|_{S^{m-1}\times\partial P}) = \deg(\widetilde{\alpha}|_{S^{m-1}\times S^{m-1}(r_2)}) - \deg(\widetilde{\alpha}|_{S^{m-1}\times S^{m-1}(r_1)}).$$

According to [6, Theorem 4.2], we get that

$$\deg(\widetilde{\alpha}|_{S^{m-1}\times S^{m-1}(r_i)}) = I(g|_{S^{m-1}(r_i)}),$$

for $i = 1, 2$. Then applying Proposition 2 we get that $\zeta((\omega, g)|_P) = -\frac{1}{2}\deg(\widetilde{\alpha}|_{S^{m-1}\times\partial P})$. And so

$$\zeta((\omega, g)|_P) = I(g|_{S^{m-1}(r_2)}) - I(g|_{S^{m-1}(r_1)}). \qquad \square$$

Corollary 1. *If* $g|_{S^{m-1}(r)}$ *is an immersion, then*

$$I(g|_{S^{m-1}(r)}) = \zeta((\omega, g)|_{\bar{B}^m(r)}).$$

Remark 1. If the only singular points of $(\omega, g)|_{\bar{B}^m(r)}$ are cross-caps, then the intersection number of an immersion $g|_{S^{m-1}(r)}$ is equal to the algebraic sum of cross-caps of $(\omega, g)|_{\bar{B}^m(r)}$. Also, in generic case,the difference between intersection numbers of immersions $g|_{S^{m-1}(r_1)}$ and $g|_{S^{m-1}(r_2)}$, is equal to the algebraic sum of cross-caps of (ω, g) appearing in P.

Remark 2. If (ω, g) has finite number of singular points, and all of them are cross-caps, then for any $R > 0$ big enough, $g|_{S^{m-1}(R)}$ is an immersion with the same intersection number equal to the algebraic sum of cross-caps of (ω, g).

Example 3. Take $g : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$g = (-3y^2 + 5yz - x + 2, -4x^2 + z^2 + 9y - 6z + 5,$$

$$4x^2z - 2x^2 + 2xy - y - 3, 3y^2z + xy - 4yz + 4x - 5y - 5),$$

and $\omega = x^2 + y^2 + z^2$. In the same way as in Section 3, one may check that the only singular points of (ω, g) are cros-s-caps, moreover (ω, g) has 8 cross-caps, 5 of them are positive and 3 negative. According to previous results $g|_{S^2(r)}$ is an immersion for all $r > 0$, except at most 8 values of r. And if $g|_{S^2(r)}$ is an immersion, then

$$-3 \leqslant I(g|_{S^2(r)}) \leqslant 5.$$

Moreover for $R > 0$ big enough $g|_{S^2(R)}$ is an immersion with

$$I(g|_{S^2(R)}) = 2.$$

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