Algebra and Discrete Mathematics
Volume **24** (2017). Number 1, pp. 169–180

RESEARCH ARTICLE

Jacobsthal-Lucas series and their applications

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Communicated by A. P. Petravchuk

Abstract. In this paper we study the properties of positive series such that its terms are reciprocals of the elements of Jacobsthal-Lucas sequence ($J_{n+2} = 2J_{n+1} + J_n$, $J_1 = 2$, $J_2 = 1$). In particular, we consider the properties of the set of incomplete sums as well as their applications. We prove that the set of incomplete sums of this series is a nowhere dense set of positive Lebesgue measure. Also we study singular random variables of Cantor type related to Jacobsthal-Lucas sequence.

Introduction

Today mathematicians heavily research structural, topological, metric and fractal properties of the set of incomplete sums (subsums) of absolutely convergent series. Despite essential progress for some series, the problem is quite difficult in general case. In this context scientists focus on series such that their terms are elements of some sequences with some condition of homogeneity (depend on finite numbers of parameters and defined by a formula for general term or some recurrence relation).

In this research article we investigate series

$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1} + (-1)^{n-1}}.$$

2010 MSC: 11B83, 11B39, 60G50.

Key words and phrases: Jacobsthal-Lucas sequence, the set of incomplete sums, singular random variable, Hausdorff-Besicovitch dimension.

It seems that this series is a “simple perturbation” of classic binary series and changes of properties of the set of subsums are insignificant. However this is not true.

1. Jacobsthal-Lucas sequence

Definition 1. The sequence of real numbers $(u_n) \equiv (u_n)_{n=1}^{\infty}$ having the property

$$u_{n+2} = pu_{n+1} + su_n, \tag{1}$$

where u_1, u_2, p, s are fixed real numbers, is called a generalized Fibonacci sequence.

Let p and s be fixed real numbers ($p^2 + s^2 \neq 0$), and let $F_{p,s}$ be a set of all sequences satisfying condition (1). Consider linear operations on this set defined by formulas

$$(a_n) \oplus (b_n) = (a_n + b_n) \ \text{ and } \ \lambda(a_n) = (\lambda a_n), \ n \in N, \ \lambda \in R.$$

Then $F_{p,s}$ forms a two-dimensional vector space with respect to these operations. It is easy to introduce different mathematical structures in this space [4].

If $p = 1, s = 2$, then generalized Fibonacci sequence is called Jacobsthal sequence. Thus, Jacobsthal sequences form a two-dimensional vector space containing geometric progression with common ratios 2 and -1. This family does not contain infinitesimal sequences except for zero sequence. The general term of Jacobsthal sequence is equal to

$$u_n = \frac{(u_2 + u_1)2^{n-1} + (2u_1 - u_2)(-1)^{n-1}}{3}.$$

These sequences can be used for representing real numbers [2], modeling of objects with complicated local structure (sets, functions, random variables, etc. [3]). One particular case (for $u_2 = p = 1$, $u_1 = s = 2$) of generalized Fibonacci sequence is Jacobsthal-Lucas sequence defined as follows

$$(J_n) = (2, 1, 5, 7, 17, 31, 65, \ldots J_n, \ldots), \quad J_n = J_{n-1} + 2J_{n-2}, \quad n \geqslant 3. \tag{2}$$

Lemma 1. *The general term of Jacobsthal-Lucas sequence is determined by formula*

$$J_n = 2^{n-1} + (-1)^{n-1}. \tag{3}$$

Proof. Finding a general term of the sequence (2) is equivalent to solving homogeneous difference equation of order 2

$$y(x+2) - y(x+1) - 2y(x) = 0. \tag{4}$$

Characteristic equation of (4) has the form $\lambda^2 - \lambda - 2 = 0$. Numbers 2 and -1 are solutions of characteristic equation. Functions $y_1(x) = 2^x$ and $y_2(x) = (-1)^x$ are solutions of equation (4). Hence, general solution can be written as a function $y(x) = c_1 2^{x-1} + c_2(-1)^{x-1}$. Taking into account initial conditions $\begin{cases} c_1 y_1(1) + c_2 y_2(1) = 2, \\ c_1 y_1(2) + c_2 y_2(2) = 1, \end{cases}$ we can find constants c_1 and c_2. It is easy to see that $c_1 = c_2 = 1$.

So, general term of sequence (2) has form (3). □

Theorem 1. *Jacobsthal-Lucas sequence has the following properties:*

1. $\sum\limits_{n=1}^{k} J_n = \begin{cases} 2^k & \text{if } k \text{ is odd}, \\ 2^k - 1 & \text{if } k \text{ is even}; \end{cases}$
2. $\sum\limits_{n=1}^{\frac{k+1}{2}} J_{2n-1} = \frac{J_{k+2} - 2}{3} + \frac{k+1}{2}$; 3. $\sum\limits_{n=1}^{\frac{k}{2}} J_{2n} = \frac{2(J_{k+1} - 2)}{3} - \frac{k}{2}$;
4. $\sum\limits_{n=1}^{k} J_n^2 = \begin{cases} \frac{J_{2k+1} + 2J_{k+1} + 2}{3} + k & \text{if } k \text{ is odd}, \\ \frac{J_{2k+1} - 2J_{k+1} + 2}{3} + k & \text{if } k \text{ is even}; \end{cases}$
5. $\sum\limits_{n=1}^{k} (-1)^{n-1} J_n = \begin{cases} \frac{-J_{k+1} + 2}{3} + k & \text{if } k \text{ is odd}, \\ \frac{J_{k+1} + 2}{3} + k & \text{if } k \text{ is even}. \end{cases}$

2. Jacobsthal-Lucas series

Consider the series of the reciprocals of the Jacobsthal-Lucas numbers

$$r = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{J_n} = \frac{1}{2} + \frac{1}{1} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{J_n} + \ldots. \tag{5}$$

This is convergent positive series, and its terms form a monotonic decreasing sequence, starting with the second term. Furthermore, it is known [8] that infinite sum $\sum\limits_{n=1}^{\infty} \frac{t^n}{A\alpha^n + B\beta^n}$ is an irrational number if α, β are positive integers and $A \cdot B \neq 0$, $|\alpha| > |t|$, $|A \cdot B \cdot t^2| < |\alpha|$. Hence, the sum of series (5) is also an irrational number.

Using equality (3), we have the formula for general term of the sequence of reciprocal Jacobsthal-Lucas numbers:

$$u_n = \frac{1}{2^{n-1} + (-1)^{n-1}}. \tag{6}$$

Lemma 2. *For series* (5)*, the following system of inequalities holds:*

$$\begin{cases} u_n > r_n & \text{if } n \text{ is even,} \\ u_n < r_n & \text{if } n \text{ is odd.} \end{cases} \tag{7}$$

Proof. Using (6), for even numbers n, we have $u_n = \dfrac{1}{2^{n-1} - 1}$. Since

$$\frac{1}{2^k + 1} + \frac{1}{2^{k+1} - 1} < \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

for any $k \in N$, we obtain

$$r_n = \frac{1}{2^n + 1} + \frac{1}{2^{n+1} - 1} + \frac{1}{2^{n+2} + 1} + \cdots < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \frac{1}{2^{n-1}}.$$

So, $r_n < \dfrac{1}{2^{n-1}} < \dfrac{1}{2^{n-1} - 1} = u_n$. Hence $u_n > r_n$ for even numbers n.

Similarly, using (6), for odd numbers n, we have $u_n = \dfrac{1}{2^{n-1} + 1}$. Since

$$\frac{1}{2^k - 1} + \frac{1}{2^{k+1} + 1} > \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

for any $k \in N$, we obtain

$$r_n = \frac{1}{2^n - 1} + \frac{1}{2^{n+1} + 1} + \frac{1}{2^{n+2} - 1} + \cdots > \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \frac{1}{2^{n-1}}.$$

So, $r_n > \dfrac{1}{2^{n-1}} > \dfrac{1}{2^{n-1} + 1} = u_n$. Hence $u_n < r_n$ for odd numbers n. □

Lemma 3. *For remainders of series* (5)*, the following system of inequalities holds:*

$$\begin{cases} \dfrac{1}{2^n + 1} + \dfrac{1}{2^n} < r_n < \dfrac{1}{2^{n-1}} & \text{if } n \text{ is even,} \\ \dfrac{1}{2^{n-1}} < r_n < \dfrac{1}{2^n - 1} + \dfrac{1}{2^n} & \text{if } n \text{ is odd.} \end{cases} \tag{8}$$

Proof. For even number n, we have

$$r_n = \frac{1}{2^n+1} + \frac{1}{2^{n+1}-1} + \dots + \frac{1}{2^{n+k}+(-1)^{n+k}} + \dots$$
$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{n+k}} + \dots = \frac{1}{2^{n-1}}.$$

On the other hand,

$$r_n > \frac{1}{2^n+1} + \left[\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^{n+k+1}} + \dots\right] = \frac{1}{2^n+1} + \frac{1}{2^n}.$$

Hence, for even number n, the following inequalities hold:

$$\frac{1}{2^n+1} + \frac{1}{2^n} < r_n < \frac{1}{2^{n-1}}.$$

For odd number n, we have

$$r_n = \frac{1}{2^n-1} + \frac{1}{2^{n+1}+1} + \dots + \frac{1}{2^{n+k}+(-1)^{n+k}} + \dots$$
$$> \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{n+k}} + \dots = \frac{1}{2^{n-1}}.$$

On the other hand,

$$r_n < \frac{1}{2^n-1} + \left[\frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^{n+k+1}} + \dots\right] = \frac{1}{2^n-1} + \frac{1}{2^n}.$$

Hence, for odd number n, the following inequalities hold:

$$\frac{1}{2^{n-1}} < r_n < \frac{1}{2^n-1} + \frac{1}{2^n}.$$ □

Lemma 4. *For any positive integer k, the following inequalities hold:*

$$u_{2k+2}+u_{2k+3}+\ldots+u_{4k} < u_{2k+1} < u_{2k+2}+u_{2k+3}+\ldots+u_{4k}+(u_{4k+1}+u_{4k+2}).$$

To prove this lemma, we estimate the difference of the right-hand and left-hand side of each inequality and take into account

$$\frac{1}{2^k+1} + \frac{1}{2^{k+1}-1} < \frac{1}{2^k} + \frac{1}{2^{k+1}} < \frac{1}{2^k-1} + \frac{1}{2^{k+1}+1}.$$

3. The set of incomplete sums of the series

Definition 2. Let $M \in 2^N$ that is $M \subseteq N$. Then number

$$x = x\left(M\right) = \sum_{n \in M} u_n = \sum_{n=1}^{\infty} \varepsilon_n u_n, \tag{9}$$

where $\varepsilon_n = \begin{cases} 1 & \text{if } n \in M, \\ 0 & \text{if } n \notin M, \end{cases}$ is called the incomplete sum of series $\sum u_n$. By Δ' we denote the set of all incomplete sums of series (5).

The set of incomplete sums of convergent positive series such that inequality $u_n \leqslant r_n$ ($u_n > r_n$) holds only finitely many times was investigated in paper [7]. Moreover, it is well known [6] that the set of all subsums of any convergent positive series always belongs to one of the following three types: a finite union of closed intervals, a Cantor type set or an M-Cantorval. However, the question about type and properties of the set of subsums of series (5) is still open because inequalities $u_n < r_n$ and $u_n > r_n$ hold infinitely many times for this series.

Definition 3. The set $\Delta'_{c_1 \dots c_k}$ of all incomplete sums

$$\sum_{n=1}^{k} c_n u_n + \sum_{n=k+1}^{\infty} \varepsilon_n u_n, \text{ where } \varepsilon_n \in \{0, 1\},$$

of series (5) is called the cylinder of rank k with base $c_1 \dots c_k (c_i \in \{0, 1\})$.

Definition 4. The closed interval $\Delta_{c_1 c_2 \dots c_k} = \left[\inf \Delta'_{c_1 \dots c_k}, \sup \Delta'_{c_1 \dots c_k}\right]$ is called the cylindrical interval of rank k with base $c_1 \dots c_k (c_i \in \{0, 1\})$.

It is possible that $\Delta'_{c_1 \dots c_k}$ and $\Delta_{c_1 \dots c_k}$ coincide or not, depending on properties of series and sequence $(c_1 \dots c_k)$. However $\Delta'_{c_1 \dots c_k} \subset \Delta_{c_1 \dots c_k}$ in any case.

Lemma 5. *The cylindrical sets have the following properties:*

1. $\Delta_{c_1 c_2 \dots c_k} = \left[\sum_{i=1}^{k} c_i u_i, \sum_{i=1}^{k} c_i u_i + r_k\right]$;
2. $|\Delta_{c_1 c_2 \dots c_k}| = r_k \to 0$ *as* $k \to \infty$;
3. $\Delta_{c_1 c_2 \dots c_k} \subset \Delta_{c_1 c_2 \dots c_k 0} \bigcup \Delta_{c_1 c_2 \dots c_k 1}$, $\Delta'_{c_1 c_2 \dots c_k} = \Delta'_{c_1 c_2 \dots c_k 0} \bigcup \Delta'_{c_1 c_2 \dots c_k 1}$;
4. $\inf \Delta_{c_1 c_2 \dots c_k} = \inf \Delta_{c_1 c_2 \dots c_k 0} < \inf \Delta_{c_1 c_2 \dots c_k 1}$,
   $\sup \Delta_{c_1 c_2 \dots c_k} = \sup \Delta_{c_1 c_2 \dots c_k 1} > \sup \Delta_{c_1 c_2 \dots c_k 0}$;

5. $\bigcap\limits_{k=1}^{\infty} \Delta_{c_1c_2\ldots c_k} = \bigcap\limits_{k=1}^{\infty} \Delta'_{c_1c_2\ldots c_k} \equiv \Delta_{c_1c_2\ldots c_k\ldots} = x \subset [0, r]$;

6. $\dfrac{|\Delta_{c_1c_2\ldots c_k c}|}{|\Delta_{c_1c_2\ldots c_k}|} = \dfrac{r_{k+1}}{r_{k+1}+u_{k+1}} = \dfrac{1}{\delta_{k+1}+1}$, *where* $\delta_{k+1} = \dfrac{u_{k+1}}{r_{k+1}}$;

7. $\Delta_{c_1c_2\ldots c_k} = \Delta_{s_1s_2\ldots s_k}$ *if and only if* $c_i = s_i$, $i = \overline{1,k}$;

8. $$O^{k+1}_{c_1\ldots c_k}(1,0) = \Delta_{c_1c_2\ldots c_k1} \bigcap \Delta_{c_1c_2\ldots c_k0} = \begin{cases} \left[\sum\limits_{n=1}^{k} c_n u_n + u_{n+1}, \sum\limits_{n=1}^{k} c_n u_n + r_{n+1}\right] & \text{if } k \text{ is even,} \\ \varnothing & \text{if } k \text{ is odd;} \end{cases}$$

9. *For any even number* k,
$$\Delta_{c_1c_2\ldots c_k1} \bigcap \Delta_{c_1c_2\ldots c_k0} = \Delta_{c_1c_2\ldots c_k10} \bigcap \Delta_{c_1c_2\ldots c_k01};$$

10. $$G^{k+1}_{c_1\ldots c_k}(1,0) = \Delta_{c_1c_2\ldots c_k} \setminus \left(\Delta_{c_1c_2\ldots c_k1} \bigcup \Delta_{c_1c_2\ldots c_k0}\right) = \begin{cases} \left(\sum\limits_{n=1}^{k} c_n u_n + r_{n+1}, \sum\limits_{n=1}^{k} c_n u_n + u_{n+1}\right) & \text{if } k \text{ is odd,} \\ \varnothing & \text{if } k \text{ is even;} \end{cases}$$

11. *For any positive integer* $k > 2$,
$$G^{k+2}_{c_1\ldots c_k}(01,00) \bigcap G^{k+2}_{c_1\ldots c_k}(10,11) = \varnothing;$$

12. *For any positive integer* $k > 2$,
$$\left(G^{k+2}_{c_1\ldots c_k}(01,00) \bigcup G^{k+2}_{c_1\ldots c_k}(10,11)\right) \bigcap O^{k+1}_{c_1\ldots c_k}(1,0) = \varnothing.$$

Lemma 6. *For any odd number* n*, the following relations hold:*
$$O^{n}_{c_1\ldots c_{n-1}}(0,1) = \Delta_{c_1\ldots c_{n-1}0\underbrace{1\ldots1}_{m}} \bigcap \Delta_{c_1\ldots c_{n-1}1\underbrace{0\ldots0}_{m}} \quad \text{if } m < n-2,$$
$$O^{n}_{c_1\ldots c_{n-1}}(0,1) \neq \Delta_{c_1\ldots c_{n-1}0\underbrace{1\ldots1}_{m}} \bigcap \Delta_{c_1\ldots c_{n-1}1\underbrace{0\ldots0}_{m}} \quad \text{if } m > n.$$

Lemma 7. *For any nonempty* $O^{n+1}_{c_1\ldots c_n}(0,1)$ *there exist a positive integer* m *and sequence* $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$*, where* $\alpha_i \in \{0,1\}$*,* $\beta_i \in \{0,1\}$*,* $i = \overline{1,m}$*, such that*
$$G^{n+m+1}_{c_1\ldots c_n1\alpha_1\ldots\alpha_m}(0,1) \bigcap G^{n+m+1}_{c_1\ldots c_n0\beta_1\ldots\beta_m}(0,1) \in O^{n+1}_{c_1\ldots c_n}(0,1).$$

Proof. Let us show that there exist a positive integer m and sequence $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_m$, where $\alpha_i \in \{0,1\}$, $\beta_i \in \{0,1\}$, $i = \overline{1,m}$, such that

$$\min G^{n+m+1}_{c_1\ldots c_n 0\beta_1\ldots\beta_m} < \min G^{n+m+1}_{c_1\ldots c_n 1\alpha_1\ldots\alpha_m} < \max G^{n+m+1}_{c_1\ldots c_n 0\beta_1\ldots\beta_m},$$

$$0 < u_{n+1} + \sum_{i=1}^{m} (\alpha_i - \beta_i) u_{n+1+i} < u_{n+m+2} - r_{n+m+2}.$$

Taking into account $u_{n+1} < r_{n+1}$, we can find the number $\widetilde{m} > n+1$ such that $u_{n+1} - u_{n+2} - u_{n+3} - \cdots - u_{n+\widetilde{m}} < 0$. Since $u_{n+m+2} - r_{n+m+2} > 0$, we see that there exist numbers $\alpha_1, \ldots, \alpha_m$, $\beta_1, \ldots, \beta_m$, where $\alpha_i \in \{0,1\}$ and $\beta_i \in \{0,1\}$, $i = \overline{1,m}$, such that

$$0 < u_{n+1} + \sum_{i=1}^{m} (\alpha_i - \beta_i) u_{n+1+i} < u_{n+m+2} - r_{n+m+2}. \qquad \square$$

Corollary 1 (Lemmas 7, 8)**.** *Any intersection of cylinders of the same rank is not completely contained in the set of subsums of series* (5)*.*

Theorem 2. *The set of incomplete sums of series* (5) *is a perfect nowhere dense set of positive Lebesgue measure.*

Proof. It is easy to prove that the set of incomplete sums of any positive series is a perfect set (i.e., closed set without isolated points).

By G we denote the sum of all gaps in the form $G^{n+1}_{c_1\ldots c_n}(0,1)$ between cylinders of even ranks. Then Lebesgue measure of Δ' is greater than or equal to some number: $L(\Delta') \geqslant \sum\limits_{n=1}^{\infty} u_n - G$.

Using properties 10, 11, 12 of cylindrical sets, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} u_n - G &= \sum_{n=1}^{\infty} u_n - 3[u_1 - r_2] - 8[u_4 - r_4] - \ldots - 2\cdot 4^{n-1}[u_{2n} - r_{2n}] - \ldots \\
&= 4u_3 - 4u_4 + 12u_5 - 20u_6 + 44u_7 - \ldots \\
&\quad + u_{n+2}[2^2 - 2^3 + 2^4 - \ldots + (-2)^{n+1}] + \ldots \\
&= \sum_{n=1}^{\infty} \left(u_{2n+1} \cdot \frac{4 + 2^{2n+1}}{3} + u_{2n+2} \cdot \frac{4 - 2^{2n+2}}{3} \right) = \sum_{n=1}^{\infty} A_n.
\end{aligned}$$

Now we show that $A_n > 0$ for all $n \in N$. Taking into account equality (6), we see that

$$A_n = \frac{4 + 2^{2n+1}}{3\,(2^{2n}+1)} + \frac{4 - 2^{2n+2}}{3\,(2^{2n+1}-1)} = \frac{2^{2n+1}}{(2^{2n}+1)(2^{2n+1}-1)} > 0.$$

Using approximate calculation, we can conclude that Lebesgue measure of Δ' is greater than some positive number:

$$L(\Delta') > \sum_{n=1}^{100} \frac{2^{2n+1}}{(2^{2n}+1)(2^{2n+1}-1)} \approx 0,3099984859\ldots > 0.$$

Finally, we show that the set of incomplete sums is nowhere dense. Suppose that there exists some closed interval $[a, b] \subset \Delta'$. It is obvious that we can find numbers $c_1, c_2, \ldots, c_k$ such that

$$a < \sum_{n=1}^{k} c_n u_n < b, \quad \max\{u_k, r_k\} < b - \sum_{n=1}^{k} c_n u_n.$$

So, there exists some cylindrical interval $\Delta_{c_1\ldots c_k} \subset [a, b]$.

If k is odd, then from the properties of cylindrical sets it follows that $G^{k+1}_{c_1\ldots c_k}(0,1) \subset \Delta_{c_1\ldots c_k}$. So there exists some gap $G^{k+1}_{c_1\ldots c_k}(0,1)$ such that it is a subset of $[a, b]$ but is not contained in the set of subsums.

If k is even, then from the properties of cylindrical sets it follows that $O^{k+1}_{c_1\ldots c_k}(0,1) \subset \Delta_{c_1\ldots c_k} \subset [a, b]$. According to Lemmas 7 and 8, intersection $O^{k+1}_{c_1\ldots c_k}(0,1)$ cannot be contained in the set of subsums. This contradiction proves the theorem. □

4. Distribution of random incomplete sum

Let us consider random variable $\xi = \sum\limits_{k=1}^{\infty} \dfrac{\xi_k}{J_k}$, where (ξ_k) is a sequence of independent random variables taking the values 0 and 1 with probabilities $P\{\xi_k = 0\} = p_{0k} \geqslant 0$, $P\{\xi_k = 1\} = p_{1k} \geqslant 0$ respectively, $p_{0k} + p_{1k} = 1$, and (J_k) is Jacobsthal-Lucas sequence.

Properties of distribution of ξ depend both on infinite stochastic matrix $||p_{ik}||$ and series (5). According to the Jessen-Wintner theorem [1], the random variable ξ has a pure distribution (pure discrete, pure absolutely continuous or pure singular). Criterion for the discreteness of ξ follows from the P. Lévy theorem (see [5]): the random variable ξ has a discrete distribution if and only if $M = \prod_{k=1}^{\infty} \max\{p_{0k}, p_{1k}\} > 0$.

It is easy to prove that the spectrum S_ξ of the distribution of random variable ξ is the set $S_\xi = \{x : x = \Delta_{c_1 c_2 \ldots c_k \ldots}, p_{c_k k} > 0,\ \forall k \in N\}$ and S_ξ is a subset of the set of incomplete sums of series (5). We understand topological, metric and fractal properties of distribution of random variable ξ as topological, metric and fractal properties of its spectrum.

Theorem 3. *If*

$$\begin{cases} p_{0(2m)}p_{1(2m)} = 0, \\ p_{0(2m-1)}p_{1(2m-1)} \neq 0, \end{cases} \quad or \quad \begin{cases} p_{0(2m)}p_{1(2m)} \neq 0, \\ p_{0(2m-1)}p_{1(2m-1)} = 0, \end{cases}$$

then random variable ξ has a singular distribution of Cantor type. The Hausdorff-Besicovitch dimension of the spectrum of ξ is equal to $0,5$.

Proof. We prove the theorem if ξ satisfies condition $p_{0(2m)} \cdot p_{1(2m)} = 0$, $p_{0(2m-1)} \cdot p_{1(2m-1)} \neq 0$. The proof is similar in second case.

The spectrum of ξ coincides with the set of incomplete sums of series

$$\sum_{n=1}^{\infty} t_n = \frac{1}{J_1} + \frac{1}{J_3} + \cdots + \frac{1}{J_{2n-1}} + \cdots = \sum_{n=1}^{\infty} u_{2n-1}. \tag{10}$$

It is easy to see that $t_k > r_k(t) = \sum\limits_{n=k+1}^{\infty} t_n$ for any positive integer k. Then the set of subsums of series (10) is a perfect nowhere dense set [7]. By (3), we have $t_n = \frac{1}{4^{n-1}+1}$.

The Lebesgue measure of the set of incomplete sums of series (10) $\Delta'(t)$ can be computed by formula $\lambda(\Delta'(t)) = \lim_{n\to\infty} 2^n r_n(t)$. Since $r_n(t) = \sum\limits_{k=n+1}^{\infty} t_k < \frac{1}{4^n} + \frac{1}{4^{n+1}} + \cdots = \frac{1}{3 \cdot 4^{n-1}} < \frac{1}{4^{n-1}}$, we see that

$$\lambda(\Delta'(t)) < \lim_{n\to\infty} \frac{2^n}{4^{n-1}} = 0.$$

So, we can conclude that the Lebesgue measure of the set of incomplete sums of series (10) is equal to zero. However, there is still open question about Hausdorff-Besicovitch dimension of spectrum of ξ. It is well known that

$$H^{\alpha}(E) = \lim_{k\to\infty} 2^k r_k^{\alpha}(t) = \lim_{k\to\infty} (2r_k^{\frac{\alpha}{k}}(t))^k = \begin{cases} 0 & \text{if } 2r_k^{\frac{\alpha}{k}}(t) < 1, \\ 1 & \text{if } 2r_k^{\frac{\alpha}{k}}(t) = 1, \\ \infty & \text{if } 2r_k^{\frac{\alpha}{k}}(t) > 1. \end{cases}$$

Let us consider the case $2r_k^{\frac{\alpha}{k}}(t) = 1$. We have $\alpha(k) = \dfrac{-k\ln 2}{\ln r_k(t)}$. Since α depends on k in the last equality, we see that

$$\alpha = \lim_{k\to\infty} \alpha(k) = \lim_{k\to\infty} \frac{-k\ln 2}{\ln r_k(t)}.$$

For the remainders of series (10), the following inequalities hold:

$$\frac{1}{4^{k+1}} < r_k(t) < \frac{1}{4^{k-1}}.$$

Hence, we have $\ln\left(\frac{1}{4^{k+1}}\right) < \ln r_k(t) < \ln\left(\frac{1}{4^{k-1}}\right)$,

$$\frac{\ln 2}{\ln\left(\frac{1}{4^{k-1}}\right)} < \frac{\ln 2}{\ln r_k(t)} < \frac{\ln 2}{\ln\left(\frac{1}{4^{k+1}}\right)},$$

$$\frac{-k\cdot\ln 2}{-(k-1)\cdot\ln 4} < \frac{-k\cdot\ln 2}{\ln r_k(t)} < \frac{-k\cdot\ln 2}{-(k+1)\cdot\ln 4}.$$

We can find the limit of the sequence on the left and right side of the above inequality respectively:

$$\lim_{k\to\infty}\frac{-k\cdot\ln 2}{-(k-1)\cdot\ln 4} = \frac{1}{2} \quad\text{and}\quad \lim_{k\to\infty}\frac{-k\cdot\ln 2}{-(k+1)\cdot\ln 4} = \frac{1}{2}.$$

From the squeeze theorem it follows that

$$\lim_{k\to\infty}\alpha(k) = \lim_{k\to\infty}\frac{-k\cdot\ln 2}{\ln r_k(t)} = \frac{1}{2}.$$

So, the Hausdorff-Besicovitch dimension of the set of incomplete sums of series (10) is equal to $0,5$. □

Theorem 4. *Let pairs* $(\xi_{2k-1}\xi_{2k})$ *of consecutive independent random variables take values from the set* $\{(0,0),(1,1)\}$ *with probabilities* $p_{0k}\geqslant 0$, $p_{1k}\geqslant 0$ *respectively,* $\prod_{k=1}^{\infty}\max\{p_{0k},p_{1k}\}=0$, *and stochastic matrix* $\|p_{ik}\|$ *has finite numbers of zeroes. Then random variable* ξ *has a singular probability distribution of Cantor type; and the Hausdorff-Besicovitch dimension of the spectrum of* ξ *is equal to* $0,5$.

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Received by the editors: 12.09.2016
and in final form 29.03.2017.