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RESEARCH ARTICLE

Free n-dinilpotent doppelsemigroups

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Abstract. A doppelalgebra is an algebra defined on a vector space with two binary linear associative operations. Doppelalgebras play a prominent role in algebraic K-theory. In this paper we consider doppelsemigroups, that is, sets with two binary associative operations satisfying the axioms of a doppelalgebra. We construct a free n-dinilpotent doppelsemigroup and study separately free n-dinilpotent doppelsemigroups of rank 1. Moreover, we characterize the least n-dinilpotent congruence on a free doppelsemigroup, establish that the semigroups of the free n-dinilpotent doppelsemigroup are isomorphic and the automorphism group of the free n-dinilpotent doppelsemigroup is isomorphic to the symmetric group. We also give different examples of doppelsemigroups and prove that a system of axioms of a doppelsemigroup is independent.

1. Introduction

The notion of a doppelalgebra was considered by Richter [1] in the context of algebraic K-theory. She defined this notion as a vector space over a field equipped with two binary linear associative operations $\dashv$ and $\vdash$ satisfying the axioms $(x \dashv y) \vdash z = x \dashv (y \vdash z)$, $(x \vdash y) \dashv z = x \vdash (y \dashv z)$. Observe that any doppelalgebra gives rise to a Lie algebra by $[x, y] = x \vdash y + x \dashv y - y \vdash x - y \dashv x$ and conversely, any

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Lie algebra has a universal enveloping doppelalgebra (see [1]). Moreover, for any doppelalgebra a new operation $\cdot$ defined by $x \cdot y = x \vdash y + x \dashv y$ is associative and so, there exists a functor from the category of doppelalgebras to the category of associative algebras. Later Pirashvili [2] considered duplexes which are sets equipped with two binary associative operations and constructed a free duplex of an arbitrary rank. He also considered duplexes with operations satisfying the axioms of a doppelalgebra denoting obtained category by Duplexes$_2$. Such algebraic structures are called doppelsemigroups [3]. A free doppelsemigroup of rank 1 is given in [1] (see also [2]). Operations of the free doppelsemigroup of rank 1 are used in [4]. Doppelalgebras appeared in [5] as algebras over some operad. Doppelalgebras and doppelsemigroups have relationships with interassociativity for semigroups originated by Drouzy [6] and investigated in [7–11], strong interassociativity for semigroups introduced by Gould and Richardson [12] and dimonoids introduced by Loday [13] (see also [14–18], [20–22]). Doppelsemigroups are a generalization of semigroups and all results obtained for doppelsemigroups can be applied to doppelalgebras. For further details and background see [1].

The free product of doppelsemigroups, the free doppelsemigroup, the free commutative doppelsemigroup and the free n-nilpotent doppelsemigroup were constructed in [3]. The paper [17] gives a classification of relatively free dimonoids, in particular, therein the free n-dinilpotent dimonoid [18] is presented. In this paper we continue researches from [3, 18] developing the variety theory of doppelsemigroups. The main focus of our paper is to study dinilpotent doppelsemigroups.

In Section 3 we present different examples of doppelsemigroups.

In Section 4 we prove that a system of axioms of a doppelsemigroup is independent.

In Section 5 we construct a free n-dinilpotent doppelsemigroup of an arbitrary rank and consider separately free n-dinilpotent doppelsemigroups of rank 1. We also establish that the semigroups of the free n-dinilpotent doppelsemigroup are isomorphic and the automorphism group of the free n-dinilpotent doppelsemigroup is isomorphic to the symmetric group.

In the final section we characterize the least n-dinilpotent congruence on a free doppelsemigroup.

2. Preliminaries

Recall that a doppelalgebra [1, 2] is a vector space V over a field equipped with two binary linear operations $\dashv$ and $\vdash$: $V \otimes V \to V$, satisfying the axioms

$$(x \dashv y) \vdash z = x \dashv (y \vdash z), \tag{D1}$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z), \tag{D2}$$
$$(x \dashv y) \dashv z = x \dashv (y \dashv z), \tag{D3}$$
$$(x \vdash y) \vdash z = x \vdash (y \vdash z). \tag{D4}$$

A nonempty set with two binary operations $\dashv$ and $\vdash$ satisfying the axioms (D1)–(D4) is called a doppelsemigroup [3].

Given a semigroup $(D, \dashv)$, consider a semigroup $(D, \vdash)$ defined on the same set. Recall that $(D, \vdash)$ is an interassociate of $(D, \dashv)$ [6], if the axioms (D1) and (D2) hold. Strong interassociativity [12] is defined by the axioms (D1) and (D2) along with

$$x \vdash (y \dashv z) = x \dashv (y \vdash z). \tag{D5}$$

Thus, we can see that in any doppelsemigroup $(D, \dashv, \vdash)$, $(D, \vdash)$ is an interassociate of $(D, \dashv)$, and conversely, if a semigroup $(D, \vdash)$ is an interassociate of a semigroup $(D, \dashv)$, then $(D, \dashv, \vdash)$ is a doppelsemigroup [3]. Moreover, a semigroup $(D, \vdash)$ is a strong interassociate of a semigroup $(D, \dashv)$ if and only if $(D, \dashv, \vdash)$ is a doppelsemigroup satisfying the axiom (D5).

Descriptions of all interassociates of a monogenic semigroup and of the free commutative semigroup are presented in [7] and [8, 10], respectively. More recently, the paper [11] was devoted to studying interassociates of the bicyclic semigroup. Methods of constructing interassociates for semigroups were developed in [19].

Recall the definition of a k-nilpotent semigroup (see also [14, 17, 18]). As usual, $\mathbb{N}$ denotes the set of all positive integers. A semigroup S is called nilpotent, if $S^{n+1} = 0$ for some $n \in \mathbb{N}$. The least such n is called the nilpotency index of S. For $k \in \mathbb{N}$ a nilpotent semigroup of nilpotency index $\leqslant k$ is called k-nilpotent.

An element 0 of a doppelsemigroup $(D, \dashv, \vdash)$ is called zero [3], if $x * 0 = 0 = 0 * x$ for all $x \in D$ and $* \in \{\dashv, \vdash\}$. A doppelsemigroup $(D, \dashv, \vdash)$ with zero will be called dinilpotent, if $(D, \dashv)$ and $(D, \vdash)$ are nilpotent semigroups. A dinilpotent doppelsemigroup $(D, \dashv, \vdash)$ will be called n-dinilpotent, if $(D, \dashv)$ and $(D, \vdash)$ are n-nilpotent semigroups. If ρ is a congruence on a doppelsemigroup $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is an n-dinilpotent doppelsemigroup, we say that ρ is an n-dinilpotent congruence.

Note that operations of any 1-dinilpotent doppelsemigroup coincide and it is a zero semigroup. The class of all n-dinilpotent doppelsemigroups forms a subvariety of the variety of doppelsemigroups. It is not difficult to check that the variety of n-nilpotent doppelsemigroups [3] is a subvariety of the variety of n-dinilpotent doppelsemigroups. A doppelsemigroup which is free in the variety of n-dinilpotent doppelsemigroups will be called a free n-dinilpotent doppelsemigroup.

Lemma 1 ([3], Lemma 3.1)**.** *In a doppelsemigroup* $(D, \dashv, \vdash)$ *for any* $n > 1, n \in \mathbb{N}$*, and any* $x_i \in D$*,* $1 \leqslant i \leqslant n+1$*, and* $*_j \in \{\dashv, \vdash\}$*,* $1 \leqslant j \leqslant n$*, any parenthesizing of*

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1}$$

gives the same element from D*.*

The free doppelsemigroup is given in [3]. Recall this construction.

Let X be an arbitrary nonempty set and let ω be an arbitrary word in the alphabet X. The length of ω will be denoted by l_ω. Let further $F[X]$ be the free semigroup on X, T the free monoid on the two-element set $\{a, b\}$ and $\theta \in T$ the empty word. By definition, the length l_θ of θ is equal to 0. Define operations $\dashv$ and $\vdash$ on $F = \{(w, u) \in F[X] \times T \,|\, l_w - l_u = 1\}$ by

$$(w_1, u_1) \dashv (w_2, u_2) = (w_1 w_2, u_1 a u_2),$$
$$(w_1, u_1) \vdash (w_2, u_2) = (w_1 w_2, u_1 b u_2)$$

for all $(w_1, u_1), (w_2, u_2) \in F$. The algebra $(F, \dashv, \vdash)$ is denoted by $\mathrm{FDS}(X)$.

Theorem 1 ([3], Theorem 3.5)**.** $\mathrm{FDS}(X)$ *is the free doppelsemigroup.*

If $f : D_1 \to D_2$ is a homomorphism of doppelsemigroups, the corresponding congruence on D_1 will be denoted by Δ_f. Denote the symmetric group on X by $\Im[X]$ and the automorphism group of a doppelsemigroup M by $\mathrm{Aut}\, M$.

3. Some examples

In this section we give different examples of doppelsemigroups.

a) Every semigroup can be considered as a doppelsemigroup (see [3]).

b) Recall that a dimonoid [13–18, 20–22] is a nonempty set equipped with two binary operations $\dashv$ and $\vdash$ satisfying the axioms (D2)–(D4) and

$$(x \dashv y) \dashv z = x \dashv (y \vdash z),$$
$$(x \dashv y) \vdash z = x \vdash (y \vdash z).$$

A dimonoid is called commutative [20], if both its operations are commutative. The following assertion gives relationships between commutative dimonoids and doppelsemigroups (this assertion was formulated without the proof in [3] and [16]).

Proposition 1. *Every commutative dimonoid is a doppelsemigroup.*

Proof. Let $(D, \dashv, \vdash)$ be a commutative dimonoid. Then, by definition, $(D, \dashv, \vdash)$ satisfies the axioms (D2)–(D4). From Lemma 2 of [20] it follows that $(D, \dashv, \vdash)$ satisfies the axiom (D1). So, it is a doppelsemigroup. □

Examples of commutative dimonoids can be found in [14, 20].

c) Let $(D, \dashv, \vdash)$ be a doppelsemigroup and $a, b \in D$. Define operations $\dashv_a$ and $\vdash_b$ on D by

$$x \dashv_a y = x \dashv a \dashv y, \qquad x \vdash_b y = x \vdash b \vdash y$$

for all $x, y \in D$. By a direct verification $(D, \dashv_a, \vdash_b)$ is a doppelsemigroup. We call the doppelsemigroup $(D, \dashv_a, \vdash_b)$ a variant of $(D, \dashv, \vdash)$, or, alternatively, the sandwich doppelsemigroup of $(D, \dashv, \vdash)$ with respect to the sandwich elements a and b, or the doppelsemigroup with deformed multiplications.

d) The direct product $\prod_{i\in I} D_i$ of doppelsemigroups D_i, $i \in I$, is, obviously, a doppelsemigroup.

e) Now we give a new class of doppelsemigroups with zero.

Let $\overline{D} = (D, \dashv, \vdash)$ be an arbitrary doppelsemigroup and I an arbitrary nonempty set. Define operations $\dashv'$ and $\vdash'$ on $D' = (I \times D \times I) \cup \{0\}$ by

$$(i, a, j) *' (k, b, t) = \begin{cases} (i, a * b, t), & j = k, \\ 0, & j \neq k, \end{cases}$$

$$(i, a, j) *' 0 = 0 *' (i, a, j) = 0 *' 0 = 0$$

for all $(i, a, j), (k, b, t) \in D' \backslash \{0\}$ and $* \in \{\dashv, \vdash\}$. The algebra $(D', \dashv', \vdash')$ will be denoted by $B(\overline{D}, I)$.

Proposition 2. *$B(\overline{D}, I)$ is a doppelsemigroup with zero.*

Proof. The proof is similar to the proof of Proposition 1 from [21]. □

Observe that if operations of a doppelsemigroup $\overline{D}$ coincide and it is a group G, then any Brandt semigroup [23] is isomorphic to some semigroup $B(G, I)$. So, $B(\overline{D}, I)$ generalizes the semigroup $B(G, I)$. We call the doppelsemigroup $B(\overline{D}, I)$ a Brandt doppelsemigroup.

4. Independence of axioms of a doppelsemigroup

In this section for a doppelsemigroup we prove the following theorem.

Theorem 2. *A system of axioms* (D1)–(D4) *as defined above is independent.*

Proof. Let X be an arbitrary nonempty set, $|X| > 1$. Define operations $\dashv$ and $\vdash$ on X by

$$x \dashv y = x, \quad x \vdash y = y$$

for all $x, y \in X$. The model $(X, \dashv, \vdash)$ satisfies the axioms (D2)–(D4) but does not satisfy (D1). Indeed, for all $x, y, z \in X$,

$$(x \vdash y) \dashv z = y = x \vdash (y \dashv z),$$
$$(x \dashv y) \dashv z = x = x \dashv (y \dashv z),$$
$$(x \vdash y) \vdash z = z = x \vdash (y \vdash z).$$

Since $|X| > 1$, there is $x, z \in X$ such that $x \neq z$. Consequently, for all $y \in X$,

$$(x \dashv y) \vdash z = z \neq x = x \dashv (y \vdash z).$$

Put

$$x \dashv y = y, \quad x \vdash y = x$$

for all $x, y \in X$. As in the previous case, we can show that $(X, \dashv, \vdash)$ satisfies the axioms (D1), (D3), (D4) but does not satisfy (D2).

Let $\mathbb{N}^0$ be the set of all positive integers with zero and let

$$x \dashv y = 2x, \quad z \dashv 0 = 0 = 0 \dashv z, \quad z \vdash c = 0$$

for all $x, y \in \mathbb{N}$ and $z, c \in \mathbb{N}^0$. In this case the model $(\mathbb{N}^0, \dashv, \vdash)$ satisfies the axioms (D1), (D2), (D4) but does not satisfy (D3). Indeed, for all $z, c, a \in \mathbb{N}^0$,

$$(z \dashv c) \vdash a = 0 = z \dashv (c \vdash a),$$
$$(z \vdash c) \dashv a = 0 = z \vdash (c \dashv a),$$
$$(z \vdash c) \vdash a = 0 = z \vdash (c \vdash a).$$

In addition, for all $x, y, b \in \mathbb{N}$ we get

$$(x \dashv y) \dashv b = 2x \dashv b = 4x \neq 2x = x \dashv 2y = x \dashv (y \dashv b).$$

Put

$$z \dashv c = 0, \quad x \vdash y = 2y, \quad z \vdash 0 = 0 = 0 \vdash z$$

for all $z, c \in \mathbb{N}^0$ and $x, y \in \mathbb{N}$. As in the previous case, we can show that $(\mathbb{N}^0, \dashv, \vdash)$ satisfies the axioms (D1)–(D3) but does not satisfy (D4). $\square$

5. Constructions

In this section we construct a free n-dinilpotent doppelsemigroup of an arbitrary rank and consider separately free n-dinilpotent doppelsemigroups of rank 1. We also establish that the semigroups of the free n-dinilpotent doppelsemigroup are isomorphic and the automorphism group of the free n-dinilpotent doppelsemigroup is isomorphic to the symmetric group.

As in Section 2, let $F[X]$ be the free semigroup on X, T the free monoid on the two-element set $\{a, b\}$ and $\theta \in T$ the empty word. For $x \in \{a, b\}$ and all $u \in T$, the number of occurrences of an element x in u is denoted by $d_x(u)$. Obviously, $d_x(\theta) = 0$. Fix $n \in \mathbb{N}$ and assume

$$M_n = \{(w, u) \in F[X] \times T \mid l_w - l_u = 1,\, d_x(u) + 1 \leqslant n, x \in \{a, b\}\} \cup \{0\}.$$

Define operations $\dashv$ and $\vdash$ on M_n by

$$(w_1, u_1) \dashv (w_2, u_2) = \begin{cases} (w_1w_2, u_1au_2), & d_x(u_1au_2) + 1 \leqslant n, x \in \{a, b\}, \\ 0, & \text{in all other cases,} \end{cases}$$

$$(w_1, u_1) \vdash (w_2, u_2) = \begin{cases} (w_1w_2, u_1bu_2), & d_x(u_1bu_2) + 1 \leqslant n, x \in \{a, b\}, \\ 0, & \text{in all other cases,} \end{cases}$$

$$(w_1, u_1) * 0 = 0 * (w_1, u_1) = 0 * 0 = 0$$

for all $(w_1, u_1), (w_2, u_2) \in M_n \backslash \{0\}$ and $* \in \{\dashv, \vdash\}$. The obtained algebra will be denoted by $\mathrm{FDDS}_n(X)$.

Theorem 3. $\mathrm{FDDS}_n(X)$ *is the free n-dinilpotent doppelsemigroup.*

Proof. First prove that $\mathrm{FDDS}_n(X)$ is a doppelsemigroup. Let (w_1, u_1), (w_2, u_2), $(w_3, u_3) \in M_n \backslash \{0\}$. For $x, y, z \in \{a, b\}$ it is clear that

$$d_x(u_1yu_2zu_3) + 1 \leqslant n$$

implies

$$d_x(u_1yu_2) + 1 \leqslant n, \tag{1}$$

$$d_x(u_2zu_3) + 1 \leqslant n. \tag{2}$$

Let $d_x(u_1au_2au_3) + 1 \leqslant n$ for all $x \in \{a, b\}$. Then, using (1), (2), we get

$$\begin{aligned} ((w_1, u_1) \dashv (w_2, u_2)) \dashv (w_3, u_3) &= (w_1w_2, u_1au_2) \dashv (w_3, u_3) \\ &= (w_1w_2w_3, u_1au_2au_3) \\ &= (w_1, u_1) \dashv (w_2w_3, u_2au_3) \\ &= (w_1, u_1) \dashv ((w_2, u_2) \dashv (w_3, u_3)). \end{aligned}$$

If $d_x(u_1au_2au_3)+1>n$ for some $x\in\{a,b\}$, then, obviously,

$$((w_1,u_1)\dashv(w_2,u_2))\dashv(w_3,u_3)=0=(w_1,u_1)\dashv((w_2,u_2)\dashv(w_3,u_3)).$$

So, the axiom (D3) of a doppelsemigroup holds.

If $d_x(u_1au_2bu_3)+1\leqslant n$ for all $x\in\{a,b\}$, then, using (1), (2), obtain

$$\begin{aligned}((w_1,u_1)\dashv(w_2,u_2))\vdash(w_3,u_3)&=(w_1w_2,u_1au_2)\vdash(w_3,u_3)\\&=(w_1w_2w_3,u_1au_2bu_3)\\&=(w_1,u_1)\dashv(w_2w_3,u_2bu_3)\\&=(w_1,u_1)\dashv((w_2,u_2)\vdash(w_3,u_3)).\end{aligned}$$

Let $d_x(u_1au_2bu_3)+1>n$ for some $x\in\{a,b\}$. Then, clearly,

$$((w_1,u_1)\dashv(w_2,u_2))\vdash(w_3,u_3)=0=(w_1,u_1)\dashv((w_2,u_2)\vdash(w_3,u_3)).$$

Thus, the axiom (D1) of a doppelsemigroup holds. Similarly, one can check the axioms (D2) and (D4). Thus, $\mathrm{FDDS}_n(X)$ is a doppelsemigroup.

Take arbitrary elements $(w_i,u_i)\in M_n\backslash\{0\}$, $1\leqslant i\leqslant n+1$. It is clear that

$$d_a(u_1au_2a\ldots au_{n+1})+1>n.$$

From here

$$(w_1,u_1)\dashv(w_2,u_2)\dashv\ldots\dashv(w_{n+1},u_{n+1})=0.$$

At the same time, assuming $y^0=\theta$ for $y\in\{a,b\}$, for any $(x_i,\theta)\in M_n\backslash\{0\}$, where $x_i\in X$, $1\leqslant i\leqslant n$, get

$$(x_1,\theta)\dashv(x_2,\theta)\dashv\ldots\dashv(x_n,\theta)=(x_1x_2\ldots x_n,a^{n-1})\neq 0.$$

From the last arguments we conclude that $(M_n,\dashv)$ is a nilpotent semigroup of nilpotency index n. Analogously, we can prove that $(M_n,\vdash)$ is a nilpotent semigroup of nilpotency index n. So, by definition, $\mathrm{FDDS}_n(X)$ is an n-dinilpotent doppelsemigroup.

Let us show that $\mathrm{FDDS}_n(X)$ is free in the variety of n-dinilpotent doppelsemigroups.

Obviously, $\mathrm{FDDS}_n(X)$ is generated by $X\times\{\theta\}$. Let $(K,\dashv',\vdash')$ be an arbitrary n-dinilpotent doppelsemigroup. Let $\beta:X\times\{\theta\}\to K$ be an arbitrary map. Consider a map $\alpha:X\to K$ such that $x\alpha=(x,\theta)\beta$ for all $x\in X$ and define a map

$$\pi:\mathrm{FDDS}_n(X)\to(K,\dashv',\vdash')$$

by

$$\omega\pi = \begin{cases} x_1\alpha\widetilde{y}_1x_2\alpha\widetilde{y}_2\ldots\widetilde{y}_{s-1}x_s\alpha, & \text{if } \omega = (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1}), \\ & \quad x_d \in X, 1 \leqslant d \leqslant s, y_p \in \{a,b\}, \\ & \quad 1 \leqslant p \leqslant s-1, s > 1, \\ x_1\alpha, & \text{if } \omega = (x_1, \theta), x_1 \in X, \\ 0, & \text{if } \omega = 0, \end{cases}$$

where

$$\widetilde{y}_p = \begin{cases} \dashv', & y_p = a, \\ \vdash', & y_p = b \end{cases}$$

for all $1 \leqslant p \leqslant s-1$, $s > 1$. According to Lemma 1 π is well-defined.

To show that π is a homomorphism we will use the axioms of a doppelsemigroup and the identities of an n-dinilpotent doppelsemigroup.

If $s = 1$, we will regard the sequence $y_1y_2\ldots y_{s-1} \in T$ as θ. For arbitrary elements

$$(w_1, u_1) = (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1}),$$
$$(w_2, u_2) = (z_1z_2\ldots z_k, c_1c_2\ldots c_{k-1}) \in \mathrm{FDDS}_n(X),$$

where $x_d, z_i \in X$, $1 \leqslant d \leqslant s$, $1 \leqslant i \leqslant k$, $y_p, c_j \in \{a,b\}$, $1 \leqslant p \leqslant s-1$, $1 \leqslant j \leqslant k-1$, in the case $d_x(u_1au_2) + 1 \leqslant n$ for all $x \in \{a,b\}$, we get

$$\begin{aligned} &((x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1}) \dashv (z_1z_2\ldots z_k, c_1c_2\ldots c_{k-1}))\pi \\ &\quad = (x_1\ldots x_sz_1\ldots z_k, y_1\ldots y_{s-1}ac_1\ldots c_{k-1})\pi \\ &\quad = x_1\alpha\widetilde{y}_1\ldots\widetilde{y}_{s-1}x_s\alpha\widetilde{a}z_1\alpha\widetilde{c}_1\ldots\widetilde{c}_{k-1}z_k\alpha \\ &\quad = (x_1\alpha\widetilde{y}_1\ldots\widetilde{y}_{s-1}x_s\alpha) \dashv' (z_1\alpha\widetilde{c}_1\ldots\widetilde{c}_{k-1}z_k\alpha) \\ &\quad = (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1})\pi \dashv' (z_1z_2\ldots z_k, c_1c_2\ldots c_{k-1})\pi. \end{aligned}$$

If $d_x(u_1au_2) + 1 > n$ for some $x \in \{a,b\}$, then

$$((x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1}) \dashv (z_1z_2\ldots z_k, c_1c_2\ldots c_{k-1}))\pi = 0\pi = 0.$$

Since $(K, \dashv', \vdash')$ is n-dinilpotent, we have

$$\begin{aligned} 0 &= x_1\alpha\widetilde{y}_1\ldots\widetilde{y}_{s-1}x_s\alpha\widetilde{a}z_1\alpha\widetilde{c}_1\ldots\widetilde{c}_{k-1}z_k\alpha \\ &= (x_1\alpha\widetilde{y}_1\ldots\widetilde{y}_{s-1}x_s\alpha) \dashv' (z_1\alpha\widetilde{c}_1\ldots\widetilde{c}_{k-1}z_k\alpha) \\ &= (x_1x_2\ldots x_s, y_1y_2\ldots y_{s-1})\pi \dashv' (z_1z_2\ldots z_k, c_1c_2\ldots c_{k-1})\pi. \end{aligned}$$

So,

$$((w_1, u_1) \dashv (w_2, u_2))\pi = (w_1, u_1)\pi \dashv' (w_2, u_2)\pi$$

for all $(w_1, u_1), (w_2, u_2) \in \mathrm{FDDS}_n(X)$.

Similarly for $\vdash$. So, π is a homomorphism. Clearly, $(x, \theta)\pi = (x, \theta)\beta$ for all $(x, \theta) \in X \times \{\theta\}$. Since $X \times \{\theta\}$ generates $\mathrm{FDDS}_n(X)$, the uniqueness of such homomorphism π is obvious. Thus, $\mathrm{FDDS}_n(X)$ is free in the variety of n-dinilpotent doppelsemigroups. □

Now we construct a doppelsemigroup which is isomorphic to the free n-dinilpotent doppelsemigroup of rank 1.

Fix $n \in \mathbb{N}$ and assume

$$\overline{\Phi}_n = \{u \in T \ \mid \ d_x(u) + 1 \leqslant n, x \in \{a, b\}\} \cup \{0\}.$$

Define operations $\dashv$ and $\vdash$ on $\overline{\Phi}_n$ by

$$u_1 \dashv u_2 = \begin{cases} u_1 a u_2, & d_x(u_1 a u_2) + 1 \leqslant n, x \in \{a, b\}, \\ 0, & \text{in all other cases,} \end{cases}$$

$$u_1 \vdash u_2 = \begin{cases} u_1 b u_2, & d_x(u_1 b u_2) + 1 \leqslant n, x \in \{a, b\}, \\ 0, & \text{in all other cases,} \end{cases}$$

$$u_1 * 0 = 0 * u_1 = 0 * 0 = 0$$

for all $u_1, u_2 \in \overline{\Phi}_n \backslash \{0\}$ and $* \in \{\dashv, \vdash\}$. The obtained algebra will be denoted by Φ_n. Obviously, Φ_n is a doppelsemigroup.

Lemma 2. *If $|X| = 1$, then $\Phi_n \cong \mathrm{FDDS}_n(X)$.*

Proof. Let $X = \{r\}$. One can show that a map $\gamma : \Phi_n \to \mathrm{FDDS}_n(X)$, defined by the rule

$$u\gamma = \begin{cases} (r^{l_u + 1}, u), & u \in \overline{\Phi}_n \backslash \{0\}, \\ 0, & u = 0, \end{cases}$$

is an isomorphism. □

The following lemma establishes a relationship between semigroups of the free n-dinilpotent doppelsemigroup $\mathrm{FDDS}_n(X)$.

Lemma 3. *The semigroups $(M_n, \dashv)$ and $(M_n, \vdash)$ are isomorphic.*

Proof. Let $\widehat{a} = b$, $\widehat{b} = a$ and define a map $\sigma : (M_n, \dashv) \to (M_n, \vdash)$ by putting

$$t\sigma = \begin{cases} (w, \widehat{y_1}\widehat{y_2}\ldots\widehat{y_m}), & t = (w, y_1y_2\ldots y_m) \in M_n\backslash\{0\}, \\ & y_p \in \{a,b\},\ 1 \leqslant p \leqslant m, \\ t, & \text{in all other cases.} \end{cases}$$

An immediate verification shows that σ is an isomorphism.

□

Since the set $X \times \{\theta\}$ is generating for $\mathrm{FDDS}_n(X)$, we obtain the following description of the automorphism group of the free n-dinilpotent doppelsemigroup.

Lemma 4. $\mathrm{Aut}\,\mathrm{FDDS}_n(X) \cong \Im[X]$.

6. The least n-dinilpotent congruence on a free doppelsemigroup

In this section we present the least n-dinilpotent congruence on a free doppelsemigroup.

Let $\mathrm{FDS}(X)$ be the free doppelsemigroup (see Section 2) and $n \in \mathbb{N}$. Define a relation $\mu_{(n)}$ on $\mathrm{FDS}(X)$ by

$$(w_1, u_1)\mu_{(n)}(w_2, u_2) \quad \text{if and only if} \quad (w_1, u_1) = (w_2, u_2) \text{ or}$$
$$\begin{cases} d_x(u_1) + 1 > n & \text{for some } x \in \{a, b\}, \\ d_y(u_2) + 1 > n & \text{for some } y \in \{a, b\}. \end{cases}$$

Theorem 4. *The relation $\mu_{(n)}$ on the free doppelsemigroup* $\mathrm{FDS}(X)$ *is the least n-dinilpotent congruence.*

Proof. Define a map $\varphi : \mathrm{FDS}(X) \to \mathrm{FDDS}_n(X)$ by

$$(w, u)\varphi = \begin{cases} (w, u), & \text{if } d_x(u) + 1 \leqslant n \text{ for all } x \in \{a, b\}, \\ 0, & \text{in all other cases} \end{cases}$$

$((w, u) \in \mathrm{FDS}(X))$. Show that φ is a homomorphism.

Let $(w_1, u_1), (w_2, u_2) \in \mathrm{FDS}(X)$ and $d_x(u_1au_2)+1 \leqslant n$ for all $x \in \{a, b\}$. From the last inequality it follows that $d_x(u_1) + 1 \leqslant n$ and $d_x(u_2) + 1 \leqslant n$ for all $x \in \{a, b\}$. Then

$$\begin{aligned} ((w_1, u_1) \dashv (w_2, u_2))\varphi &= (w_1w_2, u_1au_2)\varphi = (w_1w_2, u_1au_2) \\ &= (w_1, u_1) \dashv (w_2, u_2) = (w_1, u_1)\varphi \dashv (w_2, u_2)\varphi. \end{aligned}$$

If $d_x(u_1au_2)+1>n$ for some $x\in\{a,b\}$, then

$$((w_1,u_1)\dashv(w_2,u_2))\varphi=(w_1w_2,u_1au_2)\varphi=0=(w_1,u_1)\varphi\dashv(w_2,u_2)\varphi.$$

Let further $d_x(u_1bu_2)+1\leqslant n$ for all $x\in\{a,b\}$. Then $d_x(u_1)+1\leqslant n$, $d_x(u_2)+1\leqslant n$ for all $x\in\{a,b\}$ and

$$\begin{aligned}((w_1,u_1)\vdash(w_2,u_2))\varphi&=(w_1w_2,u_1bu_2)\varphi=(w_1w_2,u_1bu_2)\\&=(w_1,u_1)\vdash(w_2,u_2)=(w_1,u_1)\varphi\vdash(w_2,u_2)\varphi.\end{aligned}$$

If $d_x(u_1bu_2)+1>n$ for some $x\in\{a,b\}$, then

$$((w_1,u_1)\vdash(w_2,u_2))\varphi=(w_1w_2,u_1bu_2)\varphi=0=(w_1,u_1)\varphi\vdash(w_2,u_2)\varphi.$$

Thus, φ is a surjective homomorphism. By Theorem 3 $\mathrm{FDDS}_n(X)$ is the free n-dinilpotent doppelsemigroup. Then Δ_φ is the least n-dinilpotent congruence on $\mathrm{FDS}(X)$. From the definition of φ it follows that $\Delta_\varphi=\mu_{(n)}$. □

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