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RESEARCH ARTICLE

Indecomposable and irreducible *t*-monomial matrices over commutative rings

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Abstract. We introduce the notion of the defining sequence of a permutation indecomposable monomial matrix over a commutative ring and obtain necessary conditions for such matrices to be indecomposable or irreducible in terms of this sequence.

Introduction

Let K be a commutative ring (with unity). By a *monomial matrix* $M = (m_{ij})$ over K we mean a quadratic $n \times n$ matrix, in each row and each column of which there is exactly one non-zero element. With such matrix M one can associate the directed graph $\Gamma(M)$ with n vertices numbered from 1 to n and arrows $i \to j$ for all $m_{ij} \neq 0$. Obviously, $\Gamma(M)$ is the disjoint union of cycles, each of which has the same direction of arrows. If there is only one cycle, the monomial matrix M is called *cyclic* (in other words, it is a permutation indecomposable monomial matrix). A cyclic matrix of the form

$$M = \begin{pmatrix} 0 & \dots & 0 & m_{1n} \\ m_{21} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & m_{n,n-1} & 0 \end{pmatrix}$$

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we call *canonically cyclic*. The sequence

$$v = v(M) = (m_{21}, \dots, m_{n-1,n}, m_{1n})$$

we call the *defining sequence of* M, and write

$$M = M(v) = M(m_{21}, \dots, m_{n-1,n}, m_{1n}).$$

The sequence $v^* = v^*(M) = (m_{1n}, m_{n-1,n}, \dots, m_{21})$ is called *dual to* v and the matrix $M^* = M(v^*)$ *dual to* M.

When all elements m_{ij} of a monomial (respectively, cyclic or canonically cyclic) matrix M are of the form $t^{s_{ij}}$ $(t \in K)$, where $s_{ij} \geqslant 0$, the matrix M is called *t-monomial* (respectively, *t-cyclic* or *canonically t-cyclic*); obviously, then $t^{s_i} \neq 0$ for all i.

The most interesting cases are, obviously, those when the element t is non-invertible.

Matrices of such form were studied by the authors in [1], and in this paper we continue our investigation.

1. Defining sequences and indecomposability

Through this section K denotes a commutative local ring with maximal ideal $R = Rad\,K \neq 0$ and $t \in R$. All matrices are considered over K. By E_s one denotes the identity $s \times s$ matrix.

1.1. Permutation similarity. Let

$$M = \begin{pmatrix} 0 & \dots & 0 & t^{s_n} \\ t^{s_1} & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & t^{s_{n-1}} & 0 \end{pmatrix} \qquad (*)$$

be a canonically t-cyclic matrix. A permutation of the members $x_i = t^{s_i}$ of $v(M)$ of the form $x_i, x_{i+1}, \dots, x_m, x_1, x_2, \dots, x_{i-1}$, is called a *cyclic permutation*. Two matrices $M(v)$ and $M(v')$ is called *cyclically similar* if v can be obtained from v' by a such permutation.

It is easy to prove the following statement[1].

Proposition 1. *a) Two canonically t-cyclic matrices is permutation similar if and only if they are cyclically similar.*

b) The matrix transpose to a canonically t-cyclic matrix is permutation similar to the dual one.

[1]This statement is valid for matrices with elements from any set if one modifies the definitions accordingly.

1.2. Conjecture. Let $M(v)$ be a canonically t-cyclic $n\times n$ matrix with defining sequence $v=(x_1,x_2,\dots,x_n)$, where $x_i=t^{s_i}$ (see $(*)$).

The sequence v is called *periodic* with a period $0<p<n$ if $p|n$ and $x_{s+p}=x_s$ for any $1\leqslant s\leqslant n-p$, and *non-periodic* if otherwise. In the case of v to be periodic, the matrix $M(v)$ can be reduced by a permutation of its rows and column to the following block-monomial form: $N=(N_{ij})_{i,j=1}^m$, $m=n/p$, where $N_{21}=x_2E_m, N_{32}=x_3E_m,\dots,N_{n,n-1}=x_nE_m$ and $N_{1m}=x_1M(1,1,\dots,1)$ with 1 to occur m times[2]. The $m\times m$ matrix $M(1,1,\dots,1)$ can be indecomposable or decomposable depending on properties of the ring K, and therefore so can be the matrix $M(v)$.

Conjecture 6 (V. M. Bondarenko, Private Communication)**.** *Any canonically t-cyclic matrix over K with non-periodic defining sequence is indecomposable.*

It is obvious that the idea of a proof of this conjecture basing on decomposition of a given matrix M into a direct sum of others ones, with a final contradiction, is futile. It is most likely the best idea is to use the simple fact that M is indecomposable if any idempotent matrix X such that $MX=XM$ is identity or zero. The main difficulty in this way is that the non-periodicity of the defining sequence of M is not determined by its local properties. In the next subsection we consider a special case in which the condition of the non-periodicity is satisfied automatically.

1.3. 2-homogeneous defining sequences. Let $M(v)$ be again a canonically t-cyclic $n\times n$ matrix with defining sequence $v=(x_1,x_2,\dots,x_n)$, where $x_i=t^{s_i}$. The sequence v is said to be 2-homogeneous if it is translated by a cyclic permutation on that of the form $(a,a,\dots,a,b,b,\dots,b)$, where a and $b\neq a$ both actually occur.

The following theorem proves Conjecture 1 for this case.

Theorem 1. *Any canonically t-cyclic matrix over K with 2-homogeneous defining sequence is indecomposable.*

Proof. Let $M=M(v)$ be a canonically t-cyclic $n\times n$ matrix and let v has s coordinates equal to $a=t^p$ and the other ones equal to $b=t^q$. Assume without loss of generality that $v=(a,a,\dots,a,b,b,\dots,b)$ and $p<q$. So $v=t^pv_0$, where $v_0=(1,1,\dots,1,t^{q-p},t^{q-p},\dots,t^{q-p})$. After replacing v

[2]To do it, one is to arrange the rows and columns in the order 2, 2+p, 2+2p,. . . , 2+(m-1)p, 3, 3+p, 3+2p,. . . , 3+(m-1)p,. . . , p, 2p, 3p,. . . , mp, p+1, (p+1)+p, (p+1)+2p,. . . , (p+1)+(m-2)p,1.

by v_0 and t^{q-p} by t (again without loss of generality), we come to the situation where

$$M = \begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 & t \\ 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & t & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & t & 0 \end{pmatrix}$$

with s elements to equal 1 and $k = n - s$ elements equal to t.

Arrange the rows and columns of the matrix M in the order $1, 2, \dots,$ $n-k, n, n-1, \dots, n-k+2, n-k+1$, denoting the new matrix by N:

$$N = \left(\begin{array}{c|c} N_{11} & N_{12} \\ \hline N_{21} & N_{22} \end{array}\right) = \left(\begin{array}{cccc|cccc} 0 & \dots & 0 & 0 & t & 0 & \dots & 0 \\ 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & \dots & 0 & 0 & 0 & t & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots & t \\ 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right).$$

Prove that there is not a non-trivial idempotent matrix commuting with N (see the previous subsection).

Let C be an $n \times n$ matrix such that $NC = CN$, where

$$C = \left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right) =$$

$$= \left(\begin{array}{ccc|ccc} c_{11} & \cdots & c_{1,n-k} & c_{1,n-k+1} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n-k,1} & \cdots & c_{n-k,n-k} & c_{n-k,n-k+1} & \cdots & c_{n-k,n} \\ \hline c_{n-k+1,1} & \cdots & c_{n-k+1,n-k} & c_{n-k+1,n-k+1} & \cdots & c_{n-k+,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{n,n-k} & c_{n,n-k+1} & \cdots & c_{nn} \end{array}\right)$$

with the same partition as N.

We denote by (i,j) the scalar equality $(NC)_{ij} = (CN)_{ij}$. All comparisons below are considered modulo the maximal ideal $R = Rad\,K$ (that is, they are equalities over the residue field K/R). We use by default the following simple fact: $tx = ty$ implies $x \equiv y$ $(x, y \in K)$.

Since every jth column of CN with $j > n-k$ consists of elements from tK, we have from the equations (i,j) for $i = 2, 3, \ldots, n-k$ and $i = n$ that

$$C \equiv \left(\begin{array}{c|c} C_{11} & 0 \\ \hline C_{21} & C_{22} \end{array}\right) =$$

$$= \left(\begin{array}{ccc|ccc} c_{11} & \ldots & c_{1,n-k} & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n-k,1} & \ldots & c_{n-k,n-k} & 0 & \ldots & 0 \\ \hline c_{n-k+1,1} & \ldots & c_{n-k+1,n-k} & c_{n-k+1,n-k+1} & \ldots & c_{n-k+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & \ldots & c_{n,n-k} & c_{n,n-k+1} & \ldots & c_{nn} \end{array}\right).$$

The equations (i,j) for $i, j = 1, 2, \ldots, n-k$ mean that the matrix C_{11} commutes modulo R with the lower Jordan block N_{11} and hence the matrix C_{11} is lower triangular modulo R (see, e. g., [2, Chap. VIII] or [3, Theorem 3.2.4.2]).

Further, from the equlities

$$(n-k+i, n-k+j+1) : tC_{n-k+i+1,n-k+j+1} = tC_{n-k+i,n-k+j}$$

for $1 \leqslant i < j \leqslant k-1$ it follows that all elements of the matrix C_{22} belonging to its lth upper diagonal[3] are pairwise comparable modulo R, $1 \leqslant l \leqslant k-2$. But since the equalities

$$(n-k+i, n-k) : tC_{n-k+i+1,n-k} = C_{n-k+i,n},$$

$1 \leqslant i \leqslant k-1$, imply that the last elements of all (mentioned above) upper diagonals are comparable with 0, we have eventually that the matrix C_{22} is, as well as C_{11} (see above), upper triangular modulo R.

Finally, from the equalities

I. $(2,1) :\ c_{11} = c_{22}, \quad (3,2) :\ c_{22} = c_{33}, \ldots,$
$(n-k, n-k-1) :\ c_{n-k-1,n-k-1} = c_{n-k,n-k},$

II. $(n-k+1, n-k+2) :\ tc_{n-k+1,n-k+1} = tc_{n-k+2,n-k+2},$
$(n-k, n-k+1) :\ tc_{n-k,n-k} = tc_{n-k+1,n-k+1}, \ldots,$
$(n-1, n) :\ tc_{n-1,n-1} = tc_{n,n},$

[3]The lth upper diagonal of a matrix $M = (m_{ij})$, where $l \geqslant 1$, is the collection of elements $m_{i,i+l}$.

III. $(n, n-k) : c_{n-k,n-k} = c_{n,n}$,
it follows that $c_{11} \equiv c_{22} \equiv \cdots \equiv c_{nn}$.

Thus we prove that the matrix C is comparable to an upper triangular one with the same elements on the main diagonal. It easily follows that if $C^2 \equiv C$, then $C \equiv E_n$ or $C \equiv 0$, and, consequently, because the comparisons are modulo the only maximal ideal of K, $C = E_n$ or $C = 0$, respectively. □

1.4. Applications in the representation theory of groups. Through this subsection K is as above and of characteristic p^s (p is simple, $s \geqslant 1$). All groups G are assumed to be finite of order $|G| > 1$. The number of nonequivalent indecomposable matrix K-representations of degree n of a group G is denoted by $\mathrm{ind}_K(G, n)$.

From [4] it follows that $\mathrm{ind}_K(G, n) \geqslant |K/R|$ for any p-group G of order $|G| > 2$ and $n > 1$. Here we strengthen this result in the case of both cyclic groups and radicals.

Theorem 2. *Let $R = tK \neq 0$ with t being nilpotent. Then, for a cyclic p-group G of some order N (hence of greater order), $\mathrm{ind}_K(G, n) \geqslant n - 1$ for any $n > 1$.*

Proof. Let S be an $n \times n$ matrix over K that is nilpotent modulo R; then $S^n \equiv 0\,(mod\,R)$ and $S^{2n} \equiv 0\,(mod\,R^2)$. It is easy to see that the map $\Gamma_S : a \to \Gamma_S(a) = E_n + uS$ with $u = t^{m-2}$ is a K-representation of a cyclic group $G = \langle a \rangle$ of an order $p^r \geqslant 2n$, $r \geqslant 2$. It is indecomposable if and only if so is modulo $Ann\,u := \{x \in K \mid t^{m-2}x = 0\} = R^2$, and representations Γ_S and $\Gamma_{S'}$ are equivalent if and only if the matrices S and S' are similar modulo R^2.

Consider the K-representations Γ_{M_k}, $k = 1, 2, \ldots, n-1$, with the matrices $M_k = M(1, \ldots, 1, t \ldots, t)$, where 1 occurs k times (see the previous subsection). They are non-equivalent, because M_k has rank k modulo R, and indecomposable by Theorem 1 (taking into account that $Rad\,(K/Ann\,u) = R/R^2$ is a principal ideal of K/R^2 generated by $t + R^2 \neq R^2$). □

Theorem 3. *Let the characteristic of K is p and $R = tK \neq 0$ with $t^2 = 0$. Then, for any cyclic p-group G and $n \geqslant |G|$, $\mathrm{ind}_K(G, n) \geqslant |G| - 2$.*

Proof. We use the notation of the proof of Theorem 2. It is easy to see that, for $0 < k < n$, the map $\Lambda_k : a \to \Lambda(a) = E_n + M_k$ is a K-representation of a cyclic group $G = \langle a \rangle$ if $k + 2 \leqslant |G|$; in particular, if $0 < k < |G| - 1 \leqslant n - 1$. By the last condition, their number is equal to

$|G| - 2$. The representations Λ_k are indecomposable by Theorem 1, and non-equivalent (see the previous proof). □

2. Defining sequences and irreducibility

Through this section K also denotes a commutative local ring with maximal ideal $R = Rad\, K \neq 0$; $t \neq 0$ is any element of R unless otherwise stated.

For an $n \times n$ matrix A over K, we denote by $[i \xrightarrow{a} j]^+$ (resp. $[i \xrightarrow{a} j]^-$) the following similarity transformation of A: adding ith row (resp. column), multiplied by a, to jth row (resp. column), and then subtracting jth column (resp. rows), multiplied by a, from ith column (resp. row).

2.1. Theorems on irreducible canonically *t*-cyclic matrices. Let $M = M(v)$ be a canonically t-cyclic $n \times n$ matrix with defining sequence $v = (x_1, x_2, \ldots, x_n)$, where $x_i = t^{s_i}$ (see $(*)$). We call the sequence $v_0 = v_0(M) = (s_1, s_2, \ldots, x_n)$ the *weighted sequence of* M, and the number $w = w(M) = s_1 + s_2 + \cdots + s_n$ the *weight of* M[4].

From the results of [1] it follows the next theorem[5].

Theorem 4. *If the matrix $M(v)$ is irreducible, then its weight is prime to n.*

Corollary 1. *If $t^2 = 0$ and the matrix $M(v)$ is irreducible, then its rank modulo R is prime to n.*

By a *connected subsequence of the length* $1 \leqslant l \leqslant n$ of a defining sequence $v = (x_1, x_2, \ldots, x_n)$ we mean any subsequence which maps by a cyclic permutation on one of the form $x_1, x_2, \ldots, x_l$. If it is 2-homogeneous i. e., by analogy with the above said, has the form $u = (t^i, t^i, \ldots, t^i, t^j, t^j, \ldots, t^j), t^i \neq t^j$, where t^i and t^j both actually occur, then the pair (p, q), consisting of the numbers of occurrences of t^i and t^j we call the *type of* u.

We shall obtain the following theorem as a consequence of statements in more general situations.

Theorem 5. *If the matrix $M(v)$ with 2-homogeneous defining sequence v is irreducible and $n > 5$, $t^2 = 0$, then its weight is equal $1, 2$ or $n - 1$.*

Concerning the cases when the weight is equal $1, n-1$ see [1, Introduction].

[4] One can write $M(t, v_0)$ instead of $M(v)$ (as in [1]).

[5] A quadratic matrix is irreducible if it is not similar to a 2×2 upper block triangular matrix with quadratic diagonal blocks.

2.2. Defining sequences with subsequences of type $(2,4)$ and $(4,2)$. We are interested in the cases when a 2-homogeneous subsequence has the form $(t^s,t^s,t^s,t^s,1,1)$ or $(1,1,t^s,t^s,t^s,t^s)$. Since they are mutually dual (see Proposition 1), we consider only the first case.

Proposition 2. *If $t^m=0$ and a canonically t-cyclic $n\times n$ matrix $M(v)$ is irreducible, then the sequence v does not contain a subsequence of the form $(t^s,t^s,t^s,t^s,1,1)$ with $m\leqslant 2s<2m$.*

We prove a more general statement replacing $(t^s,t^s,t^s,t^s,1,1)$ by $(t^i,t^j,t^p,t^q,1,1)$ with $0<i,j,p,q<m$, $i+j\geqslant m, p+q\geqslant m$, assuming (by Proposition 1) that the subsequence is the beginning of v and (by Theorem 4) that $n>6$.

This follows from the following: if we perform with the reducible matrix

$$N=\left(\begin{array}{cccccc|ccc}
0 & 0 & \dots & 0 & 0 & -t^j & 1 & 0 & 0\\
1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \alpha_1 & \dots & 0 & 0 & 0 & 0 & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & \alpha_k & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \dots & 0 & t^i & 0 & 0 & 0 & 0\\
\hline
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & t^q\\
0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & t^p & 0
\end{array}\right)$$

($k=n-6$) the transformation $[(n-2)\xrightarrow{t^j}(n-3)]^-$, $[1\xrightarrow{-1}(n-1)]^+$, $[2\xrightarrow{-t^p}n]^+$, and arrange the rows and columns of the resulting monomial matrix

$$N'=\left(\begin{array}{cccccc|ccc}
0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \alpha_1 & \dots & 0 & 0 & 0 & 0 & 0 & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \dots & \alpha_k & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \dots & 0 & t^i & 0 & 0 & 0 & 0\\
\hline
0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & t^q\\
0 & 0 & \dots & 0 & 0 & t^j & 0 & 0 & 0\\
0 & 0 & \dots & 0 & 0 & 0 & 0 & t^p & 0
\end{array}\right)$$

in the order $n-4,n-3,n-1,n,n-2,1,2,\dots,n-5$, we get the matrix $M(t^i,t^j,t^p,t^q,1,1,\alpha_1,\dots,\alpha_k)$.

2.3. Defining sequences with subsequences of type $(3,3)$. We consider the cases, when a 2-homogeneous subsequence has the form $(t^s,t^s,t^s,1,1,1)$ or $(1,1,1,t^s,t^s,t^s)$, in the same way as those in subsection 2.2; therefore we present only the main part and do not repeat similar assumptions and comments.

Proposition 3. *If $t^m=0$ and a canonically t-cyclic $n\times n$ matrix $M(v)$ is irreducible, then the sequence v does not contain a subsequence of the form $(t^s,t^s,t^s,1,1,1)$ with $m\leqslant 2s<2m$.*

We prove a more general statement replacing $(t^s,t^s,t^s,1,1,1)$ by $(t^i,t^j,t^p,1,1,1)$ with $0<p,q,j<m$, $i+j\geqslant m$, $2p\geqslant m$.

This follows from the following: if we perform with the reducible matrix

$$N=\left(\begin{array}{ccccccc|cc} 0 & 0 & 0 & \dots & 0 & 0 & -t^j & 1 & 0 \\ 1 & 0 & -t^p & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & t^i & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & t^p \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$(k=n-6)$ the transformation $[(n-1)\overset{t^j}{\to}(n-2)]^-$, $[1\overset{-1}{\to}n]^+$, $[2\overset{-t^p}{\to}(n-1)]^+$, $[3\overset{-t^p}{\to}1]^+$, and arrange the rows and columns of the resulting monomial matrix

$$N'=\left(\begin{array}{ccccccc|cc} 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & t^i & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & t^p \\ 0 & 0 & 0 & \dots & 0 & 0 & t^j & 0 & 0 \end{array}\right)$$

in the order $n-3,n-2,n,n-1,1,2,\dots,n-4$, we get the matrix $M(t^i,t^j,t^p,1,1,1,\alpha_1,\alpha_2,\dots,\alpha_k)$.

2.4. Proof of Theorem 5. The proof follows from Propositions 2 and 3. Indeed, the first proposition implies at once that $M = M(v)$ is reducible, if $w(M) = n-2$, and the second one that $M = M(v)$ is reducible, if $2 < w(M) < n-2$ (in both the cases it need to take $m = 2, s = 1$).

In conclusion, we note that in the cases $n = 2, 3$ the theorem is trivial, in the case $n = 4$ it follows from Theorem 4 and in the case $n = 5$ there is the only exception, namely the matrix $M(1, 1, t, t, t,)$ of weight 3 is irreducible.

The theorem is proved.

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