On nilpotent Lie algebras
of derivations of fraction fields

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ABSTRACT. Let $\mathbb{K}$ be an arbitrary field of characteristic zero
and A an integral $\mathbb{K}$-domain. Denote by R the fraction field of A and
by $W(A) = RDer_{\mathbb{K}}A$, the Lie algebra of $\mathbb{K}$-derivations on R obtained
from $Der_{\mathbb{K}}A$ via multiplication by elements of R. If $L \subseteq W(A)$ is a
subalgebra of $W(A)$ denote by $rk_R L$ the dimension of the vector
space RL over the field R and by $F = R^L$ the field of constants of
L in R. Let L be a nilpotent subalgebra $L \subseteq W(A)$ with $rk_R L \leqslant 3$.
It is proven that the Lie algebra FL (as a Lie algebra over the field
F) is isomorphic to a finite dimensional subalgebra of the triangular
Lie subalgebra $u_3(F)$ of the Lie algebra $Der F[x_1, x_2, x_3]$, where
$u_3(F) = \{f(x_2, x_3)\frac{\partial}{\partial x_1} + g(x_3)\frac{\partial}{\partial x_2} + c\frac{\partial}{\partial x_3}\}$ with $f \in F[x_2, x_3], g \in F[x_3], c \in F$.

Introduction

Let $\mathbb{K}$ be an arbitrary field of characteristic zero and A an asso-
ciative commutative $\mathbb{K}$-algebra that is a domain. The set $Der_{\mathbb{K}}A$ of all
$\mathbb{K}$-derivations of A is a Lie algebra over $\mathbb{K}$ and an A-module in a natural
way: given $a \in A, D \in Der_{\mathbb{K}}A$, the derivation aD sends any element $x \in A$
to $a \cdot D(x)$. The structure of the Lie algebra $Der_{\mathbb{K}}A$ is of great interest
because in case $\mathbb{K} = \mathbb{R}$ and $A = \mathbb{R}[[x_1, \ldots, x_n]]$, the ring of formal power
series, the Lie algebra of all $\mathbb{K}$-derivations of the form

$$D = f_1\frac{\partial}{\partial x_1} + \cdots + f_n\frac{\partial}{\partial x_n}, \ f_i \in \mathbb{R}[[x_1, \ldots, x_n]]$$

can be considered as the Lie algebra of vector fields on $\mathbb{R}^n$ with formal power series coefficients. Such Lie algebras with polynomial, formal power series, or analytical coefficients were studied by many authors. Main results for fields $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$ in case $n = 1$ and $n = 2$ were obtained in [7] [4], [5] (see also [1], [3], [9], [10]).

One of the important problems in Lie theory is to describe finite dimensional subalgebras of the Lie algebra $\overline{W_3}(\mathbb{C})$ consisting of all derivations on the ring $\mathbb{C}[[x_1, x_2, x_3]]$ of the form

$$a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3}, a_i \in \mathbb{C}[[x_1, \dots, x_n]].$$

In order to characterize nilpotent subalgebras of the Lie algebra $\overline{W_3}(\mathbb{C})$ we consider more general situation. Let $R = \mathrm{Frac}(A)$ be the field of fractions of an integral domain A and $W(A) = R\mathrm{Der}_{\mathbb{K}}(A)$ the Lie algebra of derivations of the field R obtained from derivations on A by multiplying by elements of the field R (obviously $\mathrm{Der}_{\mathbb{K}}A \subseteq W(A)$). For a subalgebra L of the Lie algebra $W(A)$ let us define $\mathrm{rk}_R(L) = \dim_R RL$ and denote by $F = R^L = \{r \in R \mid D(r) = 0, \forall D \in L\}$ the field of constants of the Lie algebra L. The $\mathbb{K}$-space FL is a vector space over the field F and a Lie algebra over F. If L is a nilpotent subalgebra of $W(A)$, then FL is finite dimensional over F (by Lemma 5).

The main result of the paper: If L is a nilpotent subalgebra of rank $k \leqslant 3$ over R from the Lie algebra $W(A)$, then FL is isomorphic to a finite dimensional subalgebra of the triangular Lie algebra $u_k(F)$ (Theorem 2). Triangular Lie algebras were studied in [1] and [2], they are locally nilpotent but not nilpotent, the structure of their ideals was described in these papers.

We use standard notation, the ground field is arbitrary of characteristic zero. The quotient field of the integral domain A under consideration is denoted by R. Any derivation D of A can be uniquely extended to a derivation of R by the rule: $D(a/b) = (D(a)b - aD(b))/b^2$. If F is a subfield of the field R and $r_1, \dots, r_k \in R$, then the set of all linear combinations of these elements with coefficients in F is denoted by $F\langle r_1, \dots, r_k \rangle$, it is a subspace of the F-space R. The triangular subalgebra $u_n(\mathbb{K})$ of the Lie algebra $W_n(\mathbb{K}) = \mathrm{Der}(\mathbb{K}[x_1, \dots, x_n])$ consists of all the derivations on the ring $\mathbb{K}[x_1, \dots, x_n]$ of the form $D = f_1(x_2, \dots x_n)\frac{\partial}{\partial x_1} + \dots + f_{n-1}(x_n)\frac{\partial}{\partial x_{n-1}} + f_n\frac{\partial}{\partial x_n}$, where $f_i \in \mathbb{K}[x_{i+1}, \dots x_n], f_n \in \mathbb{K}$.

1.   Some properties of nilpotent subalgebras of $W(A)$

We will use some statements about derivations and nilpotent Lie algebras of derivations from the paper [8]. The next statement can be immediately checked.

Lemma 1. *Let $D_1, D_2 \in W(A)$ and $a, b \in R$. Then it holds:*
1. $[aD_1, bD_2] = ab[D_1, D_2] + aD_1(b)D_2 - bD_2(a)D_1$.
2. *If $a, b \in R^{D_1} \cap R^{D_2}$, then $[aD_1, bD_2] = ab[D_1, D_2]$.*

Let L be a subalgebra of rank k over R of the Lie algebra $W(A)$ and $F = R^L$ its field of constants. Denote by RL the set of all linear combinations over $\mathbb{K}$ of elements aD, where $a \in R$ and $D \in L$. The set FL is defined analogously.

Lemma 2 ([8], Lemma 2)**.** *Let L be a nonzero subalgebra of $W(A)$ and let FL, RL be $\mathbb{K}$-spaces defined as above. Then:*

1. FL and RL are $\mathbb{K}$-subalgebras of the Lie algebra $W(A)$. Moreover, FL is a Lie algebra over the field F.

2. If the algebra L is abelian, nilpotent, or solvable then the Lie algebra FL has the same property, respectively.

Lemma 3 ([8], Lemma 3)**.** *Let L be a subalgebra of finite rank over R of the Lie algebra $W(A)$, $Z = Z(L)$ the center of L, and $F = R^L$ the field of constants of L. Then $\mathrm{rk}_R Z = \dim_F FZ$ and FZ is a subalgebra of the center $Z(FL)$. In particular, if L is abelian, then FL is an abelian subalgebra of $W(A)$ and $\mathrm{rk}_R L = \dim_F FL$.*

Lemma 4 ([8], Lemma 4)**.** *Let L be a subalgebra of the Lie algebra $W(A)$ and I be an ideal of L. Then the vector space $RI \cap L$ (over $\mathbb{K}$) is also an ideal of L.*

Lemma 5 ([8], Proposition 1, Theorem 1)**.** *Let L be a nilpotent subalgebra of $W(A)$ and $F = R^L$ be its field of constants. Then:*

1. If $\mathrm{rk}_R L < \infty$, then $\dim_F FL < \infty$.

2. If $\mathrm{rk}_R L = 1$, then L is abelian and $\dim_F FL = 1$.

3. If $\mathrm{rk}_R L = 2$, then there exist elements $D_1, D_2 \in FL$ and $a \in R$ such that
$$FL = F\langle D_1, aD_1, \ldots, \tfrac{a^k}{k!}D_1, D_2 \rangle, \; k \geqslant 0$$
$$(if \; k = 0, \; then \; put \; FL = F\langle D_1, D_2 \rangle),$$
where $[D_1, D_2] = 0$, $D_1(a) = 0$, $D_2(a) = 1$.

Lemma 6. *Let $D_1, D_2, D_3 \in W(A)$ and $a \in R$ be such elements that $D_1(a) = D_2(a) = 0$, $D_3(a) = 1$ and let $F = \cap_{i=1}^{3} R^{D_i}$. If there exists an element $b \in R$ such that $D_1(b) = D_2(b) = 0$, $D_3(b) \in F\langle 1, a, \ldots, a^s/s! \rangle$ for some $s \geqslant 0$, then $b \in F\langle 1, a, \ldots, a^{s+1}/(s+1)! \rangle$.*

Proof. Write down $D_3(b) = \sum_{i=0}^{s} \beta_i a^i/i!$ with $\beta_i \in F$ and take the element $c = \sum_{i=0}^{s} \beta_i a^{i+1}/(i+1)!$ of the field R. It holds obviously $D_3(b-c) = 0$ and (by the conditions of Lemma) $D_1(b - c) = 0$ and $D_2(b - c) = 0$. Then we have $b - c \in \cap_{i=1}^{3} R^{D_i} = F$, and therefore $b = \gamma + \sum_{i=0}^{s} \beta_i a^{i+1}/(i + 1)!$ for some element $\gamma \in F$. The latter means that $b \in F\langle 1, a, \ldots, a^{s+1}/(s+1)! \rangle$. $\square$

Lemma 7. *Let $D_1, D_2, D_3 \in W(A)$ and $a, b \in R$ be such elements that $D_1(a) = D_1(b) = 0$, $D_2(a) = 1$ $D_2(b) = 0$, $D_3(a) = 0$, $D_3(b) = 1$ and let $F = \cap_{i=1}^{3} R^{D_i}$. If there exists an element $c \in R$ such that*

$$D_1(c) = 0, \quad [D_2, D_3](c) = 0,$$

$$D_2(c) \in F\langle \{ \tfrac{a^i b^j}{i! j!} \}, 0 \leqslant i \leqslant m - 1, 0 \leqslant j \leqslant k \rangle,$$

$$D_3(c) \in F\langle \{ \tfrac{a^i b^j}{i! j!} \}, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant k - 1 \rangle,$$

then $c \in F\langle \{ \tfrac{a^i b^j}{i! j!} \}, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant k \rangle$.

Proof. The elements $D_2(c)$ and $D_3(c)$ can be written (by conditions of the lemma) in the form $D_2(c) = f(a, b)$, $D_3(c) = g(a, b)$ where $f, g \in F[u, v]$ are some polynomials of u, v. Since $[D_2, D_3](c) = 0$, it holds $D_2(g) = D_3(f)$. It follows from the relations $D_2(g) = \frac{\partial}{\partial a} g(a, b)$, $D_3(f) = \frac{\partial}{\partial b} f(a, b)$ that $\frac{\partial}{\partial a} g(a, b) = \frac{\partial}{\partial b} f(a, b)$. Hence there exists a polynomial $h(a, b) \in F[a, b]$ (the potential of the vector field $f(a, b) \frac{\partial}{\partial a} + g(a, b) \frac{\partial}{\partial b}$) such that $D_3(h(a, b)) = g$, $D_2(h(a, b)) = f$. The polynomial $h(a, b)$ is obtained from the polynomials f, g by formal integration on a and on b, so we have $h(a, b) \in F\langle \{ \tfrac{a^i b^j}{i! j!} \}, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant k \rangle$. Further, using properties of the element $h(a, b)$ we get $D_2(h - c) = D_3(h - c) = 0$. Besides, it holds $D_1(h - c) = 0$. The latter means that $h - c \in F = \cap_{i=1}^{3} R^{D_i}$. But then $c = \gamma + h$ for some $\gamma \in F$ and therefore $c \in F\langle \{ \tfrac{a^i b^j}{i! j!} \}, 0 \leqslant i \leqslant m, 0 \leqslant j \leqslant k \rangle$. $\square$

2.   On nilpotent subalgebras of small rank of $W(A)$

Lemma 8. *Let L be a nilpotent subalgebra of rank 3 over R from the Lie algebra $W(A)$ and $F = R^L$ be its field of constants. If the center $Z(L)$*

of the algebra L is of rank 2 over R and $\dim_F FL \geqslant 4$, then there exist $D_1, D_2, D_3 \in L, a \in R$ such that the Lie algebra FL is contained in a nilpotent Lie algebra $\widetilde{L}$ of the Lie algebra $W(A)$ of the form

$$\widetilde{L} = F\langle D_3, D_1, aD_1, \ldots, (a^n/n!)D_1, D_2, aD_2, \ldots, (a^n/n!)D_2\rangle$$

for some $n \geqslant 1$, with $[D_i, D_j] = 0$, $i, j = 1, 2, 3$, $D_1(a) = D_2(a) = 0$, $D_3(a) = 1$.

Proof. Take any elements $D_1, D_2 \in Z(L)$ that are linearly independent over R and denote $I = (RD_1 + RD_2) \cap L$. Then I is an ideal of the Lie algebra L (by Lemma 4). Take an arbitrary element $D \in I$ and write down $D = a_1 D_1 + a_2 D_2$ for some elements $a_i \in R$. Since $[D_i, D] = 0 = D_i(a_1)D_1 + D_i(a_2)D_2, i = 1, 2$ we get $D_i(a_j) = 0, i, j = 1, 2$. It follows easily that for any element $D' \in I$ it holds the equality $[D, D'] = 0$, so the ideal I is abelian. The Lie algebra FL is finite dimensional over F and $\dim_F FL/FI = 1$ by Lemma 5. Take any element $D_3 \in L \setminus I$. Then $FL = FI + FD_3$ and D_1, D_2, D_3 are linearly independent over R.

Since $\mathrm{rk}_R Z(L) = 2$, (by conditions of the lemma) we have $\dim_F FZ(L) = 2$ by Lemma 3. The ideal I of the Lie algebra L is abelian by the above proven, so the ideal FI of the Lie algebra FL over the field F is also abelian. Since $FL = FI + FD_3$, there exists a basis of the F-space FI in which the nilpotent linear operator $\mathrm{ad}D_3$ has a matrix consisting of two Jordan blocks. Let J_1 and J_2 be the correspondent Jordan bases; without loss of generality one can assume that $D_1 \in J_1, D_2 \in J_2$ and the elements D_1, D_2 are the first members of the bases J_1 and J_2 respectively.

If $\dim_F F\langle J_1\rangle = \dim_F F\langle J_2\rangle = 1$ then $FL = F\langle D_3, D_1, D_2\rangle$ is of dimension 3 over F which contradicts the conditions of the lemma. So, we may assume that $\dim_F F\langle J_1\rangle \geqslant \dim_F F\langle J_2\rangle$ and $\dim_F F\langle J_1\rangle = n+1, n \geqslant 1$. Denote the elements of the basis J_1 by $D_1, a_1 D_1 + b_1 D_2, \ldots, a_n D_1 + b_n D_2$, where the elements a_i, b_i belong to R and put for convenience $a = a_1$. Let us prove by induction on i that $a_i, b_i \in F\langle 1, a, \ldots, a^i/i!\rangle$. If $i = 1$, then $a_1 = a \in F\langle 1, a\rangle$ by definition. It follows from the relation $[D_3, a_1 D_1 + b_1 D_2] = D_1 = D_3(a_1)D_1 + D_3(b_1)D_2$ that $D_3(b_1) = 0$. Since FI is abelian (by the above proven), we have $D_1(b_1) = D_2(b_1) = 0$. The latter means that $b_1 \in F \subset F\langle 1, a\rangle$.

Further, the relation

$$[D_3, a_i D_1 + b_i D_2] = a_{i-1}D_1 + b_{i-1}D_2 = D_3(a_i)D_1 + D_3(b_i)D_2$$

gives the equalities $D_3(a_i) = a_{i-1}$ and $D_3(b_i) = b_{i-1}$. By the inductive assumption, $a_{i-1}, b_{i-1} \in F\langle 1, a, \ldots, a^{i-1}/(i-1)!\rangle$ and taking into account

the relations $D_j(a_i) = D_j(b_i) = 0, j = 1, 2$ (they hold because FI is abelian) we get by Lemma 6 that $a_i, b_i \in F\langle 1, \ldots, a^i/i!\rangle$. The latter relation means that the F-subspace $F\langle J_1 \rangle$ of FI lies in the subalgebra $\widetilde{L}$ from the conditions of the lemma.

Now let

$$J_2 = \{D_2, c_1 D_1 + d_1 D_2, \ldots, c_k D_1 + d_k D_2\}$$

be a basis corresponding to the second Jordan block. The relation $[D_3, c_1 D_1 + d_1 D_2] = D_2$ implies the equality $D_3(d_1) = 1$ and therefore $D_3(a - d_1) = 0$. Since $D_1(a - d_1) = D_2(a - d_1) = 0$, we get $a - d_1 \in F$, i.e. $d_1 = a + \gamma$ for some $\gamma \in F$. Applying the above considerations to the Jordan basis J_2 we obtain that $F\langle J_2 \rangle \subset \widetilde{L}$. But then the Lie algebra L is entirely contained in $\widetilde{L}$. □

Lemma 9. *Let L be a nilpotent subalgebra of rank 3 over R from the Lie algebra $W(A)$ and $F = R^L$ be its field of constants. If the center $Z(L)$ of the algebra L is of rank 1 over R and $\dim_F FL \geqslant 4$, then the Lie algebra FL is contained either in the nilpotent Lie algebra $\widetilde{L}$ from the conditions of Lemma 8 or in a subalgebra $\overline{L}$ of $W(A)$ of the form*

$$\overline{L} = F\langle D_3, D_2, aD_2, \ldots, (a^n/n!)D_2, \{\tfrac{a^i b^j}{i!j!} D_1\}, 0 \leqslant i, j \leqslant m\rangle$$

where $n \geqslant 0, m \geqslant 1, D_i \in L, [D_i, D_j] = 0$, for $i, j = 1, 2, 3$, and $a, b \in R$ such that $D_1(a) = D_2(a) = 0, D_3(a) = 1, D_1(b) = D_3(b) = 0, D_2(b) = 1$.

Proof. Take any nonzero element $D_1 \in Z(L)$ and denote $I_1 = RD_1 \cap L$. Then I_1 is an abelian ideal of the algebra Lie L and $\mathrm{rk}_R I_1 = 1$ by Lemma 3. Choose any nonzero element $D_2 + I_1$ in the center of the quotient Lie algebra L/I_1 and denote $I_2 = (RD_1 + RD_2) \cap L$. By the same Lemma 3, I_2 is an ideal of the Lie algebra L and $\mathrm{rk}_R I_2 = 2$. Further, take any element $D_3 \in L \setminus I_2$. Since $\dim_F FL/FI_2 = 1$ by Lemma 5 from the paper [8], we have $FL = FI_2 + FD_3$.

<u>Case 1.</u> The ideal I_2 is abelian. Let us show that FL is contained in the Lie algebra $\widetilde{L}$ from the conditions of Lemma 8. It is obvious that FI_2 is an abelian ideal of codimension 1 of the Lie algebra FL over the field F. By Lemma 3, $\mathrm{rk}_R Z(L) = \dim_F FZ(L)$ and by the conditions of the lemma, we see that $\dim_F FZ(L) = 1$. The linear operator $\mathrm{ad}D_3$ acts on the F-space FI_2 and $\dim_F \mathrm{Ker}(\mathrm{ad}D_3) = \dim_F FZ(L)$. Therefore $\dim_F \mathrm{Ker}(\mathrm{ad}D_3) = 1$ and there exists a basis of FI_2 in which $\mathrm{ad}D_3$ has a matrix in the form of a single Jordan block. The same is true for the

action of $\text{ad}D_3$ on the vector space FI_1 (since $[D_3, I_1] \subseteq I_1$, the ideal FI_1 is invariant under $\text{ad}D_3$). The subalgebra $FI_1 + FD_3$ is of rank 2 over R. If $\dim_F FI_1 > 1$, then the center of the Lie algebra $F_1 + FD_3$ is of dimension 1 over F. By Lemma 5, there exists a Jordan basis in FI_1 of the form

$$\{D_1, aD_1, \ldots, (a^s/s!)D_1\}, \text{ where } s \geqslant 0, D_3(a) = 1, [D_3, D_1] = 0.$$

If $\dim_F FI_1 = 1$, then $s = 0$ and the desired basis of F_1 is of the form $\{D_1\}$.

Let first $s > 0$. Since $(FD_2 + FI_1)/FI_1$ is a central ideal of the quotient algebra FL/FI_1, we have $[D_3, D_2] \in FI_1$ and hence one can write $[D_3, D_2] = \gamma_0 D_1 + \ldots + (\gamma_s a^s/s!)D_1$ for some $\gamma_i \in F$. Taking $D_2 - \sum_{i=0}^{s-1}(\gamma_i a^{i+1}/(i+1)!)D_1$ instead D_2 we may assume that $[D_3, D_2] = \gamma_s a^s/s!D_1$. Note that $\gamma_s \neq 0$. Really, in the opposite case $[D_3, D_2] = 0$ and therefore $D_2 \in Z(L)$. Then $\text{rk}_R Z(L) = 2$ which is impossible because of the conditions of the lemma. After changing D_3 by $\gamma_s^{-1}D_3$ we may assume that $[D_3, D_2] = (a^s/s!)D_1$.

Since the linear operator $\text{ad}D_3$ has in a basis of the F-space FI_2 a matrix, consisting of a single Jordan block, the same is true for the linear operator $\text{ad}D_3$ on the vector space FI_2/FI_1. Let $\dim_F FI_2/FI_1 = k$ and $\{\overline{S}_1, \ldots, \overline{S}_k\}$ be a Jordan basis for $\text{ad}D_3$ on FI_2/FI_1, where $\overline{S}_i = (c_i D_1 + d_i D_2) + FI_1$, $i = 1, \ldots, k, c_i, d_i \in R$. The representatives $c_i D_1 + d_i D_2$ of the cosets $\overline{S}_i$ can be chosen in such a way that $[D_3, c_i D_1 + d_i D_2] = c_{i-1}D_1 + d_{i-1}D_2, i = 2, \ldots, k$ and

$$[D_3, c_1 D_1 + d_1 D_2] = \sum_{i=0}^{s} \beta_i(a^i/i!)D_1 \tag{1}$$

for some $\beta_i \in F$. Let us show by induction on i that the relations hold:

$$d_i \in F\langle 1, \ldots, a^{i-1}/(i-1)!\rangle, \quad c_i \in F\langle 1, \ldots, a^{s+i}/(s+i)!\rangle. \tag{2}$$

Really, for $i = 1$ it follows from the relation (1) that

$$[D_3, c_1 D_1 + d_1 D_2] = \sum_{i=0}^{s} \beta_i a^i/i!D_1 =$$

$$= D_3(c_1)D_1 + (d_1 a^s/s!)D_1 + D_3(d_1)D_2. \tag{3}$$

It follows from (3) that $D_3(d_1) = 0$, and since the ideal FI_2 is abelian, it holds $D_1(d_1) = D_2(d_1) = 0$. The latter means that $d_1 \in F = F\langle 1\rangle$.

We also get from (3) that $D_3(c_1) \in F\langle 1, \ldots, a^s/s! \rangle$ and obviously it holds $D_1(c_1) = D_2(c_1) = 0$. Then, by Lemma 6, $c_1 \in F\langle 1, a, \ldots, a^{s+1}/(s+1)! \rangle$ and the relations (2) hold for $i = 1$. Assume they hold for $i - 1$. Let us prove that the relations (2) hold for i. Using the equalities $[D_3, c_i D_1 + d_i D_2] = c_{i-1} D_1 + d_{i-1} D_2$ and $[D_3, D_2] = a^s/s! D_1$ we get $D_3(d_i) = d_{i-1}$, $D_3(c_i) = d_i a^s/s! + c_{i-1}$. By the inductive assumption, we have $d_{i-1} \in F\langle 1, a, \ldots, a^{i-2}/(i-2)! \rangle$, hence $d_i \in F\langle 1, a, \ldots, a^{i-1}/(i-1)! \rangle$ by Lemma 6. Analogously, by the inductive assumption it holds

$$c_{i-1} \in F\langle 1, a, \ldots, a^{s+i-1}/(s+i-1)! \rangle$$

and therefore $D_3(c_i) \in F\langle 1, a, \ldots, a^{s+i-1}/(s+i-1)! \rangle$. Since $D_1(c_i) = D_2(c_i) = 0$ we get by Lemma 6 that $c_i \in F\langle 1, a, \ldots, a^{s+i}/(s+i)! \rangle$. But then we have inclusion

$$FI_2 \subseteq F\langle D_1, aD_1, \ldots, (a^{s+k}/(s+k)!) D_1, D_2, aD_2, \ldots, (a^k/k!) D_2 \rangle.$$

The last subalgebra of the Lie algebra $W(R)$ is contained in the subalgebra of the form

$$F\langle D_1, aD_1, \ldots, (a^{s+k}/(s+k)!) D_1, D_2, aD_2, \ldots, (a^{s+k}/(s+k)!) D_2 \rangle.$$

But then the Lie algebra L is contained in the subalgebra $\widetilde{L}$ from the conditions Lemma 8.

Let now $s = 0$. Then $FI_1 = FD_1$ and without loss of generality we may assume that $[D_3, D_2] = D_1$. Repeating the above considerations we can build a Jordan basis $\{(c_i D_1 + d_i D_2) + FI_1, i = 1, \ldots, k\}$ of the quotient algebra FI_2/FI_1 with $[D_3, c_i D_1 + d_i D_2] = c_{i-1} D_1 + d_{i-1} D_2, i = 2, \ldots, k$ and $[D_3, c_1 D_1 + d_1 D_2] = \alpha D_1$ for some $\alpha \in F$. It follows from the last equality that $D_3(d_1) = 0$ and taking into account the equalities $D_1(d_1) = 0$ and $D_2(d_1) = 0$ we see that $d_1 \in F$. Since $a_1 D_1 + d_1 D_2 \notin FI_1$, we have $d_1 \neq 0$. By conditions of the lemma, $\dim_F FL > 3$, so we have $k \geqslant 2$ and the relation $[D_3, c_2 D_1 + d_2 D_2] = c_1 D_1 + d_1 D_2$ implies the equality $D_3(d_2) = d_1$. But then $D_3(d_2 d_1^{-1}) = 1$ and multiplying all the elements of the Jordan basis considered above by d_1^{-1} we may assume that $D_3(d_2) = 1$. Denoting $a = d_2$ and repeating the considerations from the subcase $s > 0$ we see that the Lie algebra L is contained in the subalgebra $\widetilde{L}$ from the conditions of Lemma 8.

Case 2. The ideal I_2 is nonabelian. We may assume without loss of generality that I_1 coincides with its centralizer in L, i.e. $C_L(I_1) = I_1$. Really, let $C_L(I_1) \supset I_1$ with strong containment. Choose a one-dimensional

(central) ideal $(D_4 + I_1)/I_1$ in the ideal $C_L(I_1)/I_1$ of the quotient algebra L/I_1. Then $I_4 := (RD_1 + RD_4) \cap L$ is an abelian ideal of rank 2 of the algebra L and $\dim_F FL/FI_4 = 1$ by Lemma 5 from [8]. Thus the problem is reduced to the case 1 (one should take FI_4 instead of FI_2). So, we assume that $C_L(I_1) = I_1$. It follows from this equality that $C_{FL}(FI_1) = FI_1$.

As in the case 1 we write $FL = FI_2 + FD_3$ and $[D_3, D_2] = rD_1$ for some $r \in R$. Since the ideal FI_1 is abelian, the linear operator $\text{ad}[D_3, D_2] = \text{ad}(rD_1)$ acts trivially on the vector space FI_1, and therefore the linear operators $\text{ad}D_2$ and $\text{ad}D_3$ commute on FI_1. Denote by M_2 the kernel $\text{Ker}(\text{ad}D_2)$ on the F-space FI_1. It is obvious that M_2 is an abelian subalgebra of FI_1 and M_2 is invariant under the action of $\text{ad}D_3$. Since $[D_1, M_2] = [D_2, M_2] = 0$ the linear operator $\text{ad}D_3$ has on the F-space FI_1 the kernel of dimension 1 (in other case the center of the Lie algebra FL would have dimension $\geqslant 2$ over F which contradicts our assumption). Using Lemma 5 one can easily show that

$$M_2 = F\langle D_1, aD_1, \dots, (a^k/k!)D_1 \rangle$$

for some $a \in R$ with $D_1(a) = 0, D_2(a) = 0, D_3(a) = 1$ (if $k = 0$, then put $M_2 = FD_1$). Further denote $M_3 = \text{Ker}(\text{ad}D_3)$ on the vector space FI_1. As above one can prove that M_3 is invariant under action of $\text{ad}D_2$, this linear operator has one-dimensional kernel on M_3, and

$$M_3 = F\langle D_1, bD_1, \dots, (b^m/m!)D_1 \rangle$$

for some $b \in R$ with $D_1(b) = D_3(b) = 0$ and $D_2(b) = 1$ (if $m = 0$ put $M_3 = FD_1$).

Take now any element cD_1 of the ideal $FI_1, c \in R$. Since the linear operators $\text{ad}D_2$ and $\text{ad}D_3$ act nilpotently on FI_1, there exist the least positive integers k_0 and m_0 (depending on the element cD_1) such that $(\text{ad}D_2)^{k_0}(cD_1) = 0, (\text{ad}D_3)^{m_0}(cD_1) = 0$. Let us show by induction on $s = m_0 + k_0$ that the element cD_1 is a linear combination (with coefficients from F) of elements of the form $\frac{a^i b^j}{i! j!} D_1 \in W(A)$ for some $0 \leqslant i \leqslant k_0 - 1, 0 \leqslant j \leqslant m_0 - 1$ (note that the elements $\frac{a^i b^j}{i! j!} D_1$ can be outside of FI_1). If $s = 2$ (obviously $s \geqslant 2$), then we must only consider the case $m_0 = 1, k_0 = 1$. In this case, we have $[D_3, cD_1] = 0, [D_2, cD_1] = 0$. These equalities imply that $cD_1 \in Z(FL) = FD_1$ and all is done. Let $s \geqslant 3$. The element $[D_2, cD_1]$ can be written by the inductive assumption in the form

$$[D_2, cD_1] = \sum_{i=0}^{k_0-2} \sum_{j=0}^{m_0-1} \gamma_{ij} \frac{a^i b^j}{i! j!} D_1 \text{ for some } \gamma_{ij} \in F.$$

Analogously we get

$$[D_3, cD_1] = \sum_{i=0}^{k_0-1} \sum_{j=0}^{m_0-2} \delta_{ij} \frac{a^i b^j}{i! j!} D_1 \text{ for some } \delta_{ij} \in F.$$

It follows from the previous two equalities that

$$D_2(c) = \sum_{i=0}^{k_0-2} \sum_{j=0}^{m_0-1} \gamma_{ij} \frac{a^i b^j}{i! j!}, \quad D_3(c) = \sum_{i=0}^{k_0-1} \sum_{j=0}^{m_0-2} \delta_{ij} \frac{a^i b^j}{i! j!}.$$

Note that $[D_3, D_2](c) = rD_1(c) = 0$ and therefore by Lemma 7 $c \in F\langle \frac{a^i b^j}{i! j!}, \ 0 \leqslant i \leqslant k_0 - 1, 0 \leqslant j \leqslant m_0 - 1 \rangle$. Since cD_1 is arbitrarily chosen we have $FI_1 \subseteq F\langle \frac{a^i b^j}{i! j!} D_1, 0 \leqslant i \leqslant k_0 - 1, 0 \leqslant j \leqslant m_0 - 1 \rangle$. One can straightforwardly check that $k_0 \leqslant k$, where $k = \dim M_2 - 1$ and analogously $m_0 \leqslant m = \dim M_3 - 1$. Let, for example, $m \geqslant n$. Then $FI_1 \subseteq F\langle \frac{a^i b^j}{i! j!} D_1, 0 \leqslant i, j \leqslant m \rangle$.

Further, by the above proven, the linear operator adD_3 on the vector space FI_2/FI_1 has a matrix in a basis in the form of a single Jordan block. This basis can be chosen in the form $(u_1 D_1 + v_1 D_2) + FI_1, \ldots, (u_t D_1 + v_t D_2) + FI_1$ such that

$$[D_3, u_i D_1 + v_i D_2] = u_{i-1} D_1 + v_{i-1} D_2, i \geqslant 2,$$

$$[D_3, u_1 D_1 + v_1 D_2] = f D_1 \qquad (4)$$

for some element f, $f \in F\langle \frac{a^i b^j}{i! j!}, 0 \leqslant i, j \leqslant m \rangle$. Let us show by induction on s that

$$u_s \in F\langle \frac{a^i b^j}{i! j!}, \ 0 \leqslant i, j \leqslant m + s \rangle, v_s \in F\langle 1, \ldots, a^{s-1}/(s-1)! \rangle.$$

If $s = 1$, then the equalities

$$[D_3, u_1 D_1 + v_1 D_2] = f D_1 = D_3(u_1) D_1 + D_3(v_1) D_2 + v_1 r D_1 \qquad (5)$$

imply $D_3(v_1) = 0$ (let us recall here that $[D_3, D_2] = rD_1$). Taking into account the relations $[D_1, u_1 D_1 + v_1 D_2] = 0$ and $[D_2, u_1 D_1 + v_1 D_2] \in FI_1$ we obtain that $v_1 \in \cap_{i=1}^3 R^{D_i} = F$, that is $v_1 \in F\langle 1 \rangle$. It follows from the relations (4) that $D_3(u_1) + v_1 r \in F\langle \{ \frac{a^i b^j}{i! j!} \}, \ 0 \leqslant i, j \leqslant m. \rangle$. Analogously the inclusion $[D_2, u_1 D_1 + v_1 D_2] \in FI_1$ implies the relation

$$D_2(u_1) \in F\langle \{ \frac{a^i b^j}{i! j!} \}, \ 0 \leqslant i, j \leqslant m. \rangle.$$

Since $[D_3, D_2] = rD_1$ and $rD_1(u_1) = 0$, we see (using Lemma 7) that $u_1 \in F\langle\{\frac{a^i b^j}{i!j!}\}, \ 0 \leqslant i, j \leqslant m+1.\rangle$. By inductive assumption, we have

$$u_{s-1} \in F\langle\{\tfrac{a^i b^j}{i!j!}\}, \ 0 \leqslant i, j \leqslant m+s-1\rangle, \quad v_{s-1} \in F\langle 1, \ldots, \tfrac{a^{s-2}}{(s-2)!}\rangle.$$

Note that the relations (4) imply the equalities

$$D_3(u_s) = u_{s-1} - rv_s, \ D_3(v_s) = v_{s-1}$$

(here $[D_3, D_2] = rD_1$). Analogously it follows from the relation

$$[D_2, u_s D_1 + v_s D_2] \in FI_1$$

that

$$D_2(v_s) = 0, D_2(u_s) \in F\langle\{\tfrac{a^i b^j}{i!j!}D_1\}, \ 0 \leqslant i, j \leqslant m\rangle.$$

Since $D_1 \in Z(L)$, we have the equalities $D_1(u_s) = D_1(v_s) = 0$. Therefore we get by Lemma 6 that $v_s \in F\langle 1, \ldots, a^{s-1}/(s-1)!\rangle$. By Lemma 7, $u_s \in F\langle\{\frac{a^i b^j}{i!j!}\}, \ 0 \leqslant i, j \leqslant m+s\rangle$ (since $D_3(u_s) \in F\langle\{\frac{a^i b^j}{i!j!}\}, \ 0 \leqslant i, j \leqslant m+s-1\rangle$ by the relations (4)). Since $rD_1 \in FI_1$, we have $rv_s \in F\langle\{\frac{a^i b^j}{i!j!}\}, \ 0 \leqslant i, j \leqslant m+s-1\rangle$. But then by Lemma 7 $u_s \in F\langle\{\frac{a^i b^j}{i!j!}\}, \ 0 \leqslant i, j \leqslant m+s\rangle$. So, we have proved that the Lie algebra L is contained in the subalgebra $\overline{L}$ from the conditions of the lemma. To finish with the proof we must prove that the element D_2 can be chosen in $W(A)$ in such a way that $[D_3, D_2] = 0$. Take the element $D_2 - r_0 D_1$ instead D_2, where the element r_0 is obtained from r by formal integration on variable a (recall that $r \in F\langle\{\frac{a^i b^j}{i!j!}\}, \ 0 \leqslant i, j \leqslant m\rangle$). Then $[D_3, D_2] = 0$. The proof is complete. $\square$

Collect now the results about nilpotent Lie algebras into the next statement.

Theorem 1. *Let $\mathbb{K}$ be a field of characteristic zero, A an integral $\mathbb{K}$-domain with fraction field R. Denote by $W(A)$ the subalgebra $R\mathrm{Der}_{\mathbb{K}}A$ of the Lie algebra $\mathrm{Der}_{\mathbb{K}}R$. Let L be a nilpotent subalgebra of rank 3 over R from $W(A)$ and $F = R^L$ its field of constants. If $\dim_F FL \geqslant 4$, then there exist integers $n \geqslant 0, m \geqslant 0$, elements $D_1, D_2, D_3 \in FL$ such that $[D_i, D_j] = 0, i, j = 1, 2, 3$ and the Lie algebra FL is contained in one of the following subalgebras of the Lie algebra $W(A)$:*

1) $L_1 = F\langle D_3, D_1, aD_1, \ldots, (a^n/n!)D_1, D_2, aD_2, \ldots, (a^n/n!)D_2\rangle$, where $a \in R$ is such that $D_1(a) = D_2(a) = 0, D_3(a) = 1$.

2) $L_2 = F\langle D_3, D_2, aD_2, \ldots, (a^n/n!)D_2, \{\frac{a^i b^j}{i!j!}D_1\}, 0 \leqslant i, j \leqslant m\rangle$ where $a, b \in R$ are such that $D_1(a) = D_2(a) = 0, D_3(a) = 1, D_1(b) = D_3(b) = 0, D_2(b) = 1$.

As a corollary we get the next characterization of nilpotent Lie algebras of rank $\leqslant 3$ from the Lie algebra $W(A)$.

Theorem 2. *Under conditions of Theorem 1, every nilpotent subalgebra L of rank $k \leqslant 3$ over R from the Lie algebra $W(A)$ is isomorphic to a finite dimensional subalgebra of the triangular Lie algebra $u_k(F)$.*

Proof. If $k = 1$ then the Lie algebra FL is one-dimensional over F and therefore is isomorphic to $u_1(F) = F\frac{\partial}{\partial x_1}$. In the case $k = 2$, the Lie algebra FL is (by Lemma 4) of the form

$$FL = F\langle D_1, aD_1, \ldots, \tfrac{a^k}{k!}D_1, D_2 \rangle, \ k \geqslant 0$$

$$(\text{if } k = 0, \text{ then put } FL = F\langle D_1, D_2 \rangle),$$

where $[D_1, D_2] = 0$, $D_1(a) = 0$, $D_2(a) = 1$. The Lie algebra FL is isomorphic to a suitable subalgebra of the triangular Lie algebra $u_2(F) = \{f(x_2)\frac{\partial}{\partial x_1} + F\frac{\partial}{\partial x_2}\}$: the correspondence $D_i \mapsto \frac{\partial}{\partial x_i}, i = 1, 2$ and $a \mapsto x_2$ can be extended to an isomorphism between FL and a subalgebra of $u_2(F)$. Let now $k = 3$. If $\dim_F FL = 3$, then FL is either abelian or has a basis D_1, D_2, D_3 with multiplication rule $[D_3, D_2] = D_1, [D_2, D_1] = [D_3, D_1] = 0$. In the first case, FL is isomorphic to the subalgebra $F\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \rangle$, in the second case it is isomorphic to the subalgebra $F\langle \frac{\partial}{\partial x_1}, x_3\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \rangle$ of the triangular Lie algebra $u_3(F)$.

Let now $\dim_F FL \geqslant 4$. The Lie algebra FL is contained (by Theorem 1) in one of the Lie algebras L_1 or L_2 from the statement of that theorem. Note that the Lie algebra L_1 is isomorphic to the subalgebra $\overline{L}_1$ of the Lie algebra $u_3(F)$ of the form

$$\overline{L}_1 = F\langle \tfrac{\partial}{\partial x_3}, \tfrac{\partial}{\partial x_1}, \ldots, (x_3^n/n!)\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial x_2}, \ldots, (x_3^n/n!)\tfrac{\partial}{\partial x_2} \rangle,$$

Analogously the Lie algebra L_2 is isomorphic the the subalgebra $\overline{L}_2$ of $u_3(F)$ of the form

$$\overline{L}_2 = F\langle \tfrac{\partial}{\partial x_3}, \tfrac{\partial}{\partial x_2}, \ldots, (x_3^n/n!)\tfrac{\partial}{\partial x_2}, \{\tfrac{x_2^i x_3^j}{i!j!}\tfrac{\partial}{\partial x_1}\}, 0 \leqslant i, j \leqslant m \rangle. \qquad \square$$

Corollary 1. *Let L be a nilpotent subalgebra of the Lie algebra $W_3(\mathbb{K}) = \mathrm{Der}(\mathbb{K}[x_1, x_2, x_3])$ and F the field of constants for the Lie algebra L in the field $\mathbb{K}(x, x_2, x_3)$. Then the Lie algebra FL (over the field F) is isomorphic to a finite dimensional subalgebra of the triangular Lie algebra $u_3(F)$.*

Remark 1. If L is a nilpotent subalgebra of rank 3 over R from the Lie algebra $W_3(\mathbb{K})$, then L being isomorphic to a subalgebra of the triangular Lie algebra $u_3(\mathbb{K})$ can be not conjugated (by an automorphism of $W_3(\mathbb{K})$) with any subalgebra of $u_3(\mathbb{K})$. Indeed, the subalgebra $L = \mathbb{K}\langle x_1\frac{\partial}{\partial x_1}, x_2\frac{\partial}{\partial x_2}, x_3\frac{\partial}{\partial x_3}\rangle$ is nilpotent but not conjugated with any subalgebra of $u_3(\mathbb{K})$ (L is selfnormalized in $W_3(\mathbb{K})$, but any finite dimensional subalgebra of $u_3(\mathbb{K})$ is not, because of locally nilpotency of the Lie algebra $u_3(\mathbb{K})$).

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