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(G, ϕ)-crossed product on (G, ϕ)-quasiassociative algebras*

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Abstract. The notions of (G, ϕ)-crossed product and quasicrossed system are introduced in the setting of (G, ϕ)-quasiassociative algebras, i.e., algebras endowed with a grading by a group G, satisfying a "quasiassociative" law. It is presented two equivalence relations, one for quasicrossed systems and another for (G, ϕ)-crossed products. Also the notion of graded-bimodule in order to study simple (G, ϕ)-crossed products is studied.

1. Introduction

The (G, ϕ)-quasiassociative algebras were introduced by H. Albuquerque and S. Majid about a decade ago [4], and during the last years have been studied with some collaborators (see [2] and the references therein). Inspired by the theory of graded rings and graded algebras ([9–12]), in the present paper we extend the concepts of crossed product and crossed

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system to the context of (G,ϕ)-quasiassociative algebras. The division (G,ϕ)-quasiassociative algebras are (G,ϕ)-crossed products, as well as some notable nonassociative algebras such as the twisted group algebras like Cayley algebras. Among them, we stand out the octonions with potential relevance to many interesting fields of mathematics, namely spinors, Bott periodicity, projective and Lorentzian geometry, Jordan algebras, and the exceptional Lie groups. We also refer its applications to physics, such as, the foundations of quantum mechanics and string theory. We prove some basic results about (G,ϕ)-crossed products and quasicrossed systems, emphasizing the case of twisted group algebras. Our work extends the study on unital antiassociative quasialgebras with semisimple even part presented in [7].

In Section 2 we introduce the basic definitions and properties related to (G,ϕ)-quasiassociative algebras. Section 3 is devoted to present some results about the set of the units of this class of algebras. In Section 4 the (G,ϕ)-crossed products and quasicrossed systems are defined, and a correspondence between them is presented with some examples. Then, in Section 5, we present two equivalence relations, one for quasicrossed systems and another for (G,ϕ)-crossed products, and in the end of this section we relate them in a suitable way. In Section 6 we study some compatibilities between (G,ϕ)-crossed products and the Cayley-Dickson process. It is shown that the quasicrossed system corresponding to the twisted group algebra obtained from the Cayley-Dickson process applied to a twisted group algebra is related to the quasicrossed system corresponding to the initial algebra. Section 7 is dedicated to simple (G,ϕ)-crossed products. The definition of representation of a (G,ϕ)-quasiassociative algebra is introduced and described in a commutative diagram. Some examples of graded modules over (G,ϕ)-quasiassociative algebras are included.

2. Preliminaries

Throughout this work, A denotes an algebra with identity element 1 over an algebraically closed field $\mathbb{K}$ with characteristic zero and G a multiplicative group with neutral element e.

Definition 2.1. A *grading* by a group G of an algebra A is a decomposition $A = \bigoplus_{g\in G} A_g$ as a direct sum of vector subspaces $\{A_g \neq 0 : g \in G\}$ of A indexed by the elements of G satisfying

$$A_g A_h \subset A_{gh} \quad \text{for any } \; g, h \in G,$$

where we denote by A_gA_h the set of all finite sums of products x_gx_h with $x_g \in A_g$ and $x_h \in A_h$. An algebra A endowed with a grading by a group G is called a *G-graded algebra*. Moreover, if A satisfies the stronger condition

$$A_gA_h = A_{gh} \quad \text{for any } \ g,h \in G,$$

it is called a *strongly G-graded algebra*.

In this paper G is generated by the set of all the elements $g \in G$ such that $A_g \neq 0$, usually called the *support* of the grading.

The subspaces A_g (with $g \in G$) are referred to as *homogeneous components* of the grading, and a nonzero element $x_g \in A_g$ is called *homogeneous* of degree g. Any nonzero element $x \in A$ can be written uniquely in the form $x = \sum_{g\in G} x_g$, where $x_g \in A_g$ and at most finitely many elements x_g are nonzero.

Given two gradings Γ and Γ' on A, Γ is a *refinement* of Γ' if any homogeneous component of Γ' is a (direct) sum of homogeneous components of Γ. A grading is *fine* if it admits no proper refinement. Throughout this paper, the gradings will be considered fine.

A subspace $B \subseteq A$ is called a *graded subspace* if $B = \bigoplus_{g\in G}(B \cap A_g)$. Equivalently, a subspace B is *graded* if for any $x \in B$, we can write $x = \sum_{g\in G} x_g$, where x_g is a homogeneous element of degree g in B, for any $g \in G$. We say that a *graded subalgebra* is a subalgebra which is a graded subspace, and we say that a *graded ideal* $I \subset A$ is a graded subspace $I = \oplus_{g\in G} I_g$ of A such that $IA + AI \subset I$.

Definition 2.2. A map $\phi : G \times G \times G \longrightarrow \mathbb{K}^\times$ is a 3-*cocycle* (in the following just *cocycle*) if

$$\phi(h,k,l)\phi(g,hk,l)\phi(g,h,k) = \phi(g,h,kl)\phi(gh,k,l), \tag{2.1}$$

$$\phi(g,e,h) = 1, \tag{2.2}$$

hold for any $g,h,k,l \in G$, where e is the identity of G.

Next lemma lists some properties of cocycles useful in the sequel.

Lemma 2.3. *If $\phi : G \times G \times G \longrightarrow \mathbb{K}^\times$ is a cocycle then the following conditions hold for any $g,h \in G$:*

(i) $\phi(e,g,h) = \phi(g,h,e) = 1$;
(ii) $\phi(g,g^{-1},g)\phi(g^{-1},g,h) = \phi(g,g^{-1},gh)$;
(iii) $\phi(g,g^{-1},g)\phi(g^{-1},g,g^{-1}) = 1$;
(iv) $\phi(h,h^{-1},g^{-1})\phi(g,h,h^{-1}) = \phi(g,h,h^{-1}g^{-1})\phi(gh,h^{-1},g^{-1})$.

Proof. First we show (i). In (2.1) we consider $k = e$ and get

$$\phi(h,e,l)\phi(g,h,l)\phi(g,h,e) = \phi(g,h,l)\phi(gh,e,l).$$

Now by (2.2) it comes $\phi(g,h,e) = 1$. We obtain the other equality in a similar way. To show (ii) we replace in (2.1) h by g^{-1}, k by g, l by h and take in account (i). The item (iii) is a particular case of (ii) with $h = g^{-1}$. Now we prove (iv) from the definition of cocycle. For any $g, h \in G$ we have

$$\begin{aligned}\phi(h,h^{-1},g^{-1})\phi(g,h,h^{-1}) &= \phi(h,h^{-1},g^{-1})\phi(g,hh^{-1},g^{-1})\phi(g,h,h^{-1})\\ &= \phi(g,h,h^{-1}g^{-1})\phi(gh,h^{-1},g^{-1})\end{aligned}$$ □

The category of G-graded vector spaces is monoidal by way of

$$\begin{aligned}\Phi_{V,W,Z} : (V \otimes W) \otimes Z &\longrightarrow V \otimes (W \otimes Z)\\ (v_g \otimes w_h) \otimes z_k &\longmapsto \phi(g,h,k) v_g \otimes (w_h \otimes z_k),\end{aligned}$$

for any homogeneous elements v_g of degree g in V, w_h of degree h in W and z_k of degree k in Z.

Definition 2.4. A map $F : G \times G \longrightarrow \mathbb{K}^\times$ is a 2-*cochain* if

$$F(e,g) = F(g,e) = 1$$

holds for any $g \in G$.

The notion of (G,ϕ)-quasiassociative algebra was introduced in [4]. This new class of algebras includes the usual associative algebras but also some notable nonassociative examples, like the octonions.

Definition 2.5. Let $\phi : G \times G \times G \longrightarrow \mathbb{K}^\times$ be an invertible cocycle. A (G,ϕ)-*quasiassociative algebra* (or simply a *quasialgebra*) is a G-graded algebra $A = \bigoplus_{g \in G} A_g$ with product map $A \otimes A \longrightarrow A$ obeying the quasiassociative law in the sense

$$(x_g x_h) x_k = \phi(g,h,k) x_g (x_h x_k), \tag{2.3}$$

for any $x_g \in A_g, x_h \in A_h, x_k \in A_k$. Moreover, a (G,ϕ)-quasiassociative algebra A is called *coboundary* if the associated cocycle is

$$\phi(g,h,k) = \frac{F(g,h)F(gh,k)}{F(h,k)F(g,hk)},$$

for a certain 2-cochain F with $g, h, k \in G$.

Remark 2.6. If A is an unital (G,ϕ)-quasiassociative algebra then A_e is an associative algebra ($1 \in A_e$) and A_g is an associative A_e-bimodule for any $g \in A_g$.

Example 2.7. All associative graded algebras are (G,ϕ)-quasiassociative algebras (with $\phi(g,h,k) = 1$ for any $g,h,k \in G$). In particular for the group $G = \mathbb{Z}_2$, the (G,ϕ)-quasiassociative algebras admit only two types of algebras. The mentioned associative case with ϕ identically 1, and the antiassociative case with $\phi(x,y,z) = (-1)^{xyz}$, for all $x,y,z \in \mathbb{Z}_2$. The antiassociative quasialgebras were considered in [3] and recently studied in [1]. For $G = \mathbb{Z}_3$, every cocycle has the form

$$\phi_{111} = \alpha, \quad \phi_{112} = \beta, \quad \phi_{121} = \frac{1}{\omega\alpha}, \quad \phi_{122} = \frac{\omega}{\beta},$$
$$\phi_{211} = \frac{\alpha}{\beta\omega}, \quad \phi_{212} = \alpha\omega, \quad \phi_{221} = \frac{\beta}{\omega\alpha}, \quad \phi_{222} = \frac{\omega}{\alpha}$$

for some nonzero $\alpha, \beta \in \mathbb{K}$ and ω a cubic root of the unity. Here ϕ_{111} is a shorthand for $\phi(1,1,1)$, etc. $\mathbb{Z}_n$-quasialgebras are studied in [5].

Lemma 2.8. *A (G,ϕ)-quasiassociative algebra A is strongly graded if and only if $1 \in A_g A_{g^{-1}}$ for all $g \in G$.*

Proof. Suppose $1 \in A_g A_{g^{-1}}$ for all $g \in G$. For any $h \in G$ it follows that

$$A_{gh} = 1A_{gh} \subset A_g A_{g^{-1}} A_{gh} \subset A_g A_h,$$

hence $A_{gh} = A_g A_h$. The converse is obvious. □

Lemma 2.9. *Let A be a strongly graded and commutative quasialgebra, then G is an abelian group.*

Proof. Since A is strongly graded, we have that $A_g A_h = A_{gh} \neq 0$ for any $g,h \in G$. Therefore there exist $x_g \in A_g$ and $x_h \in A_h$ such that $x_g x_h \neq 0$. Since A is commutative, we have that $x_g x_h = x_h x_g \neq 0$, and this implies $gh = hg$. □

Lemma 2.10. *Let A be a strongly (G,ϕ)-quasiassociative algebra. If $x \in A$ such that $xA_g = 0$ or $A_g x = 0$, for some $g \in G$, then $x = 0$.*

Proof. Let $x \in A$ such that $xA_g = 0$ for some $g \in G$ (the another case is analogue). Then we have $xA_g A_{g^{-1}} = 0$, or equivalently $xA_e = 0$. From $1 \in A_e$, we conclude that $x = 0$. □

Remark 2.11. By the previous result we have for a strongly (G,ϕ)-quasiassociative algebra that always the support of the grading must be the entire G.

3. Units of a (G, ϕ)-quasiassociative algebra

Definition 3.1. An element u of a (G, ϕ)-quasiassociative algebra A is called a *left unit* if there exists a left inverse $u_L^{-1} \in A$ such that $u_L^{-1}u = 1$. Similarly, u is said a *right unit* if there exists a right inverse $u_R^{-1} \in A$ such that $uu_R^{-1} = 1$. By an *unit* (or *invertible element*) we mean an element $u \in A$ that has a left and right inverses. We denote by $U(A)$ the set of all units of A.

Definition 3.2. An unit u of A is *graded* if $u \in A_g$ for some $g \in G$. The set of all graded units of A is denoted by $Gr\ U(A)$ and we have $Gr\ U(A) = \bigcup_{g\in G}(U(A) \cap A_g)$.

Lemma 3.3. *Let u be a graded unit of degree g of a (G, ϕ)-quasiassociative algebra A. The following assertions hold.*

(i) *The left inverse u_L^{-1} and the right inverse u_R^{-1} of u have degree g^{-1} and are related by $u_R^{-1} = \phi(g^{-1}, g, g^{-1})u_L^{-1}$.*

(ii) *The left inverse u_L^{-1} and the right inverse u_R^{-1} of u are unique.*

(iii) *If w is another graded unit of A of degree h, then the product uw is a graded unit of degree gh such that,*

$$(uw)_L^{-1} = \frac{\phi(g^{-1}, g, h)}{\phi(h^{-1}, g^{-1}, gh)} w_L^{-1}u_L^{-1},$$
$$(uw)_R^{-1} = \frac{\phi(h, h^{-1}, g^{-1})}{\phi(g, h, h^{-1}g^{-1})} w_R^{-1}u_R^{-1}.$$

(iv) *The set $Gr\ U(A)$ is closed under product and inverse.*

Proof. (i) We show that $u_L^{-1} \in A_{g^{-1}}$ (it is similar for the right inverse). We can write $u_L^{-1} = \sum_{h\in G} u_h$, where $u_h \in A_h$ and at most finitely many elements u_h are nonzero. From $1 = u_L^{-1}u = \sum_{h\in G} u_h u$ it follows that $u_h = 0$ unless $h = g^{-1}$. Thus $u_L^{-1} = u_{g^{-1}}$ has degree g^{-1}. The quasiassociativity of A gives

$$u_R^{-1} = 1u_R^{-1} = (u_L^{-1}u)u_R^{-1} = \phi(g^{-1}, g, g^{-1})u_L^{-1}(uu_R^{-1}) = \phi(g^{-1}, g, g^{-1})u_L^{-1}$$

as desired (cf. [6, 7]).

(ii) Suppose that there exist u_L^{-1} and u'^{-1}_L two left inverses of u, meaning that $u_L^{-1}u = 1$ and $u'^{-1}_L u = 1$. Then $u_L^{-1}u = u'^{-1}_L u$. Since u is an unit of A, there exists u_R^{-1} satisfying $uu_R^{-1} = 1$. We may write $(u_L^{-1}u)u_R^{-1} = (u'^{-1}_L u)u_R^{-1}$, hence $\phi(g^{-1}, g, g^{-1})u_L^{-1}(uu_R^{-1}) = \phi(g^{-1}, g, g^{-1})u'^{-1}_L(uu_R^{-1})$ and we obtain $u_L^{-1} = u'^{-1}_L$. The case with the right unit is analogue.

(iii) As $A_gA_h \subset A_{gh}$ then uw is a homogeneous element of degree gh. Since A is quasiassociative, we get the expression of the left inverse of uw doing

$$
\begin{aligned}
&(w_L^{-1}u_L^{-1})(uw) \\
&\quad = \phi(h^{-1},g^{-1},gh)w_L^{-1}\Big(u_L^{-1}(uw)\Big) = \frac{\phi(h^{-1},g^{-1},gh)}{\phi(g^{-1},g,h)}w_L^{-1}\Big((u_L^{-1}u)w\Big) \\
&\quad = \frac{\phi(h^{-1},g^{-1},gh)}{\phi(g^{-1},g,h)}w_L^{-1}w = \frac{\phi(h^{-1},g^{-1},gh)}{\phi(g^{-1},g,h)}.
\end{aligned}
$$

In a similar way we obtain the right inverse of uw (cf. [6, 7]).

(iv) By (iii) we conclude that $Gr\ U(A)$ is closed under product. To show that $Gr\ U(A)$ is closed under inverse, meaning that whenever u is a graded unit then u_L^{-1} and u_R^{-1} are graded units too, we use (i) and observe that

$$\Big(\phi(g^{-1},g,g^{-1})u\Big)u_L^{-1} = uu_R^{-1} = 1$$

and

$$u_R^{-1}\Big(\frac{1}{\phi(g^{-1},g,g^{-1})}u\Big) = u_L^{-1}u = 1$$

completing the proof. □

Remark 3.4. From Lemma 3.3(i)-(ii), the left and right inverses of any $u \in U(A) \cap A_g$ are also graded units of A and

$$
\begin{aligned}
(u_L^{-1})_R^{-1} &= u, & (u_L^{-1})_L^{-1} &= \phi(g^{-1},g,g^{-1})u,\\
(u_R^{-1})_L^{-1} &= u, & (u_R^{-1})_R^{-1} &= \frac{1}{\phi(g^{-1},g,g^{-1})}u.
\end{aligned}
$$

Lemma 3.5. *If A is a graded associative algebra, left and right inverses are equal.*

Proof. It is easy to chek that $u_L^{-1} = u_L^{-1}(uu_R^{-1}) = (u_L^{-1}u)u_R^{-1} = u_R^{-1}$ for any $u \in U(A)$. □

Corollary 3.6. *The left and right inverses of $u \in U(A) \cap A_e$ are equal and belong to A_e. Moreover, $U(A) \cap A_e = U(A_e)$.*

Proof. By Lemma 3.3-(i), the left and right inverses of u belong to A_e and $u_R^{-1} = \phi(e,e,e)u_L^{-1} = u_L^{-1}$. Therefore $U(A) \cap A_e \subseteq U(A_e)$. The converse is trivial. □

Remark 3.7. The map $deg : Gr\ U(A) \to G$ preserves the product and the elements $u \in Gr\ U(A)$ such that $deg\, u = e$ consist in the set $U(A) \cap A_e = U(A_e)$.

Lemma 3.8. (i) *The map* $\mu : Gr\ U(A) \to Aut(A_e)$ *defined by*

$$\mu(u)(x) := uxu_R^{-1} \quad \text{for any } u \in Gr\ U(A) \text{ and } x \in A_e,$$

satisfies $\mu(uw) = \mu(u) \circ \mu(w)$ *for all* $u, w \in Gr\ U(A)$.

(ii) *The right multiplication by* $u \in A_g \cap U(A)$ *is an isomorphism* $A_e \to A_e u = A_g$ *of left* A_e*-modules.*

Proof. (i) Let u, w be two graded units of A such that $deg\, u = g$ and $deg\, w = h$. Using Lemma 2.3-(iv) we obtain for any $x \in A_e$,

$$\begin{aligned}
\mu(uw)(x) = (uw)x(uw)_R^{-1} &= (uw)x\frac{\phi(h,h^{-1},g^{-1})}{\phi(g,h,h^{-1}g^{-1})}(w_R^{-1}u_R^{-1}) \\
&= \frac{\phi(h,h^{-1},g^{-1})}{\phi(g,h,h^{-1}g^{-1})}(uw)\Big(x(w_R^{-1}u_R^{-1})\Big) \\
&= \frac{\phi(h,h^{-1},g^{-1})}{\phi(g,h,h^{-1}g^{-1})}(uw)\Big((xw_R^{-1})u_R^{-1}\Big) \\
&= \frac{\phi(h,h^{-1},g^{-1})}{\phi(g,h,h^{-1}g^{-1})\phi(gh,h^{-1},g^{-1})}\Big((uw)(xw_R^{-1})\Big)u_R^{-1} \\
&= \frac{\phi(h,h^{-1},g^{-1})\phi(g,h,h^{-1})}{\phi(g,h,h^{-1}g^{-1})\phi(gh,h^{-1},g^{-1})}\Big(u(wxw_R^{-1})\Big)u_R^{-1} \\
&= u(wxw_R^{-1})u_R^{-1} = \mu(u)\circ\mu(w)(x)
\end{aligned}$$

(ii) First we prove that the right multiplication is a monomorphism. Let $x, y \in A_e$ such that $xu = yu$. Thus $(xu)u_R^{-1} = (yu)u_R^{-1}$. Since $x, y \in A_e$ then $x(uu_R^{-1}) = y(uu_R^{-1})$ and $x = y$. To prove that it is an epimorphism, we need to see if for any $v \in A_g$ there exists $x \in A_e$ such that $xu = v$. We get it taking $x = vu_R^{-1}$. □

4. (G, ϕ)-crossed products and quasicrossed systems

In this section we introduce the concept of (G, ϕ)-crossed product in the context of (G, ϕ)-quasiassociative algebras.

Definition 4.1. Let A be a (G, ϕ)-quasiassociative algebra. We say that A is a (G, ϕ)-*crossed product* of G over A_e if for any $g \in G$ there exists $\overline{g} \in U(A) \cap A_g$, meaning that, there exists an unit $\overline{g}$ in A of any degree g.

The following examples illustrate that some important quasialgebras are (G,ϕ)-crossed products.

Example 4.2. Any division (G,ϕ)-quasiassociative algebra $A = \oplus_{g\in G} A_g$ is trivially a quasicrossed product of G over A_e, because $1 \in A_e$ and every nonzero homogeneous element is invertible.

Example 4.3. Interesting examples of division (G,ϕ)-quasiassociative algebras, so of (G,ϕ)-crossed products, are twisted group algebras $\mathbb{K}_F G$ (see [4]). We present properly this class of algebras since we will pay special attention to them in this paper. Consider the *group algebra* $\mathbb{K}G$, the set of all linear combinations of elements $\sum_{g\in G} a_g g$, where $a_g \in \mathbb{K}$ such that $a_g = 0$ for all but finitely many elements g. We define $\mathbb{K}_F G$ with the same underlying vector space as $\mathbb{K}G$ but with a modified product $g.h := F(g,h)gh$, for any $g,h \in G$, where F is a 2-cochain on G. Then $\mathbb{K}_F G$ is a coboundary graded quasialgebra. Moreover, any $\mathbb{K}_F G$ is a (G,ϕ)-crossed product. In fact, given $g \in G$ and $a_g \in \mathbb{K}^\times$ then the homogeneous element $a_g g \in (\mathbb{K}_F G)_g$ is an unit with left inverse and right inverse:

$$(a_g g)_L^{-1} = \phi(g, g^{-1}, g)(a_g g)_R^{-1} = F(g^{-1}, g)^{-1} a_g^{-1} g^{-1}.$$

There are two classes of modified group algebras particularly interesting, namely the Cayley algebras and the Clifford algebras. We mention just some well studied Cayley algebras:

1) The complex algebra $\mathbb{C}$ is a quasialgebra $\mathbb{K}_F G$ with $G = \mathbb{Z}_2$ and $F(x,y) = (-1)^{xy}$, for $x,y \in \mathbb{Z}_2$.
2) The quaternion algebra $\mathbb{H}$ is a quasialgebra $\mathbb{K}_F G$ with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $F(\overrightarrow{x}, \overrightarrow{y}) = (-1)^{x_1 y_1 + (x_1 + x_2) y_2}$, where $\overrightarrow{x} = (x_1, x_2) \in \mathbb{Z}_2 \times \mathbb{Z}_2$ is a vector notation.
3) The octonion algebra $\mathbb{O}$ is another quasialgebra $\mathbb{K}_F G$ for $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $F(\overrightarrow{x}, \overrightarrow{y}) = (-1)^{\sum_{i\leqslant j} x_i y_j + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3}$, where $\overrightarrow{x} = (x_1, x_2, x_3) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Any Clifford algebra is a quasialgebra $\mathbb{K}_F G$ for $G = (\mathbb{Z}_2)^n$ and 2-cochain $F(\overrightarrow{x}, \overrightarrow{y}) = (-1)^{\sum_{i\leqslant j} x_i y_j}$ where $\overrightarrow{x} = (x_1, ..., x_n) \in (\mathbb{Z}_2)^n$. Recall that $\mathbb{C}$ and $\mathbb{H}$ are both Cayley and Clifford algebras.

Example 4.4. Let $\mathrm{Mat}_n(\Delta)$ be the $\mathbb{Z}_2$-graded algebra of the $n \times n$ matrices over Δ with the natural $\mathbb{Z}_2$-grading inherited from Δ, where $\Delta = \Delta_{\bar{0}} \oplus \Delta_{\bar{1}}$ is a division antiassociative quasialgebra ($\simeq \langle D, \sigma, a\rangle$ see [3], where σ is an automorphism of D and a is a nonzero element of D

such that $\sigma^2 = \tau_a : d \longrightarrow ada^{-1}$ with $\sigma(a) = -a)$ and $n \in \mathbb{N}$. Consider $\mathrm{Mat}_n(\Delta) = \mathrm{Mat}_n(\Delta_{\bar{0}}) \oplus \mathrm{Mat}_n(\Delta_{\bar{0}})u$ equipped with multiplication defined by

$$A(Bu) = (AB)u, \quad (Au)B = (A\overline{B})u \quad \text{and} \quad (Au)(Bu) = aA\overline{B}$$

for all $A, B \in \mathrm{Mat}_n(\Delta_{\bar{0}})$, where the matrix $\overline{B}$ is obtained from the matrix $B = [b_{ij}]_{1\leqslant i,j\leqslant n}$ by replacing the term b_{ij} by $\sigma(b_{ij})$, for all $i, j \in \{1, \ldots, n\}$. Then the simple antiassociative quasialgebra $\mathrm{Mat}_n(\Delta)$ is clearly a (G, ϕ)-crossed product of $\mathbb{Z}_2$ over $\mathrm{Mat}_n(\Delta_{\bar{0}})$. It is clear that id is an unit in $\mathrm{Mat}_n(\Delta_{\bar{0}})$ and id u is an unit in $\mathrm{Mat}_n(\Delta_{\bar{0}})u$.

For $n \in \mathbb{N}$, the set $\widetilde{Mat}_{n,n}(D)$ of $2n \times 2n$ matrices over a division algebra D, with the chess board $\mathbb{Z}_2$-grading:

$$\widetilde{\mathrm{Mat}}_{n,n}(D)_{\bar{0}} := \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in \mathrm{Mat}_n(D), b \in \mathrm{Mat}_n(D) \right\}$$
$$\widetilde{\mathrm{Mat}}_{n,n}(D)_{\bar{1}} := \left\{ \begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix} : v \in \mathrm{Mat}_n(D), w \in \mathrm{Mat}_n(D) \right\},$$

and with multiplication given by

$$\begin{pmatrix} a_1 & v_1 \\ w_1 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & v_2 \\ w_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + v_1w_2 & a_1v_2 + v_1b_2 \\ w_1a_2 + b_1w_2 & -w_1v_2 + b_1b_2 \end{pmatrix}$$

is a (G, ϕ)-crossed product. Indeed, let $a \in Mat_n(D)$ and $b \in Mat_n(D)$ be two invertible matrices, then $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is an unit in $\widetilde{Mat}_{n,n}(D)_{\bar{0}}$ with

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_R^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_L^{-1} = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}.$$

These examples show that there are (G, ϕ)-crossed products which are not division (G, ϕ)-quasiassociative algebras.

Example 4.5. Consider the $(\mathbb{Z}_n, \phi)$-quasiassociative algebra of the deformed matrices $M_{n,\phi}(\mathbb{K})$ of the usual $n \times n$ matrices $M_n(\mathbb{K})$ with the basis elements E_{ij} of degree $j - i$, for $i, j \in \mathbb{Z}_n$, and the multiplication

$$(X \cdot Y)_{ij} = \sum_{k=1}^{n} \frac{\phi(i, -k, k - j)}{\phi(-k, k, -j)} X_{ik}Y_{kj},$$

for any $X = (X_{ij})$ and $Y = (Y_{ij})$ in $M_n(\mathbb{K})$ (cf. in [8]). This $(\mathbb{Z}_n, \phi)$-quasiassociative algebra is a (G, ϕ)-crossed product. Indeed, we easily find an invertible element in each homogeneous component of $M_{n,\phi}(\mathbb{K})$.

Remark 4.6. Observe that not all (G,ϕ)-quasiassociative algebras are (G,ϕ)-crossed products. For example, we can easily extract subalgebras of the algebra of Example 4.5 which are not (G,ϕ)-crossed products. The subalgebra $T_{n,\phi}(\mathbb{K})$ of the $(\mathbb{Z}_n,\phi)$-quasiassociative algebra $M_{n,\phi}(\mathbb{K})$ formed by the upper triangular matrices is not a (G,ϕ)-crossed product. For example, the 1-dimensional homogeneous component $(T_{n,\phi}(\mathbb{K}))_{n-1}$ with basis $\{E_{1n}\}$ does not contain an invertible element.

Definition 4.7. Assume that B is an associative algebra. Given maps $\sigma : G \to Aut(B)$, automorphism system, $\alpha : G \times G \to U(B)$, quasicrossed mapping, and a cocycle $\phi : G \times G \times G \longrightarrow \mathbb{K}^{\times}$, we say that $(G, B, \phi, \sigma, \alpha)$ is a *quasicrossed system* for G over B if the following properties hold:

$$\sigma(g)\Big(\sigma(h)(x)\Big) = \alpha(g,h)\sigma(gh)(x)\alpha(g,h)^{-1} \tag{4.1}$$

$$\alpha(g,h)\alpha(gh,k) = \phi(g,h,k)\sigma(g)(\alpha(h,k))\alpha(g,hk) \tag{4.2}$$

$$\alpha(g,e) = \alpha(e,g) = 1 \tag{4.3}$$

for any $g,h,k \in G$ and $x \in B$.

Let A be a (G,ϕ)-quasiassociative algebra which is a quasicrossed product of G over A_e. Then for any $g \in G$ there exists an unit $\overline{g} \in U(A) \cap A_g$ with $\overline{e} = 1$. Define a map $\sigma(g) : A_e \longrightarrow A_e$ by

$$\sigma(g)(x) := \overline{g}x{\overline{g}_R}^{-1} \quad \text{for any } x \in A_e. \tag{4.4}$$

Lemma 4.8. *For any $g \in G$, $\sigma(g)$ is an automorphism of A_e, meaning that for $x, y \in A_e$*

$$\sigma(g)(xy) = \sigma(g)(x)\sigma(g)(y).$$

Proof. For any $g \in G$, as $\overline{g}$ is an unit it is obvious that the map $\sigma(g)$ is bijective. Applying Lemma 3.3-(i), we obtain for any $g \in G$ and $x, y \in A_e$

$$\begin{aligned}
\sigma(g)(xy) &= \overline{g}xy\overline{g}_R^{-1} = \Big(\overline{g}x(\overline{g}_L^{-1}\overline{g})\Big)y\overline{g}_R^{-1} \\
&= \frac{1}{\phi(g,g^{-1},g)}\Big((\overline{g}x\overline{g}_L^{-1})\overline{g}\Big)y\overline{g}_R^{-1} = \frac{1}{\phi(g,g^{-1},g)}(\overline{g}x\overline{g}_L^{-1})(\overline{g}y\overline{g}_R^{-1}) \\
&= (\overline{g}x\overline{g}_R^{-1})(\overline{g}y\overline{g}_R^{-1}) = \sigma(g)(x)\sigma(g)(y)
\end{aligned}$$

as desired. □

Proposition 4.9. *Let A be a (G,ϕ)-quasiassociative algebra which is a (G,ϕ)-crossed product of G over A_e. For any $g \in G$, fix an unit $\overline{g}$ in A_g*

with $\overline{e} = 1$*. Let* $\sigma : G \to Aut(A_e)$ *be the corresponding automorphism system given by Equation* (4.4) *and* $\alpha : G \times G \to U(A_e)$ *defined by*

$$\alpha(g,h) := (\overline{g}\overline{h})(\overline{gh})_R^{-1} = \phi((gh)^{-1}, gh, (gh)^{-1})(\overline{g}\overline{h})(\overline{gh})_L^{-1}, \tag{4.5}$$

for any $g,h \in G$*. Then the following properties hold:*

(i) *A is a strongly (G,ϕ)-quasiassociative algebra with $A_g = A_e\overline{g} = \overline{g}A_e$.*

(ii) *$(G, A_e, \phi, \sigma, \alpha)$ is a quasicrossed system for G over A_e (to which we refer as corresponding to A).*

(iii) *A is a free (left or right) A_e-module freely generated by the elements $\overline{g}$, where $g \in G$.*

(iv) *For all $g,h \in G$ and $x,y \in A_e$,*

$$(x\overline{g})(y\overline{h}) = x\sigma(g)(y)\alpha(g,h)\overline{gh}. \tag{4.6}$$

Conversely, for any associative algebra B and any quasicrossed system (G,B,ϕ,σ,α) for G over B, the free B-module C freely generated by the elements $\overline{g}$, for $g \in G$, with multiplication given by Equation (4.6) *(with $x,y \in B$) is a (G,ϕ)-quasiassociative algebra (with $C_g = B\overline{g}$ for all $g \in G$) which is a (G,ϕ)-crossed product of G over $C_e = B$ and having (G,B,ϕ,σ,α) as a corresponding quasicrossed system.*

Remark 4.10. We note that Proposition 4.9 generalizes the results on quasiassociative division algebras presented by H. Albuquerque and A.P. Santana (see Theorem 1.1 in [7] and Theorem 3.2 in [8]). The quasiassociative division algebras are precisely the (G,ϕ)-crossed products over the division associative algebras. Moreover, the three identities defining the multiplication in quasiassociative division algebras are now condensed in equation (4.6).

Proof. (i) Let $g \in G$ and take $u \in U(A) \cap A_g$. By Lemma 3.3-(i),

$$u_L^{-1}, u_R^{-1} \in A_{g^{-1}}$$

and therefore $1 = u_L^{-1}u \in A_{g^{-1}}A_g$ and $1 = uu_R^{-1} \in A_gA_{g^{-1}}$. By Lemma 2.8, we conclude that A is a strongly graded quasialgebra. Applying Lemma 3.8-(iii), $A_g = A_e\overline{g}$ and the argument of this lemma applied to left multiplication shows that $A_g = \overline{g}A_e$, proving this item.

(ii) First we prove condition (4.1). Let $g,h \in G$ and $x \in A_e$. We have

$$\sigma(g)(\sigma(h)(x)) = \sigma(g)\Big(\overline{h}x\overline{h}_R^{-1}\Big)$$

$$= \Big(\overline{g}(\overline{h}(x\overline{h}_R^{-1}))\Big)\overline{g}_R^{-1} = \frac{1}{\phi(g,h,h^{-1})}\Big((\overline{g}\overline{h})(x\overline{h}_R^{-1})\Big)\overline{g}_R^{-1}$$
$$= \frac{\phi(gh,h^{-1},g^{-1})}{\phi(g,h,h^{-1})}(\overline{g}\overline{h})\Big((x\overline{h}_R^{-1})\overline{g}_R^{-1}\Big) = \frac{\phi(gh,h^{-1},g^{-1})}{\phi(g,h,h^{-1})}(\overline{g}\overline{h})x(\overline{h}_R^{-1}\overline{g}_R^{-1})$$

and we get

$$\sigma(g)(\sigma(h)(x)) = \frac{\phi(gh,h^{-1},g^{-1})}{\phi(g,h,h^{-1})}(\overline{g}\overline{h})x(\overline{h}_R^{-1}\overline{g}_R^{-1}). \tag{4.7}$$

Now, using Lemma 3.3 we observe that

$$(\overline{gh})_R^{-1}(\alpha(g,h))_R^{-1} = (\overline{gh})_R^{-1}\Big(\phi((gh)^{-1},gh,(gh)^{-1})(\overline{g}\overline{h})(\overline{gh})_L^{-1}\Big)_R^{-1}$$
$$= \frac{\phi((gh)^{-1},gh,(gh)^{-1})}{\phi((gh)^{-1},gh,(gh)^{-1})\phi(gh,(gh)^{-1},gh(gh)^{-1})}$$
$$\times (\overline{gh})_R^{-1}\Big(((\overline{gh})_L^{-1})_R^{-1}(\overline{g}\overline{h})_R^{-1}\Big)$$
$$= (\overline{gh})_R^{-1}\Big(\overline{gh}(\overline{g}\overline{h})_R^{-1}\Big) = \frac{\phi((gh)^{-1},gh,(gh)^{-1})}{\phi((gh)^{-1},gh,(gh)^{-1})}\Big((\overline{gh})_L^{-1}\overline{gh}\Big)(\overline{g}\overline{h})_R^{-1}$$
$$= \Big((\overline{gh})_L^{-1}\overline{gh}\Big)(\overline{g}\overline{h})_R^{-1} = \frac{\phi(h,h^{-1},g^{-1})}{\phi(g,h,h^{-1}g^{-1})}\overline{h}_R^{-1}\overline{g}_R^{-1}$$

and hence

$$\overline{h}_R^{-1}\overline{g}_R^{-1} = \frac{\phi(g,h,h^{-1}g^{-1})}{\phi(h,h^{-1},g^{-1})}(\overline{gh})_R^{-1}(\alpha(g,h))_R^{-1}. \tag{4.8}$$

Applying Lemma 2.3-(iii) we obtain

$$\alpha(g,h)\overline{gh} = \Big(\phi((gh)^{-1},gh,(gh)^{-1})(\overline{g}\overline{h})(\overline{gh})_L^{-1}\Big)\overline{gh}$$
$$= \phi((gh)^{-1},gh,(gh)^{-1})\phi(gh,(gh)^{-1},gh)(\overline{g}\overline{h})\Big((\overline{gh})_L^{-1}\overline{gh}\Big) = \overline{g}\overline{h}$$

then

$$\alpha(g,h)\overline{gh} = \overline{g}\,\overline{h}. \tag{4.9}$$

Returning to (4.7), using (4.8) and (4.9) we have

$$\sigma(g)(\sigma(h)(x))$$
$$= \frac{\phi(gh,h^{-1},g^{-1})\phi(g,h,h^{-1}g^{-1})}{\phi(g,h,h^{-1})\phi(h,h^{-1},g^{-1})}\alpha(g,h)\overline{gh}x(\overline{gh})_R^{-1}(\alpha(g,h))_R^{-1}$$

and by Lemma 2.3-(iv) we conclude

$$\sigma(g)(\sigma(h)(x)) = \alpha(g,h)\sigma(gh)(x)(\alpha(g,h))_R^{-1}$$

proving (4.1). For any $g,h,k \in G$, by Lemma 4.8 and using condition (4.9) we have

$$(\overline{g}\,\overline{h})\overline{k} = (\alpha(g,h)\overline{gh})\overline{k} = \alpha(g,h)(\overline{gh}\,\overline{k}) = \alpha(g,h)\alpha(gh,k)\overline{ghk}.$$

On the other hand, we obtain

$$\overline{g}(\overline{h}\,\overline{k}) = \overline{g}(\alpha(h,k)\overline{hk}) = (\overline{g}\alpha(h,k))\overline{hk}. \tag{4.10}$$

Using Lemma 2.3-(iii) we have

$$\begin{aligned}\sigma(g)(\alpha(h,k))\overline{g} &= \Big(\overline{g}\alpha(h,k)\overline{g}_R{}^{-1}\Big)\overline{g} = \phi(g^{-1},g,g^{-1})\Big(\overline{g}\alpha(h,k)\overline{g_L}{}^{-1}\Big)\overline{g}\\ &= \phi(g^{-1},g,g^{-1})\phi(g,g^{-1},g)\Big(\overline{g}\alpha(h,k)\Big)\Big(\overline{g}_L{}^{-1}\overline{g}\Big) = \overline{g}\alpha(h,k).\end{aligned}$$

Returning to (4.10)

$$\begin{aligned}\overline{g}(\overline{h}\,\overline{k}) &= (\overline{g}\alpha(h,k))\overline{hk} = \Big(\sigma(g)(\alpha(h,k))\overline{g}\Big)\overline{hk} = \sigma(g)(\alpha(h,k))(\overline{g}\,\overline{hk})\\ &= \sigma(g)(\alpha(h,k))\alpha(g,hk)\overline{ghk}.\end{aligned}$$

Since G is associative and $(\overline{g}\overline{h})\overline{k} = \phi(g,h,k)\overline{g}(\overline{h}\,\overline{k})$, we conclude that

$$\alpha(g,h)\alpha(gh,k) = \phi(g,h,k)\sigma(g)(\alpha(h,k))\alpha(g,hk)$$

proving (4.2). Because $\overline{e} = 1$ we have

$$\begin{aligned}\alpha(g,e) &= \phi((ge)^{-1},ge,(ge)^{-1})(\overline{g}\,\overline{e})(\overline{ge})_L^{-1} = \phi(g^{-1},g,g^{-1})\overline{g}\,\overline{g}_L^{-1}\\ &= \frac{\phi(g^{-1},g,g^{-1})}{\phi(g^{-1},g,g^{-1})}\overline{g}\,\overline{g}_R^{-1} = 1\end{aligned}$$

thus (4.3) is also true, proving (ii).

(iii) It is a direct consequence of (i).

(iv) Let $g,h \in G$ and $x,y \in A_e$. Using Lemma 2.3-(ii) we obtain

$$\begin{aligned}(x\overline{g})(y\overline{h}) &= (x\overline{g})\Big((y(\overline{g}_L^{-1}\overline{g}))\overline{h}\Big) = (x\overline{g})\Big(((y\overline{g}_L^{-1})\overline{g})\overline{h}\Big)\\ &= \phi(g^{-1},g,h)(x\overline{g})((y\overline{g}_L^{-1})(\overline{g}\overline{h})) = \frac{\phi(g^{-1},g,h)}{\phi(g,g^{-1},gh)}((x\overline{g})(y\overline{g}_L^{-1}))(\overline{g}\overline{h})\end{aligned}$$

$$= \frac{\phi(g^{-1},g,h)}{\phi(g,g^{-1},gh)\phi(g^{-1},g,g^{-1})}((x\overline{g})(y\overline{g}_R^{-1}))(\overline{g}\overline{h})$$
$$= x\sigma(g)(y)(\overline{g}\overline{h}) = x\sigma(g)(y)\alpha(g,h)\overline{gh}$$

proving (4.6). To prove the converse, we need only to verify that the multiplication given by (4.6) is quasiassociative. In fact, let $g,h,k \in G$ and $x,y,z \in B$. As $\sigma(g) \in Aut(B)$ we have

$$\begin{aligned}(x\overline{g})\Big((y\overline{h})(z\overline{k})\Big) &= (x\overline{g})\Big(y\sigma(h)(z)\alpha(h,k)\overline{hk}\Big)\\ &= x\sigma(g)\Big(y\sigma(h)(z)\alpha(h,k)\Big)\alpha(g,hk)\overline{ghk}\\ &= x\sigma(g)(y)\sigma(g)(\sigma(h)(z))\sigma(g)(\alpha(h,k))\alpha(g,hk)\overline{ghk}\\ &\overset{(4.1)}{=} x\sigma(g)(y)\alpha(g,h)\sigma(gh)(z)\alpha(g,h)^{-1}\sigma(g)(\alpha(h,k))\alpha(g,hk)\overline{ghk}\\ &\overset{(4.2)}{=} \frac{1}{\phi(g,h,k)}x\sigma(g)(y)\alpha(g,h)\sigma(gh)(z)\alpha(g,h)^{-1}\alpha(g,h)\alpha(gh,k)\overline{ghk}\\ &= \frac{1}{\phi(g,h,k)}x\sigma(g)(y)\alpha(g,h)\sigma(gh)(z)\alpha(gh,k)\overline{ghk}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Big((x\overline{g})(y\overline{h})\Big)(z\overline{k}) &= \Big(x\sigma(g)(y)\alpha(g,h)\overline{gh}\Big)(z\overline{k})\\ &= x\sigma(g)(y)\alpha(g,h)\sigma(gh)(z)\alpha(gh,k)\overline{ghk}\end{aligned}$$

therefore,

$$\Big((x\overline{g})(y\overline{h})\Big)(z\overline{k}) = \phi(g,h,k)(x\overline{g})\Big((y\overline{h})(z\overline{k})\Big)$$

completing the proof. □

5. Equivalence on (G,ϕ)-crossed products and on quasicrossed systems

In this section we present two equivalence relations, one for quasicrossed systems and another for (G,ϕ)-crossed products.

Definition 5.1. We say that two quasicrossed systems (G,B,ϕ,σ,α) and $(G,B,\phi,\sigma',\alpha')$ over an associative algebra B for a fixed cocycle $\phi : G\times G\times G \longrightarrow \mathbb{K}^{\times}$ are *equivalent* if there exists a map $u : G \to U(B)$ with $u(e)=1$ such that

$$\sigma'(g) = i_{u(g)} \circ \sigma(g) \qquad (5.1)$$

$$\alpha'(g,h) = u(g)\sigma(g)(u(h))\alpha(g,h)u(gh)^{-1}, \tag{5.2}$$

for any $g, h \in G$, where $i_y(x) = yxy^{-1}$ for $x \in B$ and $y \in U(B)$.

We define an equivalence relation in the class of quasicrossed systems over an associative algebra B for a fixed cocycle $\phi : G \times G \times G \longrightarrow \mathbb{K}^\times$. Assume that a (G,ϕ)-quasiassociative algebra A is a (G,ϕ)-crossed product of G over A_e. Due to Proposition 4.9, any choice of an unit $\overline{g}$ of A in A_g, for any $g \in G$, with $\overline{e} = 1$, determines a corresponding quasicrossed system $(G, A_e, \phi, \sigma, \alpha)$ for G over A_e, with σ and α given by

$$\sigma(g)(x) = \overline{g}x\overline{g}_R^{-1} \quad \text{and} \quad \alpha(g,h) = (\overline{g}\,\overline{h})(\overline{gh})_R^{-1}$$

for any $x \in A_e$ and $g, h \in G$. Now, let $\{\widetilde{g} : g \in G\}$ be another set of units and $(G, A_e, \phi, \sigma', \alpha')$ be the corresponding quasicrossed system. Because $\widetilde{g} \in A_g$, we infer from Proposition 4.9-(i) that there is a map $u : G \to U(A_e)$ with $u(e) = 1$ such that

$$\widetilde{g} = u(g)\overline{g} \quad \text{for all} \quad g \in G.$$

We note that $u(g)$ is indeed an unit of A_e with inverse $u(g)^{-1} = \overline{g}\widetilde{g}_R^{-1}$.

Lemma 5.2. *In the previous conditions we have that the quasicrossed systems $(G, A_e, \phi, \sigma, \alpha)$ and $(G, A_e, \phi, \sigma', \alpha')$ are equivalent over the associative algebra A_e.*

Proof. For $g \in G$ and $x \in A_e$ we have

$$\begin{aligned}\sigma'(g)(x) &= \widetilde{g}x\widetilde{g}_R^{-1} = u(g)\overline{g}x(u(g)\overline{g})_R^{-1} = u(g)\overline{g}x\overline{g}_R^{-1}u(g)^{-1}\\ &= u(g)(\overline{g}x\overline{g}_R^{-1})u(g)^{-1} = u(g)\sigma(g)(x)u(g)^{-1} = i_{u(g)}(\sigma(g)(x))\end{aligned}$$

proving (5.1).

For $g, h \in G$, using Lemma 2.3-(ii),(iii) we have

$$\begin{aligned}u(gh)\overline{gh} &= \widetilde{gh} = \alpha'(g,h)^{-1}\widetilde{g}\widetilde{h}\\ &= \alpha'(g,h)^{-1}u(g)\overline{g}u(h)\overline{h} = \alpha'(g,h)^{-1}(u(g)\overline{g})(u(h)(\overline{g}_L^{-1}\overline{g})\overline{h})\\ &= \frac{1}{\phi(g^{-1},g,g^{-1})}\alpha'(g,h)^{-1}(u(g)\overline{g})(u(h)(\overline{g}_R^{-1}\overline{g})\overline{h})\\ &= \frac{\phi(g^{-1},g,h)}{\phi(g^{-1},g,g^{-1})}\alpha'(g,h)^{-1}(u(g)\overline{g})(u(h)\overline{g}_R^{-1}(\overline{g}\overline{h}))\\ &= \frac{\phi(g^{-1},g,h)}{\phi(g^{-1},g,g^{-1})\phi(g,g^{-1},gh)}\alpha'(g,h)^{-1}u(g)\Big(\overline{g}u(h)\overline{g}_R^{-1}\Big)(\overline{g}\overline{h})\end{aligned}$$

$$= \alpha'(g,h)^{-1}u(g)\sigma(g)(u(h))(\overline{g}\overline{h})$$
$$= \alpha'(g,h)^{-1}u(g)\sigma(g)(u(h))\alpha(g,h)\overline{gh}$$

therefore

$$\alpha'(g,h) = u(g)\sigma(g)(u(h))\alpha(g,h)u(gh)^{-1}$$

proving (5.2). Consequently, $(G, A_e, \phi, \sigma, \alpha)$ and $(G, A_e, \phi, \sigma', \alpha')$ are equivalent as desired. □

Thus any given (G, ϕ)-quasiassociative algebra A which is a (G, ϕ)-crossed product of G over A_e defines a unique equivalence class of corresponding quasicrossed systems for G over A_e. We emphasize the independence of the choice of the sets of units used to define the quasicrossed systems.

Definition 5.3. Assume that A, A' are two (G, ϕ)-crossed products of G over A_e. We say that A and A' are *equivalent* if there is a graded isomorphism of algebras $f : A \to A'$ which is also an isomorphism of A_e-modules. The latter means that f is an isomorphism such that $f(A_g) = {A'}_g$ for all $g \in G$ and $f(x) = x$ for any $x \in A_e$.

Theorem 5.4. *Two (G, ϕ)-crossed products of G over A_e are equivalent if and only if they determine the same equivalence class of quasicrossed systems for G over A_e.*

Proof. Consider A and A' two (G, ϕ)-crossed products of G over A_e. Let $(G, A_e, \phi, \sigma, \alpha)$ and $(G, A_e, \phi, \sigma', \alpha')$ be the representatives of the corresponding equivalence classes of quasicrossed systems for G over A_e and take the sets of units $\{\overline{g} : g \in G\}$ and $\{\widetilde{g} : g \in G\}$ in A and A', respectively, which give rise to the above quasicrossed systems.

First assume that A' and A are equivalent via $f : A' \to A$. Because $f(\widetilde{g}) \in A_g$ for all $g \in G$, there is a map $u : G \to U(A_e)$ with $u(e) = 1$ such that $f(\widetilde{g}) = u(g)\overline{g}$ for any $g \in G$. We observe that for given $g \in G$,

$$1 = f(1) = f(\widetilde{g}\widetilde{g}_R^{-1}) = f(\widetilde{g})f(\widetilde{g}_R^{-1}) = u(g)\overline{g}f(\widetilde{g}_R^{-1}),$$

so $u(g)$ is an unit in A_e with inverse

$$u(g)^{-1} = \overline{g}f(\widetilde{g}_R^{-1}).$$

Consider in A and A' the product defined, respectively, by

$$(x\overline{g})(y\overline{h}) = x\sigma(g)(y)\alpha(g,h)\overline{gh} \quad \text{and} \quad (x\widetilde{g})(y\widetilde{h}) = x\sigma'(g)(y)\alpha'(g,h)\widetilde{gh}$$

for any $x, y \in A_e$ and $g, h \in G$. Given $x \in A_e$ and $g \in G$ we have $\widetilde{g}(x\widetilde{e}) = \sigma'(g)(x)\alpha'(g,e)\widetilde{ge} = \sigma'(g)(x)\widetilde{g}$. Since f is a morphism of algebras we have

$$\begin{aligned} f(\widetilde{g}(x\widetilde{e})) &= f(\widetilde{g})f(x\widetilde{e}) = f(\widetilde{g})(xf(\widetilde{e})) = (u(g)\overline{g})(xu(e)\overline{e}) = (u(g)\overline{g})(x\overline{e}) \\ &= u(g)\sigma(g)(x)\alpha(g,e)\overline{ge} = u(g)\sigma(g)(x)\overline{g} \end{aligned}$$

and

$$f(\sigma'(g)(x)\widetilde{g}) = \sigma'(g)(x)f(\widetilde{g}) = \sigma'(g)(x)u(g)\overline{g}$$

therefore $\sigma'(g)(x) = u(g)\sigma(g)(x)u(g)^{-1}$ proving (5.1). Now for $g, h \in G$ we have $\widetilde{g}\widetilde{h} = \sigma'(g)(1)\alpha'(g,h)\widetilde{gh} = \alpha'(g,h)\widetilde{gh}$. Again, since f is a morphism of algebras,

$$f(\widetilde{g}\widetilde{h}) = f(\widetilde{g})f(\widetilde{h}) = (u(g)\overline{g})(u(h)\overline{h}) = u(g)\sigma(g)(u(h))\alpha(g,h)\overline{gh}$$

and

$$f(\alpha'(g,h)\widetilde{gh}) = \alpha'(g,h)f(\widetilde{gh}) = \alpha'(g,h)u(gh)\overline{gh}$$

therefore $\alpha'(g,h) = u(g)\sigma(g)(u(h))\alpha(g,h)u(gh)^{-1}$ getting (5.2). Thus (G, A_e, σ, α) and $(G, A_e, \sigma', \alpha')$ are equivalent.

Conversely, suppose that there is a map $u : G \to U(A_e)$ with $u(e) = 1$ such that (5.1) and (5.2) are satisfied. Using again the product in A and A', it is easily seen that the A_e-linear extension of the map $f(\widetilde{g}) = u(g)\overline{g}$ for any $g \in G$, also denoted by f, provides an equivalence of A' and A. In fact, f is an algebra morphism, because for $g, h \in G$ we have

$$f(\widetilde{g}\widetilde{h}) = f(\alpha'(g,h)\widetilde{gh}) = \alpha'(g,h)f(\widetilde{gh}) = \alpha'(g,h)u(gh)\overline{gh}$$

and

$$f(\widetilde{g})f(\widetilde{h}) = (u(g)\overline{g})(u(h)\overline{h}) = u(g)\sigma(g)(u(h))\alpha(g,h)\overline{gh}$$

that are equal by (5.2). It also satisfies $f(A'_g) = A_g$. Indeed, for $x_g \in A_g$, by Proposition 4.9-(i) we may write $x_g = x\overline{g}$ for a certain $x \in A_e$. Then

$$f(xu(g)^{-1}\widetilde{g}) = xu(g)^{-1}f(\widetilde{g}) = xu(g)^{-1}u(g)\overline{g} = x\overline{g} = x_g,$$

with $xu(g)^{-1}\widetilde{g} \in A'_g$. Finally, for any $x \in A_e$ we have $f(x) = f(x\widetilde{e}) = xf(\widetilde{e}) = x$, completing the proof. □

Definition 5.5. Consider the trivial automorphism system $\sigma : G \to Aut(\mathbb{K})$, where we take the field $\mathbb{K}$ as the associative algebra B on the natural way. A quasicrossed mapping $\delta : G \times G \to \mathbb{K}^\times$ (see Definition 4.7) is called a *coboundary* if there is a function $u : G \to \mathbb{K}^\times$ such that

$$\delta(g,h) = u(g)\sigma(g)(u(h))u(gh)^{-1},$$

for any $g, h \in G$.

Proposition 5.6. *The quasicrossed systems* $(G, \mathbb{K}, \phi, \sigma, \alpha)$ *and* $(G, \mathbb{K}, \phi, \sigma, \alpha')$ *over the associative algebra* $\mathbb{K}$ *for a fixed cocycle* $\phi : G \times G \times G \longrightarrow \mathbb{K}^\times$ *and the trivial automorphism system* $\sigma : G \to Aut(\mathbb{K})$ *are equivalent if and only if* $\alpha' = \delta\alpha$ *for a certain coboundary* δ*.*

Proof. Apply Theorem 5.4 with the field $\mathbb{K}$ playing the role of the associative algebra B on the natural way. On this context, condition (5.1) is trivial and as $\mathbb{K}$ is commutative we can rewrite (5.2) as $\alpha = \delta\alpha'$ where δ is a coboundary quasicrossed mapping. □

6. Cayley (Clifford) (G, ϕ)-crossed products

Let A be a finite-dimensional (not necessarily associative) algebra with identity element 1 and an *anti-involution* $\varsigma : A \longrightarrow A$, meaning that ς is an antiautomorphism ($\varsigma(ab) = \varsigma(b)\varsigma(a)$ for all $a, b \in A$) with $\varsigma^2 = id$. Moreover, the involution ς is *strong*, that is, it satisfies the property $a + \varsigma(a), a\varsigma(a) \in \mathbb{K}1$, for all $a \in A$. The Cayley-Dickson process (that requires the involution ς to be strong) says that we can obtain a new algebra $\overline{A} = A \oplus vA$ of twice the dimension (the elements are denoted by a, va, for $a \in A$) with multiplication defined by

$$(a + vb)(c + vd) := (ac + \epsilon d\varsigma(b)) + v(\varsigma(a)d + cb),$$

and with a new involution $\overline{\varsigma}$ given by

$$\overline{\varsigma}(a + vb) = \varsigma(a) - vb,$$

for any $a, b, c, d \in A$. The symbol v is a notation device to label the second copy of A in $\overline{A}$ and ϵ is a fixed nonzero element of $\mathbb{K}$.

Proposition 6.1. *If A is a (G, ϕ)-crossed product over the group G then the algebra $\overline{A} = A \oplus vA$ resulting from the Cayley-Dickson process is a $(\overline{G}, \phi)$-crossed product over the group $\overline{G} = G \times \mathbb{Z}_2$.*

Proof. First, we note that if $A = \bigoplus_{g \in G} A_g$ is a G-graded algebra, it is easy to see that $\overline{A} = A \oplus vA$ is a $\overline{G}$-graded algebra, with $\overline{G} = G \times \mathbb{Z}_2$ (we may write $\overline{A} = \bigoplus_{g \in G} A_{(g,0)} \oplus \bigoplus_{g \in G} A_{(g,1)}$, with $A_{(g,0)} = A_g$ and $A_{(g,1)} = vA_g$). Now assume that $A = \bigoplus_{g \in G} A_g$ is a (G, ϕ)-crossed product. For any $g \in G$ there exists an unit $\overline{g}$ in A_g, so trivially we have an unit in $A_{(g,0)}$.

Moreover, $v\overline{g}$ is an unit in $A_{(g,1)}$ with $v\frac{\varsigma(\overline{g}_R^{-1})}{\epsilon}$ its left inverse and right inverse, as

$$(v\frac{\varsigma(\overline{g}_R^{-1})}{\epsilon})(v\overline{g}) = \epsilon\overline{g}\varsigma(\frac{\varsigma(\overline{g}_R^{-1})}{\epsilon}) = \epsilon\overline{g}\frac{1}{\epsilon}\varsigma^2(\overline{g}_R^{-1}) = \overline{g}\varsigma^2(\overline{g}_R^{-1}) = \overline{g}\overline{g}_R^{-1} = 1,$$
$$(v\overline{g})(v\frac{\varsigma(\overline{g}_R^{-1})}{\epsilon}) = \epsilon\frac{\varsigma(\overline{g}_R^{-1})}{\epsilon}\varsigma(\overline{g}) = \varsigma(\overline{g}_R^{-1})\varsigma(\overline{g}) = \varsigma(\overline{g}\overline{g}_R^{-1}) = \varsigma(1) = 1$$

completing the proof. □

In [4], it was proved that after applying the Cayley-Dickson process to an algebra $\mathbb{K}_F G$ we obtain another algebra $\mathbb{K}_{\overline{F}}\overline{G}$ related to the first one which properties are predictable.

Proposition 6.2. *[4] Let G be a finite abelian group, F a cochain on it ($\mathbb{K}_F G$ is a (G,ϕ)-quasiassociative algebra). For any $s : G \longrightarrow \mathbb{K}^\times$ with $s(e) = 1$ we define $\overline{G} = G \times \mathbb{Z}_2$ and on it the cochain $\overline{F}$ and function $\overline{s}$,*

$$\overline{F}(x,y) = F(x,y),\ \overline{F}(x,vy) = s(x)F(x,y),$$
$$\overline{F}(vx,y) = F(y,x),\ \overline{F}(vx,vy) = \epsilon s(x)F(y,x),$$
$$\overline{s}(x) = s(x),\ \overline{s}(vx) = -1 \quad \textit{for all } x,y \in G.$$

Here $x \equiv (x,0)$ and $vx \equiv (x,1)$ denote elements of $\overline{G}$, where $\mathbb{Z}_2 = \{0,1\}$ with operation $1+1=0$. If $\varsigma(x) = s(x)x$ is a strong involution, then $\mathbb{K}_{\overline{F}}\overline{G}$ is the algebra obtained from Cayley-Dickson process applied to $\mathbb{K}_F G$.

7. Simple (G,ϕ)-crossed products

The aim of this section is to study simple (G,ϕ)-crossed products. We recall the notion of simple (G,ϕ)-quasiassociative algebra.

Definition 7.1. A (G,ϕ)-quasiassociative algebra A is *simple* if $A^2 \neq \{0\}$ and it has no proper graded ideals, or equivalently, if the ideal generated by each nonzero homogeneous element is the whole quasialgebra.

To study simple (G,ϕ)-crossed products we introduce the definition of representation of a (G,ϕ)-quasiassociative algebra. In the following definition of modules, $A = \bigoplus_{g\in G} A_g$ is a (G,ϕ)-quasiassociative algebra with structure given by ϕ and $V = \bigoplus_{k\in G} V_k$ is a graded vector space over the same group G. We denote by μ the product defined in A. First we emphasize that the quasiassociative law in A is performed by

$\mu \circ (\mu \otimes id) = \mu \circ (id \otimes \mu) \circ \Phi_{A,A,A}$ and it can be represented by the following commutative diagram

$$\begin{array}{ccccc} A \otimes A \otimes A & \xrightarrow{\Phi_{A,A,A}} & A \otimes A \otimes A & \xrightarrow{id \otimes \mu} & A \otimes A \\ \downarrow{\scriptstyle \mu \otimes id} & & & & \downarrow{\scriptstyle \mu} \\ A \otimes A & & \xrightarrow[\mu]{\qquad\qquad} & & A \end{array}$$

Definition 7.2. Consider a degree-preserving map $\varphi : A \otimes V \longrightarrow V$ and denote $x_g.v_k := \varphi(x_g, v_k)$. We say that V is a *left graded module* over A (or a *left A-graded-module*) if

$$(x_g x_h).v_k = \phi(g,h,k)x_g.(x_h.v_k) \quad \text{and} \quad 1.v_k = v_k$$

for any homogeneous elements $x_g \in A_g, x_h \in A_h, v_k \in V_k$.

The condition of left graded module is a natural generalization of the quasiassociativity of the product on A, as we can see by the following commutative diagram

$$\begin{array}{ccccc} A \otimes A \otimes V & \xrightarrow{\Phi_{A,A,V}} & A \otimes A \otimes V & \xrightarrow{id \otimes \varphi} & A \otimes V \\ \downarrow{\scriptstyle \mu \otimes id} & & & & \downarrow{\scriptstyle \varphi} \\ A \otimes V & & \xrightarrow[\varphi]{\qquad\qquad} & & V \end{array}$$

Definition 7.3. Consider a degree-preserving map $\psi : V \otimes A \longrightarrow V$ and denote $v_k.x_g := \psi(v_k, x_g)$. If for homogeneous elements $x_g \in A_g, x_h \in A_h, v_k \in V_k$,

$$(v_k.x_g).x_h = \phi(k,g,h)v_k.(x_g x_h) \quad \text{and} \quad v_k.1 = v_k,$$

then V is called a *right graded module* over A (or a *right A-graded-module*).

Similarly, the condition of right graded module is represented in the following commutative diagram

$$\begin{array}{ccccc} V \otimes A \otimes A & \xrightarrow{\Phi_{V,A,A}} & V \otimes A \otimes A & \xrightarrow{id \otimes \mu} & V \otimes A \\ \downarrow{\scriptstyle \psi \otimes id} & & & & \downarrow{\scriptstyle \psi} \\ V \otimes A & & \xrightarrow[\psi]{\qquad\qquad} & & V \end{array}$$

Definition 7.4. If V is a left and right graded module of A and if for homogeneous elements $x_g \in A_g, x_h \in A_h, v_k \in V_k$,

$$(x_g.v_k).x_h = \phi(g,k,h)x_g.(v_k.x_h),$$

then V is called a *graded bimodule* over A (or an A-*graded-bimodule*).

Moreover, the condition of graded bimodule is represented by the following commutative diagram

$$\begin{array}{ccccc} A\otimes V\otimes A & \xrightarrow{\Phi_{A,V,A}} & A\otimes V\otimes A & \xrightarrow{id\otimes\psi} & A\otimes V \\ \downarrow{\scriptstyle \varphi\otimes id} & & & & \downarrow{\scriptstyle \varphi} \\ V\otimes A & & \xrightarrow[\psi]{\qquad\qquad\qquad} & & V \end{array}$$

Now we present some examples of graded modules over (G,ϕ)-quasi-associative algebras.

Example 7.5. Consider the antiassociative quasialgebra $A := \widetilde{Mat}_{1,1}(\mathbb{K})$ of the square matrices over the field $\mathbb{K}$ graded by the group $\mathbb{Z}_2$ such that $A_{\bar{0}} := \langle E_{11}, E_{22}\rangle$ and $A_{\bar{1}} := \langle E_{12}, E_{21}\rangle$ satisfying the multiplication

$$\begin{pmatrix} a_1 & v_1 \\ w_1 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & v_2 \\ w_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + v_1w_2 & a_1v_2 + v_1b_2 \\ w_1a_2 + b_1w_2 & -w_1v_2 + b_1b_2 \end{pmatrix}.$$

Consider A acting on the vector space $M := \langle m, n\rangle$ endowed with the grading by the group $\mathbb{Z}_2$ with $M_{\bar{0}} := \langle m\rangle$ and $M_{\bar{1}} := \langle n\rangle$ as follows:

$$\begin{gathered} mE_{11} = nE_{21} = m, \quad mE_{12} = nE_{22} = n, \\ mE_{22} = mE_{21} = nE_{11} = nE_{12} = 0; \end{gathered}$$

and on the other side,

$$\begin{gathered} E_{22}m = E_{21}n = m, \quad -E_{12}m = E_{11}n = n, \\ E_{11}m = E_{21}m = E_{22}n = E_{12}n = 0. \end{gathered}$$

We check easily that M is both a right A-graded-module and a left A-graded-module, although the two structures are not compatible, that is, M is not a graded bimodule over A (just note that $(E_{21}n)E_{12} = n$ and $E_{21}(nE_{12}) = 0$).

Example 7.6. We consider a commutative (G, ϕ)-quasiassociative algebra $\mathbb{K}_F G$ endowed with the strong involution $\sigma(x) = s(x)x$, where $s : G \longrightarrow \mathbb{K}^\times$ with $s(e) = 1$. Applying Proposition 4.5 in [4], we know that the quasialgebra obtained from $\mathbb{K}_F G$ by the Cayley-Dickson doubling process can be defined by the same cocycle graded by G being the degree of the element vx equal to the degree of x, for $x \in G$. Then the subspace $v\mathbb{K}_F G$ constitutes an example of a graded bimodule over $\mathbb{K}_F G$.

Definition 7.7. Let V be an A-graded-bimodule, a *graded submodule* $W \subset V$ is a submodule (meaning $AW \subset W$) such that $W = \oplus_{g\in G}(W \cap V_g)$. We say that a A-graded-bimodule V is *simple* if it contains no proper graded submodules.

Example 7.8. A (G, ϕ)-quasiassociative algebra A is an A-graded-bimodule acting on itself by the product map. Also, each one A_g is an A_e-graded-bimodule and a graded submodule of A, for any $g \in G$.

Definition 7.9. Consider two A-graded-bimodules V and V'. An A-linear $f : V \to V'$ is said to be a *graded morphism of degree* g if $f(V_h) \subset V'_{hg}$, for all $h \in G$.

Now we recall the definition of radical of a (G, ϕ)-quasiassociative algebra.

Definition 7.10. Let A be a (G, ϕ)-quasiassociative algebra. The radical of A is defined by

$$\mathrm{rad}(A) = \cap\{\mathrm{ann}\ M : M \text{ simple graded left } A\text{-module}\},$$

where ann M is the annihilator of M in A.

The radical of a (G, ϕ)-quasiassociative algebra A is a graded ideal of A. So $\mathrm{rad}(A) = \{0\}$ if A is simple.

Theorem 7.11. *Let A be a simple (G, ϕ)-crossed product such that it is an unital G-graded algebra with artinian null part A_e. Then A_e is a semisimple associative algebra.*

Proof. It is similar to the proof of Theorem 4.3 in [8]. Let $\mathrm{J}(A_e)$ denote the Jacobson radical of the associative algebra A_e. Given a simple graded A-module $M = \bigoplus_{g\in G} M_g$, each M_g is a simple A_e-module. Thus if $a_0 \in \mathrm{J}(A_e)$ then $a_0 M_g = 0$, $\forall g \in G$. Therefore $\mathrm{J}(A_e) \subseteq \mathrm{rad}(A) = \{0\}$ and A_e is semisimple. □

In case $G = \mathbb{Z}_2$, the classification of quasialgebras that have semisimple artinian associative null part was done in [3], so we have the following result.

Theorem 7.12. *Any simple $(\mathbb{Z}_2, \phi)$-crossed product A of $\mathbb{Z}_2$ over artinian $A_{\bar{0}}$ is isomorphic to one of the following algebras:*

(i) *$Mat_n(\Delta)$, for some n and some division antiassociative quasialgebra Δ;*

(ii) *$\widetilde{Mat}_{n,m}(D)$, for some natural numbers n and m and some division algebra D.*

Moreover, the natural numbers n and m are uniquely determined by A and so are (up to isomorphism) the division antiassociative quasialgebra Δ and the division algebra D.

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