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The Littlewood-Richardson rule and Gelfand-Tsetlin patterns

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Abstract. We give a survey on the Littlewood-Richardson rule. Using Gelfand-Tsetlin patterns as the main machinery of our analysis, we study the interrelationship of various combinatorial descriptions of the Littlewood-Richardson rule.

1. Introduction

1.1. Let us consider Schur polynomials s_μ, s_ν and s_λ in n variables labelled by partitions μ, ν and λ, respectively. *The Littlewood-Richardson (LR) coefficient* is the multiplicity $c^\lambda_{\mu,\nu}$ of s_ν in the product of s_μ and s_ν:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$$

and its description is called *the LR rule.*

The same number appears in the tensor product decomposition problem in the representation theory of the complex general linear group GL_n and Schubert calculus in the cohomology of the Grassmannians, and is also related to the eigenvalues of the sum of Hermitian matrices. For more details, we refer readers to [8, 15, 27, 29].

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1.2. The LR rule is usually stated in terms of combinatorial objects called *LR tableaux*. Recall that a Young tableau is a filling of the boxes of a Young diagram with positive integers. We shall use the English convention of drawing Young diagrams and tableaux as in [7, 26] and assume a basic knowledge of these objects.

Definition 1. A tableau T on a skew Young diagram is called a LR tableau if it satisfies the following conditions:

1) it is semistandard, that is, the entries in each row of T weakly increase from left to right, and the entries in each column strictly increase from top to bottom; and
2) its reverse reading word is a Yamanouchi word (or lattice permutation). That is, in the word $x_1x_2x_3\dots x_r$ obtained by reading all the entries of T from left to right in each row starting from the bottom one, the sequence $x_rx_{r-1}x_{r-2}\dots x_s$ contains at least as many a's as it does $(a+1)$'s for all $a \geqslant 1$.

For example, the following is a LR tableau on a skew Young diagram $(11, 7, 5, 3)/(5, 3, 1)$

					1	1	1	1	1	1
			1	2	2	2				
	2	3	3	3						
2	4	4								

and its reverse reading word is 24423331222111111.

Remark 1. (1) In this paper we assume each tableau's entries weakly increase from left to right in every row. (2) From the second condition in the above definition, which we will call the *Yamanouchi condition*, the bth row of a LR tableau does not contain any entries strictly bigger than b for all $b \geqslant 1$.

The number of LR tableaux on the skew shape λ/μ with content ν is equal to the LR number $c^{\lambda}_{\mu,\nu}$. Here, the content $\nu = (\nu_1, \dots, \nu_n)$ of a tableau means that the entry k appears ν_k times in the tableau for $k \geqslant 1$. See, for example, [24, §I.9] and [15].

1.3. In this paper, we survey variations of the semistandard and Yamanouchi conditions with an emphasis on dualities in combinatorial descriptions of the LR rule. Although many of the results in this paper can be found in the literature, we will give complete and elementary proofs of our statements.

(1) In Theorem 1 and Theorem 2, we analyse *hives*, introduced by Knutson and Tao along with their honeycomb model [21], in terms of Gelfand-Tsetlin(GT) patterns [10]. We then show how the interlacing conditions in GT patterns are intertwined to form the defining conditions of hives. For the relevant results, see for example [1–4].

(2) In Theorem 3, we show that the semistandard and Yamanouchi conditions in LR tableaux are equivalent to, respectively, the interlacing and exponent conditions in *GZ schemes* introduced by Gelfand and Zelevinsky [11]. As a corollary we obtain a correspondence between LR tableaux and hives equivalent to [18, (3.3)]. We then observe how conditions on LR tableaux, GZ schemes and hives are translated between objects by this bijection. For the relevant results, see, for example, [1, 5, 18, 19].

(3) In Theorem 4, we show that the semistandard and Yamanouchi conditions in LR tableaux are equivalent to, respectively, the exponent and semistandard conditions in their *companion tableaux* introduced by van Leeuwen [29]. Here the correspondence between conditions is obtained by taking the transpose of matrices.

As a consequence, we obtain bijections between the families of combinatorial objects counting the LR number.

1.4. In [16, 17], Howe and his collaborators constructed a polynomial model for the tensor product of representations in terms of two copies of the multi-homogeneous coordinate ring of the flag variety, and then studied its toric degeneration with the SAGBI-Gröbner method. Through the characterization of the leading monomials of highest weight vectors, their toric variety is encoded by the *LR cone* [25]. On the other hand, via toric degenerations, the flag variety may be described in terms of the lattice cone of GT patterns [13, 20, 23]. These results led us to study the LR rule in terms of two sets of interlacing or semistandard conditions and to investigate the interrelationship of various combinatorial descriptions of the LR rule with GT patterns.

2. Hives and GT patterns I

In this section, we define GT patterns, hives, and objects related to them. We also describe hives in terms of pairs of GT patterns.

2.1. We set, once and for all, three polynomial dominant weights of the complex general linear group GL_n, that is, the sequences of nonnegative integers:

$$\lambda = (\lambda_1, \dots, \lambda_n), \quad \mu = (\mu_1, \dots, \mu_n), \quad \nu = (\nu_1, \dots, \nu_n)$$

such that $\lambda_i \geqslant \lambda_{i+1}$, $\mu_i \geqslant \mu_{i+1}$, and $\nu_i \geqslant \nu_{i+1}$ for all i. We define the dual λ^* of λ to be

$$\lambda^* = (-\lambda_n, -\lambda_{n-1}, \dots, -\lambda_1),$$

and define μ^* and ν^* similarly.

2.2. Let us consider an array of integers, which we will call a *t-array*

$$T = \left(t_1^{(1)}, \dots, t_j^{(i)}, \dots, t_n^{(n)}\right) \in \mathbb{Z}^{n(n+1)/2}$$

where $1 \leqslant j \leqslant i \leqslant n$. We are particularly interested in the case when the entries of T are either all non-negative or all non-positive integers.

Definition 2. A t-array $T = (t_j^{(i)}) \in \mathbb{Z}^{n(n+1)/2}$ is called a GT pattern for GL_n if it satisfies the *interlacing conditions*:

$$\text{IC(1): } t_j^{(i+1)} \geqslant t_j^{(i)}$$
$$\text{IC(2): } t_j^{(i)} \geqslant t_{j+1}^{(i+1)}$$

for all i and j.

We shall draw a t-array in the reversed pyramid form. For example, a generic GT pattern for GL_5 is

$$\begin{array}{ccccccccc}
t_1^{(5)} & & t_2^{(5)} & & t_3^{(5)} & & t_4^{(5)} & & t_5^{(5)} \\
& t_1^{(4)} & & t_2^{(4)} & & t_3^{(4)} & & t_4^{(4)} & \\
& & t_1^{(3)} & & t_2^{(3)} & & t_3^{(3)} & & \\
& & & t_1^{(2)} & & t_2^{(2)} & & & \\
& & & & t_1^{(1)} & & & &
\end{array}$$

where the entries are weakly decreasing along the diagonals from left to right.

Then, the *dual array* $T^* = (s_j^{(i)})$ of T is the t-array obtained by reflecting T over a vertical line and then multiplying -1, i.e.,

$$s_j^{(i)} = -t_{i+1-j}^{(i)}$$

for all $1 \leqslant j \leqslant i \leqslant n$.

Definition 3. For a t-array $T = (t_j^{(i)}) \in \mathbb{Z}^{n(n+1)/2}$,

1) the kth row of T is $t^{(k)} = (t_1^{(k)}, t_2^{(k)}, \ldots, t_k^{(k)}) \in \mathbb{Z}^k$ for $1 \leqslant k \leqslant n$. The type of T is its nth row;
2) the weight of T is $(w_1, w_2, \ldots, w_n) \in \mathbb{Z}^n$ where $w_1 = t_1^{(1)}$ and

$$w_i = \sum_{k=1}^{i} t_k^{(i)} - \sum_{k=1}^{i-1} t_k^{(i-1)} \quad \text{for} \quad 2 \leqslant i \leqslant n.$$

Note that if T is of type λ and weight $w \in \mathbb{Z}^n$, then T^* is of type λ^* and weight $-w$.

GT patterns were introduced by Gelfand and Tsetlin in [10] to label the weight basis elements of an irreducible representation of the general linear group. The weight of T is exactly the weight of the basis element labelled by T in the irreducible representation V_n^μ whose highest weight is $\mu = t^{(n)}$. It follows that the dual array T^* of T corresponds to a weight vector in the contragradiant representation of V_n^μ.

2.3. Let us consider an array of nonnegative integers, which we will call a *h-array*,

$$(h_{0,0}, \ldots, h_{a,b}, \ldots, h_{n,n}) \in \mathbb{Z}^{(n+1)(n+2)/2}$$

where $0 \leqslant a \leqslant b \leqslant n$ and $h_{0,0} = 0$.

Definition 4. A hive for GL_n is a h-array $H = (h_{a,b}) \in \mathbb{Z}^{(n+1)(n+2)/2}$ satisfying the *rhombus conditions*:

$$\begin{aligned}
&\text{RC(1)}: (h_{a,b} + h_{a-1,b-1}) \geqslant (h_{a-1,b} + h_{a,b-1}) \quad \text{for} \quad 1 \leqslant a < b \leqslant n,\\
&\text{RC(2)}: (h_{a-1,b} + h_{a,b}) \geqslant (h_{a,b+1} + h_{a-1,b-1}) \quad \text{for} \quad 1 \leqslant a \leqslant b < n,\\
&\text{RC(3)}: (h_{a,b} + h_{a,b+1}) \geqslant (h_{a+1,b+1} + h_{a-1,b}) \quad \text{for} \quad 1 \leqslant a \leqslant b < n.
\end{aligned}$$

We shall draw a h-array in the pyramid form. For example, a generic hive for GL_3 is shown below.

$$\begin{array}{ccccccc}
 & & & h_{0,0} & & & \\
 & & h_{0,1} & & h_{1,1} & & \\
 & h_{0,2} & & h_{1,2} & & h_{2,2} & \\
h_{0,3} & & h_{1,3} & & h_{2,3} & & h_{3,3}
\end{array}$$

The rhombus conditions RC(1), RC(2), and RC(3) then say that, for each fundamental rhombus of one of the following forms,

$$
\begin{array}{ccc} & O' & & A' \\ A & & O & \end{array}, \qquad
\begin{array}{ccc} & A & \\ O & & O' \\ & A' & \end{array}, \qquad
\begin{array}{ccc} A' & & O & \\ & O' & & A \end{array}
$$

the sum of entries at the obtuse corners is bigger than or equal to the sum of entries at the acute corners, i.e., $O + O' \geqslant A + A'$.

For polynomial dominant weights μ, ν, and λ of GL_n, we let $\mathcal{H}(\mu, \nu, \lambda)$ denote the set of all h-arrays such that

$$
\begin{aligned}
\mu &= (h_{0,1} - h_{0,0}, h_{0,2} - h_{0,1}, \ldots, h_{0,n} - h_{0,n-1}), \\
\nu &= (h_{1,n} - h_{0,n}, h_{2,n} - h_{1,n}, \ldots, h_{n,n} - h_{n-1,n}), \\
\lambda &= (h_{1,1} - h_{0,0}, h_{2,2} - h_{1,1}, \ldots, h_{n,n} - h_{n-1,n-1}).
\end{aligned} \tag{1}
$$

That is, the three boundary sides of $H \in \mathcal{H}(\mu, \nu, \lambda)$ are fixed:

$$
\begin{aligned}
h_{0,i} &= \mu_1 + \mu_2 + \cdots + \mu_i \\
h_{i,n} &= \sum_{j=1}^{n} \mu_j + \nu_1 + \nu_2 + \cdots + \nu_i \\
h_{i,i} &= \lambda_1 + \lambda_2 + \cdots + \lambda_i
\end{aligned}
$$

for $1 \leqslant i \leqslant n$. Recall that we always set $h_{0,0} = 0$. Let $\mathcal{H}^\circ(\mu, \nu, \lambda)$ be the subset of $\mathcal{H}(\mu, \nu, \lambda)$ satisfying the rhombus conditions. This is the set of hives whose boundaries are described by (1).

Hives were introduced by Knutson and Tao in [21] along with their honeycomb model to prove the saturation conjecture. In particular, the number of hives in $\mathcal{H}^\circ(\mu, \nu, \lambda)$ is equal to the LR number $c^{\lambda}_{\mu,\nu}$. See also [5, 22, 25].

2.4. For each h-array $H = (h_{a,b}) \in \mathbb{Z}^{(n+1)(n+2)/2}$, let us define its *derived t-arrays*

$$
T_1 = (x^{(i)}_j), \quad T_2 = (y^{(i)}_j), \quad T_3 = (z^{(i)}_j)
$$

whose entries are obtained from the differences of adjacent entries of H.

More specifically, for each fundamental triangle in H,

$$
\begin{array}{ccc} & h_{a,b} & \\ h_{a,b+1} & & h_{a+1,b+1} \end{array}
$$

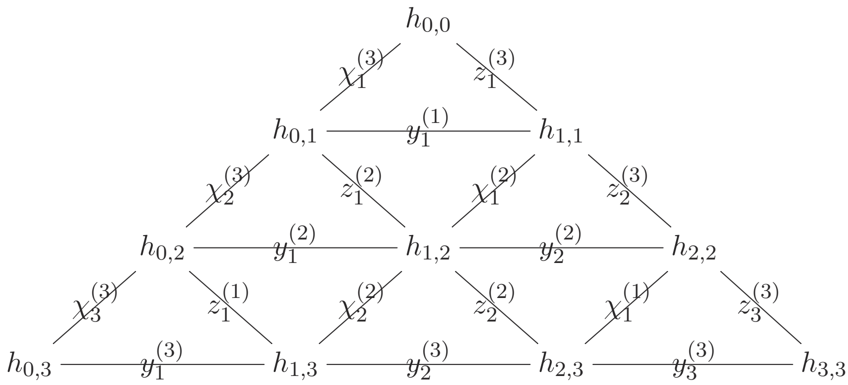


FIGURE 1. A h-array and its three derived t-arrays.

the entries of the derived t-arrays $(x_j^{(i)})$, $(y_j^{(i)})$, and $(z_j^{(i)})$ are

$$\begin{aligned} x_{b+1-a}^{(n-a)} &= h_{a,b+1} - h_{a,b} \quad \text{(SW–NE direction)} \\ y_{a+1}^{(b+1)} &= h_{a+1,b+1} - h_{a,b+1} \quad \text{(E–W direction)} \\ z_{a+1}^{(n+a-b)} &= h_{a+1,b+1} - h_{a,b} \quad \text{(SE–NW direction)} \end{aligned} \tag{2}$$

for $0 \leqslant a \leqslant b \leqslant n-1$.

This rather involved indexing is to make the entries of the derived arrays compatible with those of GT patterns. We may visualize the derived t-arrays by placing their entries between the entries of the h-array used to compute them. For example, if $n = 3$, then a h-array and its three derived t-arrays may be drawn as Figure 1.

2.5. The rhombus conditions for h-arrays are closely related to the interlacing conditions for their derived t-arrays.

Proposition 1. *Let $T_k = T_k(H)$ be a derived t-array of a h-array H for $k = 1, 2, 3$.*

1) *H satisfies RC(1) if and only if T_1 satisfies IC(2) and T_2 satisfies IC(1).*
2) *H satisfies RC(2) if and only if T_1 and T_3 satisfy IC(1).*
3) *H satisfies RC(3) if and only if T_2 and T_3 satisfy IC(2).*
4) *T_3 satisfies IC(1) if and only if T_1 satisfies IC(1).*
5) *T_3 satisfies IC(2) if and only if T_2 satisfies IC(2).*

Proof. Let us consider five adjacent entries of H of the forms

$$\begin{array}{ccccc} & & Z_1 & & \\ & Y_1 & & W_1 & \\ X_1 & & V_1 & & \end{array}, \quad \begin{array}{ccccc} & Y_2 & & W_2 & \\ X_2 & & V_2 & & U_2 \end{array}, \quad \begin{array}{cccc} & Z_3 & & \\ Y_3 & & W_3 & \\ & V_3 & & U_3 \end{array}.$$

Then, in the first and the third ones, RC(2) says that $Y_i+W_i \geqslant Z_i+V_i$ for $i=1$ and 3. This is equivalent to $Y_1-Z_1 \geqslant V_1-W_1$ and $W_3-Z_3 \geqslant V_3-Y_3$, which are IC(1) for T_1 and T_3, respectively. This proves the statement (2). The statements (1) and (3) can be shown similarly.

Next, let us consider fundamental rhombi of the following forms in H

$$\begin{array}{ccc} & K & \\ L & & N \\ & M & \end{array}, \qquad \begin{array}{cccc} P & & S & \\ & Q & & R. \end{array}$$

Note that $N-K \geqslant M-L$ if and only if $L-K \geqslant M-N$, which proves (4). Similarly, $P-Q \geqslant S-R$ if and only if $P-S \geqslant Q-R$, which proves (5). $\square$

Suppose a h-array H satisfies RC(1), RC(2), and RC(3). Then, by the statements (1) and (2) of Proposition 1, $T_1(H)$ satisfies IC(1) and IC(2). Similarly, by the statements (1) and (3), $T_2(H)$ satisfies IC(1) and IC(2). This shows that $T_1(H)$ and $T_2(H)$ are GT patterns. Conversely, if $T_1(H)$ and $T_2(H)$ are GT patterns, then, by the statements (4) and (5), $T_3(H)$ is also a GT pattern. This means all three derived t-arrays satisfy both IC(1) and IC(2), and therefore, from the statements (1), (2), and (3), H is a hive.

Theorem 1. *For a h-array $H \in \mathbb{Z}^{(n+1)(n+2)/2}$ and its derived t-arrays $T_1(H)$ and $T_2(H)$, H is a hive if and only if $T_1(H)$ and $T_2(H)$ are GT patterns for GL_n.*

We remark that, in the above result, $T_1(H)$ and $T_2(H)$ are not independent. Let $T_1 = (x_j^{(i)})$ and $T_2 = (y_j^{(i)})$ be the derived t-arrays of a h-array H. Then, for each rhombus of the form

$$\begin{array}{ccc} & B & A \\ C & D & \end{array}$$

we have $(D-C)+(C-B)=(D-A)+(A-B)$, or

$$(C-B)-(D-A)=(A-B)-(D-C)$$

which is, using (2),

$$x_{b-a}^{(n-a-1)}-x_{b+1-a}^{(n-a)}=y_{a+1}^{(b+1)}-y_{a+1}^{(b)} \tag{3}$$

for $0\leqslant a<b<n$. Note that hives (respectively, GT patterns) for GL_n with non-negative entries form a subsemigroup of $\mathbb{Z}_{\geqslant 0}^{(n+1)(n+2)/2}$ (respectively, $\mathbb{Z}_{\geqslant 0}^{n(n+1)/2}$). Theorem 1 and (3) imply that the semigroup

$$\bigcup_{(\mu,\nu,\lambda)}\mathcal{H}^\circ(\mu,\nu,\lambda)$$

of hives is a fiber product of, over $\mathbb{Z}_{\geqslant 0}^{n(n-1)/2}$, two affine semigroups S_{GT}^1 and S_{GT}^2 of GT patterns with respect to

$$\phi_k : S_{GT}^k \longrightarrow \mathbb{Z}_{\geqslant 0}^{n(n-1)/2}$$

such that, for $0\leqslant a<b<n$,

$$\begin{aligned}\phi_1(T_1)&=\left(\dots,x_{b-a}^{(n-a-1)}-x_{b+1-a}^{(n-a)},\dots\right),\\ \phi_2(T_2)&=\left(\dots,y_{a+1}^{(b+1)}-y_{a+1}^{(b)},\dots\right)\end{aligned}$$

where $T_1=(x_j^{(i)})\in S_{GT}^1$ and $T_2=(y_j^{(i)})\in S_{GT}^2$.

We also remark that by exchanging the roles of $T_1(H)$, $T_2(H)$ and $T_3(H)$, one can read the symmetry of the LR rule. See, for example, [28].

3. Hives and GT patterns II

In this section, we study the set $\mathcal{H}^\circ(\mu,\nu,\lambda)$ of hives with a given boundary condition in terms of a single GT pattern.

3.1. Gelfand and Zelevinsky counted the LR number $c_{\mu,\nu}^\lambda$ with GT patterns of type μ and weight $\lambda-\nu$ satisfying the following additional condition.

Lemma 1 ([11]). *For a t-array $T=(t_j^{(i)})\in\mathbb{Z}^{n(n+1)/2}$, we define its exponents as*

$$\varepsilon_j^{(i)}(T)=\sum_{1\leqslant h<j}(t_h^{(i+1)}-2t_h^{(i)}+t_h^{(i-1)})+(t_j^{(i+1)}-t_j^{(i)}).$$

Then the cardinality of the set $GZ(\mu, \lambda - \nu, \nu)$ of all GT patterns T of type μ with weight $\lambda - \nu$ such that, for all i and j,

$$\varepsilon_j^{(i)}(T) \leqslant \nu_i - \nu_{i+1}$$

is equal to the LR number $c_{\mu,\nu}^{\lambda}$.

The elements of $GZ(\mu, \lambda - \nu, \nu)$ will be called *GZ schemes.*

3.2. Note that, for a h-array H, since the derived t-arrays are defined from the differences of the entries in H, if the boundaries of H are fixed, then any one of the derived t-array of H uniquely determines H. Moreover, we can characterize the derived t-arrays as follows.

Theorem 2. *For a h-array H in $\mathcal{H}(\mu, \nu, \lambda)$, consider its derived t-arrays $T_1(H)$ and $T_2(H)$.*

1) *H is a hive if and only if $T_1^*(H) = (T_1(H))^*$ is a GZ scheme in $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$;*
2) *H is a hive if and only if $T_2(H)$ is a GZ scheme in $GZ(\nu, \lambda - \mu, \mu)$.*

Note that this theorem, in particular, gives bijections between hives and GZ schemes:

$$\begin{array}{ccc} \mathcal{H}^\circ(\mu, \nu, \lambda) & \longrightarrow & GZ(\mu^*, \lambda^* - \nu^*, \nu^*) \\ H & \longmapsto & T_1^*(H) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{H}^\circ(\mu, \nu, \lambda) & \longrightarrow & GZ(\nu, \lambda - \mu, \mu) \\ H & \longmapsto & T_2(H) \end{array}.$$

For the rest of this section, we will prove Theorem 2 by showing the following.

(a) $T_1^*(H)$ satisfies IC(2) if and only if $\varepsilon_j^{(i)}(T_2(H)) \leqslant \mu_i - \mu_{i+1}$;
(b) $T_1^*(H)$ satisfies IC(1) if and only if $T_2(H)$ satisfies IC(1);
(c) $T_1^*(H)$ satisfies $\varepsilon_j^{(i)}(T_1^*(H)) \leqslant \nu_i^* - \nu_{i+1}^*$ if and only if $T_2(H)$ satisfies IC(2).

The weights of the derived t arrays will also be computed.

3.3. Let us first compute the weights of $T_1(H)$ and $T_2(H)$ for $H \in \mathcal{H}(\mu, \nu, \lambda)$.

Lemma 2. *For a h-array $H = (h_{a,b}) \in \mathcal{H}(\mu, \nu, \lambda)$,*

1) *the weight of* $T_1(H)$ *is* $\nu^* - \lambda^*$, *i.e.*,

$$(\lambda_n - \nu_n, \lambda_{n-1} - \nu_{n-1}, \ldots, \lambda_1 - \nu_1)$$

therefore, the weight of $T_1^*(H)$ *is* $\lambda^* - \nu^*$;

2) *the weight of* $T_2(H)$ *is* $\lambda - \mu$, *i.e.*,

$$(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_n - \mu_n).$$

Proof. We will prove the second statement. The proof of the first case is similar. From Definition 3, (2) and the expressions for λ and μ in terms of the h-array elements it follows

$$w_1 = y_1^{(1)} = h_{1,1} - h_{0,1} = (h_{1,1} - h_{0,0}) + (h_{0,0} - h_{0,1}) = \lambda_1 - \mu_1.$$

Using the same approach for w_i, $i \geqslant 2$, we see

$$\begin{aligned} w_i &= \sum_{k=1}^{i} y_k^{(i)} - \sum_{k=1}^{i-1} y_k^{(i-1)} \\ &= \sum_{k=1}^{i} (h_{k,i} - h_{k-1,i}) - \sum_{k=1}^{i-1} (h_{k,i-1} - h_{k-1,i-1}) \\ &= (h_{i,i} - h_{0,i}) - (h_{i-1,i-1} - h_{0,i-1}) \\ &= \lambda_i - \mu_i. \end{aligned}$$

Therefore $w_i = \lambda_i - \mu_i$ for all i, and the weight of $T_2(H)$ is $\lambda - \mu$. □

3.4. Next, we study the relations between the interlacing conditions and the exponents conditions for derived arrays. Note that, from the definition of dual arrays, a t-array T satisfies IC(1) if and only if T^* satisfies IC(2), and T satisfies IC(2) if and only if T^* satisfies IC(1).

Proposition 2. *For a h-array* $H = (h_{a,b}) \in \mathcal{H}(\mu, \nu, \lambda)$ *and its derived t-arrays* $T_1(H) = (x_j^{(i)})$ *and* $T_2(H) = (y_j^{(i)})$, $T_1(H)$ *satisfies IC(1) if and only if* $\varepsilon_j^{(i)}(T_2(H)) \leqslant \mu_i - \mu_{i+1}$.

Proof. Let us assume $j > 1$. Then the exponent of $T_2(H)$,

$$\varepsilon_j^{(i)}(T_2(H)) = \sum_{1 \leqslant h < j} \Big((y_h^{(i+1)} - y_h^{(i)}) - (y_h^{(i)} - y_h^{(i-1)}) \Big) + \Big(y_j^{(i+1)} - y_j^{(i)} \Big)$$

can be rewritten in terms of the entries of $T_1(H)$. By using (3),

$$\begin{aligned}\varepsilon_j^{(i)}(T_2(H)) &= \sum_{1\leqslant h<j}\Big((x_{i-h+1}^{(n-h)}-x_{i-h+2}^{(n-h+1)})-(x_{i-h}^{(n-h)}-x_{i-h+1}^{(n-h+1)})\Big)\\ &\quad+\Big(x_{i-j+1}^{(n-j)}-x_{i-j+2}^{(n-j+1)}+y_j^{(i)}\Big)-\Big(x_{i-j}^{(n-j)}-x_{i-j+1}^{(n-j+1)}+y_j^{(i-1)}\Big)\end{aligned}$$

and we see that parts of the consecutive terms cancel to give

$$\varepsilon_j^{(i)}(T_2(H))=\Big(x_i^{(n)}-x_{i+1}^{(n)}\Big)+\Big(x_{i-j+1}^{(n-j)}-x_{i-j}^{(n-j)}+y_j^{(i)}-y_j^{(i-1)}\Big). \tag{4}$$

Now note that the interlacing condition IC(1) for $T_1(H)$ implies $x_{i-j+1}^{(n-j+1)}\geqslant x_{i-j+1}^{(n-j)}$ or equivalently, by using (3),

$$x_{i-j}^{(n-j)} \;\geqslant\; \Big(x_{i-j+1}^{(n-j)}+y_j^{(i)}-y_j^{(i-1)}\Big)$$

therefore

$$0 \;\geqslant\; \Big(x_{i-j+1}^{(n-j)}-x_{i-j}^{(n-j)}+y_j^{(i)}-y_j^{(i-1)}\Big).$$

Hence, from (4), the interlacing condition IC(1) for $T_1(H)$ is equivalent to

$$\varepsilon_j^{(i)}(T_2(H))\leqslant\Big(x_i^{(n)}-x_{i+1}^{(n)}\Big)=\mu_i-\mu_{i+1}.$$

The case $j=1$ can be shown similarly for all i. □

Proposition 3. *For a h-array $H=(h_{a,b})\in\mathcal{H}(\mu,\nu,\lambda)$ and its derived t-arrays $T_1(H)=(x_j^{(i)})$ and $T_2(H)=(y_j^{(i)})$, $T_1(H)$ satisfies IC(2) if and only if $T_2(H)$ satisfies IC(1).*

Proof. Using the equality (3),

$$\Big(x_j^{(i)}\geqslant x_{j+1}^{(i+1)}\Big)\quad\text{if and only if}\quad\Big(y_{n-i}^{(n-i+j)}\geqslant y_{n-i}^{(n-i+j-1)}\Big)$$

and therefore, by setting $i'=n-i+j-1$ and $j'=n-i$, we have

$$\Big(x_j^{(i)}\geqslant x_{j+1}^{(i+1)}\Big)\quad\text{if and only if}\quad\Big(y_{j'}^{(i'+1)}\geqslant y_{j'}^{(i')}\Big)$$

for $1\leqslant j\leqslant i\leqslant n-1$ and $1\leqslant j'\leqslant i'\leqslant n-1$. This shows that IC(2) holds for $T_1(H)$ if and only if IC(1) holds for $T_2(H)$. □

Proposition 4. *For a h-array $H=(h_{a,b})\in\mathcal{H}(\mu,\nu,\lambda)$ and its derived t-arrays $T_1(H)=(x_j^{(i)})$ and $T_2(H)=(y_j^{(i)})$, $T_1^*(H)$ satisfies $\varepsilon_j^{(i)}(T_1^*(H))\leqslant\nu_i^*-\nu_{i+1}^*$ if and only if $T_2(H)$ satisfies IC(2).*

Proof. Let us assume $j > 1$. Write the exponents of $T_1^*(H) = (s_j^{(i)})$ using $s_j^{(i)} = -x_{i+1-j}^{(i)}$.

$$\begin{aligned}\varepsilon_j^{(i)}(T_1^*(H)) &= \sum_{1\leqslant h<j} \left(-x_{i-h+2}^{(i+1)} + 2x_{i-h+1}^{(i)} - x_{i-h}^{(i-1)}\right) \\ &\quad + \left(-x_{i-j+2}^{(i+1)} + x_{i-j+1}^{(i)}\right) \\ &= \sum_{1\leqslant h<j} \left((x_{i-h+1}^{(i)} - x_{i-h+2}^{(i+1)}) - (x_{i-h}^{(i-1)} - x_{i-h+1}^{(i)})\right) \\ &\quad + \left(x_{i-j+1}^{(i)} - x_{i-j+2}^{(i+1)}\right)\end{aligned}$$

Then, using the identity (3), we can rewrite the exponents in terms of the entries of $T_2(H)$ as

$$\begin{aligned}\varepsilon_j^{(i)}(T_1^*(H)) &= \sum_{1\leqslant h<j} \left((y_{n-i}^{(n-h+1)} - y_{n-i}^{(n-h)}) - (y_{n-i+1}^{(n-h+1)} - y_{n-i+1}^{(n-h)})\right) \\ &\quad + \left(y_{n-i}^{(n-j+1)} - y_{n-i}^{(n-j)}\right) \\ &\leqslant \sum_{1\leqslant h<j} \left((y_{n-i}^{(n-h+1)} - y_{n-i}^{(n-h)}) - (y_{n-i+1}^{(n-h+1)} - y_{n-i+1}^{(n-h)})\right) \\ &\quad + \left(y_{n-i}^{(n-j+1)} - y_{n-i+1}^{(n-j+1)}\right)\end{aligned}$$

where the inequality is by IC(2): $y_{n-i}^{(n-j)} \geqslant y_{n-i+1}^{(n-j+1)}$ in $T_2(H)$. Parts of the consecutive terms in the right hand side cancel to give

$$\begin{aligned}\varepsilon_j^{(i)}(T_1^*(H)) &\leqslant \left((y_{n-i}^{(n)} - y_{n-i}^{(n-j+1)}) - (y_{n-i+1}^{(n)} - y_{n-i+1}^{(n-j+1)})\right) \\ &\quad + \left(y_{n-i}^{(n-j+1)} - y_{n-i+1}^{(n-j+1)}\right) \\ &= \left(y_{n-i}^{(n)} - y_{n-i+1}^{(n)}\right) = \nu_{n-i} - \nu_{n-i+1} = \nu_i^* - \nu_{i+1}^*.\end{aligned}$$

So the interlacing condition IC(2) for $T_2(H)$ is equivalent to

$$\varepsilon_j^{(i)}(T_1^*(H)) \leqslant \nu_i^* - \nu_{i+1}^*$$

as required. The case $j = 1$ can be shown similarly for all i. □

3.5. Suppose we have a hive H. From Lemma 2, the weights of $T_1^*(H)$ and $T_2(H)$ are $\lambda^* - \nu^*$ and $\lambda - \mu$, respectively. Theorem 1 states that H is a hive if and only if $T_1(H)$ and $T_2(H)$, and hence $T_1^*(H)$ and $T_2(H)$, satisfy both $IC(1)$ and $IC(2)$. Therefore since H is a hive, Proposition 2 and

Proposition 4 imply $T_1^*(H)$ and $T_2(H)$ satisfy the exponent conditions, and consequently they are GZ schemes in $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$ and $GZ(\nu, \lambda - \mu, \mu)$, respectively.

Conversely, if $T_1^*(H)$ is a GZ scheme from $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$ it satisfies IC(1), IC(2), and the exponent condition, thus from Propositions 2 – 4, $T_2(H)$ is a GZ scheme. In particular, $T_1(H)$ and $T_2(H)$ are GT patterns, meaning H is a hive by Theorem 1. Similarly, if $T_2(H) \in GZ(\nu, \lambda - \mu, \mu)$, then H is a hive. This proves Theorem 2.

4. LR tableaux and GT patterns I

In this section we introduce a bijection between LR tableaux and GZ schemes (Theorem 3). In proving this we will see that the semistandard and Yamanouchi conditions for tableaux are equivalent to, respectively, the interlacing and exponent conditions for t-arrays.

As an interesting consequence we then combine Theorem 3 with Theorem 2 (1) to arrive at a correspondence between LR tableaux and hives (Corollary 1). It turns out that Corollary 1 is equivalent to [18, (3.3)], so we compare the two constructions. The main difference is that our method has an intermediate GZ scheme, which is an artefact of composing Theorem 3 and Theorem 2. To conclude the section we summarise how the conditions on LR tableaux (semistandard and Yamanouchi conditions), GZ schemes (semistandard and exponent conditions) and hives (the rhombus conditions) are translated by the bijections.

The reader may find relevant results and further developments in, for example, [1–4, 6, 18, 19, 25].

4.1. A well-known bijection between semistandard tableaux and GT patterns. Our bijection between LR tableaux and GZ schemes is an extension of a well-known bijection between semistandard tableaux and GT patterns, seen in, for example, [11]. We now review this bijection and state it in the form most useful for our purposes. For this we require some relevant notation.

A non-skew semistandard tableau Y is uniquely determined by its associated matrix $(a_{i,j}(Y))$ where

$$a_{i,j}(Y) = \text{ the number of } i\text{'s in the } j\text{th row} \tag{5}$$

for all $1 \leqslant i, j \leqslant n$. Note that $a_{i,j}(Y) = 0$ for $i < j$. We also note that $\sum_{k=1}^{n} a_{k,j}(Y)$ for $1 \leqslant j \leqslant n$ give the shape of the tableau Y, and $\sum_{k=1}^{n} a_{i,k}(Y)$ for $1 \leqslant i \leqslant n$ give the content of Y. The reader is warned that these are *not* the same a_{ij} as those in [18, (3.4)]. Those label hive entries, not content of a tableau.

We remark that if Y is a semistandard tableau on the skew shape λ/μ, then the $a_{i,j}(Y)$'s are well defined, and the $a_{i,j}(Y)$'s with λ or μ uniquely define Y. It is possible to develop the theory of tableaux exclusively in terms of their associated matrices. See [6] for this direction.

Now consider a semistandard Young tableau. Removing all instances of the largest entry simultaneously yields a tableau with a new shape. Repeating this process, we would achieve a list of successively shrinking shapes, which written downwards would form the rows of a GT pattern. This process is a bijection. See Example 1.

It is easy to symbolically describe the inverse of the bijection. Given a GT pattern $T = (t_j^{(i)})$ of type λ with non-negative entries, it creates a semistandard Young tableau Y_T of shape λ whose entries are elements of $\{1, 2, \ldots, n\}$ and defined by

$$a_{i,j}(Y_T) = t_j^{(i)} - t_j^{(i-1)} \tag{6}$$

for $1 \leqslant i, j \leqslant n$ with the conventions

$$t_j^{(i)} = 0 \quad \text{for} \quad j > i \geqslant 0.$$

Manipulating (6), it follows that the bijection takes a semistandard tableau Y and creates a GT pattern $T_Y = (t_j^{(i)})$ according to the rule

$$t_j^{(i)} = \sum_{k=1}^{i} a_{k,j}(Y) \tag{7}$$

for $1 \leqslant j \leqslant i \leqslant n$. Since $a_{k,j}(Y) = 0$ for $k < j$ in every non-skew semistandard tableau Y, we can in fact write this as

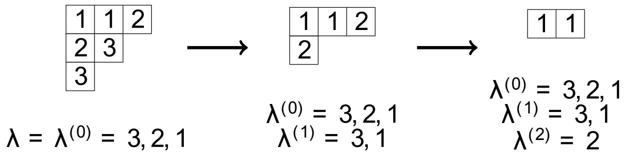
$$t_j^{(i)} = \sum_{k=j}^{i} a_{k,j}(Y). \tag{8}$$

See also, for example, [14, §8.1.2] or [20] for further background on this bijection.

Example 1. As an example we apply the bijection to the tableau

1	1	2
2	3	
3		

and list the successive shapes $\lambda^{(i)}$ as they are created.



Clearly, the shapes form a GT pattern. It is straightforward to check that the expressions (8) and (6) both hold.

Under this bijection, the content of the tableau is equal to the weight of the t-array. We also note that in this bijection, the semistandard condition on the tableau is implied by the interlacing conditions on the t-array and vice versa (cf. Remark 2).

4.2. A well-known bijection between semistandard skew tableaux and truncated GT patterns. The bijection of §4.1 can be extended to act on skew tableaux. Again, this is a well known result included in [11], [12] and [3], among others.

Lemma 3. *There is a bijection between the set of skew semistandard Young tableaux of shape λ/μ with entries from $\{1, 2, \ldots, n\}$ and the set of GT patterns for GL_{2n} whose type is $\lambda' = (\lambda_1, \ldots, \lambda_n, 0, \cdots, 0) \in \mathbb{Z}^{2n}$ and whose kth row is $(\mu_1, \mu_2, \ldots, \mu_k)$ for $1 \leqslant k \leqslant n$.*

Proof. For a given semistandard Young tableau Y of shape λ/μ, replace the i entries with $(n+i)$'s for $1 \leqslant i \leqslant n$, then fill in the empty boxes in the ℓth row of Y with ℓ's for $1 \leqslant \ell \leqslant n$. Then this process uniquely determines a non-skew semistandard Young tableau of shape λ with entries from $\{1, 2, \ldots, 2n\}$, and under the bijection given by (6), its corresponding GT pattern for GL_{2n} is the one described in the statement. □

The first half of Example 2 shows Lemma 3 applied to a skew tableau. We remark that the GT pattern for GL_n whose kth row is $(\mu_1, \mu_2, \ldots, \mu_k)$ for $1 \leqslant k \leqslant n$ corresponds to the highest weight vector of the representation V_n^μ labelled by a Young diagram μ. In fact, the GT patterns described in Lemma 3 encode the weight vectors of $V_{2n}^{\lambda'}$, which are the highest weight vector for V_n^μ under the branching of GL_{2n} down to GL_n.

The bottom $n-1$ rows of a GT pattern described by Lemma 3 hold redundant information because they are determined by μ. It is therefore convention to omit them and achieve what is called a *truncated* GT pattern. It is also common to omit the upper-right portion of this pattern, since the interlacing conditions force those entries to be zero. For example, the first half of the bijection described by [18, (3.3)] uses Lemma 3 with these conventions.

There is an excellent example of Lemma 3 and further explanation in [1, §2].

4.3. Symbolic forms of the semistandard and Yamanouchi conditions. We are almost ready to use Lemma 3 to establish the bijection between LR tableaux and GZ schemes. However, we first need symbolic forms of both the semistandard and Yamanouchi conditions.

Let us express the semistandard condition for a tableau Y in terms of the $a_{i,j}(Y)$ defined in (5). By rearranging the entries in each row if necessary, we can always make the entries of Y weakly increasing along each row from left to right. The strictly increasing condition on the columns of Y can then be rephrased as follows: the number of entries up to ℓ in the $(m+1)$th row is not bigger than the number of entries up to $(\ell-1)$ in the mth row, i.e.,

$$\sum_{k=1}^{\ell-1} a_{k,m}(Y) \geqslant \sum_{k=1}^{\ell} a_{k,m+1}(Y) \tag{9}$$

for $1 \leqslant \ell \leqslant n$ and $1 \leqslant m < n$. Here, if $\ell = 1$, then the left hand side is 0 as an empty sum and the inequality implies that $a_{1,m+1}(Y) = 0$ for $m \geqslant 1$. Inductively, we can obtain $a_{i,m+1}(Y) = 0$ for $m \geqslant i$ from the inequality with $\ell = i$. This shows that for a semistandard Young tableau Y, $a_{i,j}(Y) = 0$ for $j > i$, as we noted after (5).

Remark 2. By using the conversion formula (7), one can directly compute that IC(2) on a GT pattern T is equivalent to the semistandard condition (9) in Y_T corresponding to T. On the other hand, IC(1) in T is equivalent to a rather trivial condition $a_{i,j}(Y_T) \geqslant 0$ for all i, j.

If Y is a skew tableau of shape λ/μ, then, using the same argument as for (9), it is straightforward to see that we can make Y semistandard by rearranging elements along each row if and only if

$$\mu_{m+1} + \sum_{k=1}^{\ell} a_{k,m+1}(Y) \leqslant \mu_m + \sum_{k=1}^{\ell-1} a_{k,m}(Y) \tag{10}$$

for $1 \leqslant \ell \leqslant n$ and $1 \leqslant m < n$. The Yamanouchi condition in a LR tableau Y can be expressed as

$$\sum_{k=1}^{j} a_{i+1,k}(Y) \leqslant \sum_{k=1}^{j-1} a_{i,k}(Y) \tag{11}$$

for $1 \leqslant j \leqslant n$ and $1 \leqslant i < n$. Here, if $j = 1$, then the right hand side is 0 as an empty sum and the inequality implies that $a_{i+1,1}(Y) = 0$ for $i \geqslant 1$. Inductively, we can obtain $a_{i+1,\ell}(Y) = 0$ for $i \geqslant \ell$ from the inequality with $j = \ell$. This shows that for an LR tableau Y, $a_{i,j}(Y) = 0$ for $i > j$, as we noted in Remark 1 (2).

4.4. Bijection between LR tableaux and GZ schemes. We now establish a bijection between LR tableaux and GZ schemes using Lemma 3. After applying the lemma to an LR tableau, a center section of the resulting GT pattern is removed. Taking the dual of the removed array we get the desired GZ scheme. In doing this we observe how the conditions on the tableau become those of the scheme.

For a specific example of this bijection, see the first half of Example 2.

Theorem 3. *There is a bijection ϕ between $LR(\lambda/\mu, \nu)$ and $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$. In particular, the semistandard and Yamanouchi conditions in $L \in LR(\lambda/\mu, \nu)$ are equivalent to, respectively, the interlacing and exponent conditions in $\phi(L) \in GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$.*

Proof. Let $L \in LR(\lambda/\mu, \nu)$ be given. By applying Lemma 3 we find its corresponding GT pattern $T = (t_j^{(i)})$ for GL_{2n} and remove the bottom $n-1$ rows to achieve a truncated GT pattern of $n+1$ rows. Furthermore, the truncated pattern for L can be divided into three subtriangular arrays T_X, T_Y and T_Z, as in Figure 2. Note that these are the same size.

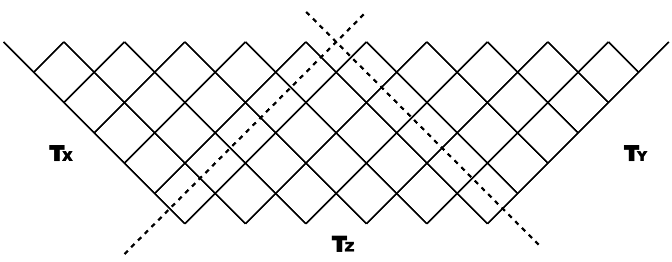


Figure 2. Dividing a truncated GT pattern into 3 subpatterns.

The upper left subarray T_X is completely determined by λ because of the Yamanouchi condition (see Remark 1 (2)). The upper right subarray T_Y contains only zeroes. Therefore, given fixed λ, μ, and ν, the LR tableau $L \in LR(\lambda/\mu, \nu)$ is uniquely determined by T_Z. We want to show that the dual array T_Z^* of T_Z is an element of $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$, and from that establish a bijection

$$\begin{array}{rcl} LR(\lambda/\mu,\nu) & \longrightarrow & GZ(\mu^*,\lambda^*-\nu^*,\nu^*) \\ L & \longmapsto & T_Z^* \end{array} .$$

Let us rewrite the middle subarray T_Z as follows by reflecting it over a horizontal line.

$$T_Z = \begin{array}{ccccccccc} t_1^{(n)} & & t_2^{(n)} & & \cdots & & t_{n-1}^{(n)} & & t_n^{(n)} \\ & t_2^{(n+1)} & & t_3^{(n+1)} & & \cdots & & t_n^{(n+1)} & \\ & & t_3^{(n+2)} & & & & t_n^{(n+2)} & & \\ & & & \ddots & & \iddots & & & \\ & & & & t_n^{(2n-1)} & & & & \end{array}$$

Then $\mu_i = t_i^{(n)}$ for $1 \leqslant i \leqslant n$ and T_Z satisfies the interlacing conditions induced from the truncated GT pattern T, which are assured by the semistandardness of L. Therefore T_Z is a GT pattern of type μ. From the fact that the weights of T_X, T_Y, and T are $(\lambda_1,\ldots,\lambda_n)$, $(0,\ldots,0)$, and $(\mu_1,\ldots,\mu_n,\nu_1,\ldots,\nu_n)$ respectively, it is easy to show that the weight of T_Z is $\nu^*-\lambda^*$. Hence its dual T_Z^* is a GT pattern (see §3.4) of type μ^* and weight $\lambda^*-\nu^*$. Next, we want to show that T_Z^* satisfies the exponent conditions.

Let $a_{i,j} = a_{i,j}(L)$, i.e., be the number of i's in the jth row of L for all i and j. Then

$$a_{i,j} = t_j^{(n+i)} - t_j^{(n+i-1)} \quad \text{and } a_{k,k} = \lambda_k - t_k^{(n+k-1)} \tag{12}$$

for $1 \leqslant i < j \leqslant n$ and $1 \leqslant k \leqslant n$. Since the content of L is ν with $\nu_q = \sum_{k=1}^n a_{q,k}$ for $1 \leqslant q \leqslant n$, we can write

$$(-\nu_{i+1}) - (-\nu_i) = \sum_{k=1}^{n} (a_{i,k} - a_{i+1,k}) \tag{13}$$

for $1 \leqslant i < n$.

On the other hand, from the Yamanouchi condition (11) in L, we have

$$\sum_{k=1}^{j} a_{i+1,k} \leqslant \sum_{k=1}^{j-1} a_{i,k} \quad \text{or equivalently,} \quad a_{i+1,j} \leqslant \sum_{k=1}^{j-1} (a_{i,k} - a_{i+1,k}).$$

Then, using this inequality, (13) becomes

$$(-\nu_{i+1}) - (-\nu_i) \geqslant \sum_{k=j+1}^{n} (a_{i,k} - a_{i+1,k}) + a_{i,j}$$

and the right hand side is, via (12), the exponents of T_Z^*. Therefore, $T_Z^* \in GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$. $\square$

4.5. A bijection between LR tableaux and hives. Composing Theorem 3 and Theorem 2 (1) gives a bijection between the set of LR tableaux and the set of hives.

$$\begin{tikzcd} & GZ(\mu^*, \lambda^* - \nu^*, \nu^*) \arrow[dl, shift left] \arrow[dr, shift left] & \\ LR(\lambda/\mu, \nu) \arrow[ur, shift left] & & \mathcal{H}^\circ(\mu, \nu, \lambda) \arrow[ul, shift left] \end{tikzcd}$$

Corollary 1. *[18, (3.3)] There is a bijection between $LR(\lambda/\mu, \nu)$ and $\mathcal{H}^\circ(\mu, \nu, \lambda)$.*

Proof. For $L \in LR(\lambda/\mu, \nu)$, we compute the corresponding truncated GT pattern and its middle subarray T_Z. Then, by Theorem 3, its dual T_Z^* belongs to $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$. Similarly, for $H \in \mathcal{H}^\circ(\mu, \nu, \lambda)$, its first derived subarray $T_1(H)$ satisfies $T_1^*(H) \in GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$ by Theorem 2. We can therefore identify a H such that $T_1(H) = T_Z$ and this gives us a bijection from $LR(\lambda/\mu, \nu)$ to $\mathcal{H}^\circ(\mu, \nu, \lambda)$. $\square$

We give an example of Corollary 1 below.

Example 2. We start by using Theorem 3 to map the LR tableau below to a GT pattern (whose dual array is a GZ scheme).

Let $\lambda = (11, 7, 5, 3)$, $\mu = (5, 3, 1, 0)$ and $\nu = (7, 5, 3, 2)$. The LR tableau from $LR(\lambda/\mu, \nu)$

					1	1	1	1	1	1
			1	2	2	2				
	2	3	3	3						
2	4	4								

considered as an object for GL_4, corresponds to the following truncated GT pattern.

$$\begin{array}{ccccccccccccccc}
11 & & 7 & & 5 & & 3 & & 0 & & 0 & & 0 & & 0 \\
& 11 & & 7 & & 5 & & 1 & & 0 & & 0 & & 0 & \\
& & 11 & & 7 & & 2 & & 1 & & 0 & & 0 & & \\
& & & 11 & & 4 & & 1 & & 0 & & 0 & & & \\
& & & & 5 & & 3 & & 1 & & 0 & & & &
\end{array}$$

Taking out the middle section, we find T_Z is

$$\begin{array}{ccccccc} 5 & & 3 & & 1 & & 0 \\ & 4 & & 1 & & 0 & \\ & & 2 & & 1 & & \\ & & & 1 & & & \end{array}$$

with a dual array T_Z^* belonging to $GZ(\mu^*, \lambda^* - \nu^*, \nu^*)$.

For the second half of the process, we apply the bijection between hives and GZ schemes (Theorem 2 (1)) to T_Z^*. We know the corresponding hive H will have boundaries given by $\mu = (5, 3, 1, 0), \nu = (7, 5, 3, 2)$ and $\lambda = (11, 7, 5, 3)$ so that it appears as follows

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 5 & & 11 & & & \\ & & 8 & & p & & 18 & & \\ & 9 & & q & & r & & 23 & \\ 9 & & 16 & & 21 & & 24 & & 26 \end{array}$$

with some inner entries p, q and r. Adding $(T_Z^*)^* = T_Z$ along the NE-SW diagonals as if it were $T_1(H)$ we find $p = 15$, $q = 16$ and $r = 20$.

Of course, there are many known bijections between LR tableaux and hives. For example, in the appendix of [5] Fulton gave a bijection between LR tableaux and hives using *contratableaux*. It is interesting to note that his first step is to construct partitions from the hive that are equivalent to the derived t-array T_1. However, that approach diverges from ours once he uses the partitions to form a contratableau.

Our Corollary 1 is simpler than most other bijections between LR tableaux and hives, such as the one by Fulton, but is in fact equivalent to [18, (3.3)]. There, the authors also take an LR tableau and compute the truncated GT pattern via Lemma 3. They then take row sums in the pattern and separate out a bottom section, which becomes the hive. This is simply our process in reverse, since we separate a section of the truncated GT pattern in Theorem 3 by removing T_Z, and then successively add those entries to the boundary of the hive in Theorem 2 (1).

The key difference, however, is that here we establish GZ schemes as an intermediate object in the bijection, which is absent from the simple and elegant treatment in [18]. This provides background as to why their simple bijection works, and also allows us to track the conditions as they move between objects (see tables 1 and 2). We also note that Corollary

1 is not the main focus of our discussion. Rather, it is an interesting consequence that appears when piecing together two sets of combinatorial theory – the derived t-arrays of hives on one hand, and the classical bijection between tableaux and t-arrays on the other.

To complete the section we combine the two approaches for some insights. Though not clear from our presentation, the elegant formula [18, (3.4)] states that the elements of the hive $H = (h_{l,m})$ are given by

$$h_{l,m} = \begin{array}{l}\text{the number of empty boxes and entries} \leqslant l \\ \text{in the first } m \text{ rows of the LR tableau.}\end{array}$$

Finally, in proving [18, Proposition 3.2], King *et al.* show that, under their bijection, conditions[1] on hives correspond to conditions on LR tableaux. Table 1 summarises these equivalences.

Hive	**LR tableau**
RC(1)	trivial
RC(2)	Semistandard condition
RC(3)	Yamanouchi

TABLE 1. Equivalent LR tableau and hive conditions in [18, (3.3)]

Using our results from Proposition 1, Theorem 2, Remark 2 and Theorem 3 we are able to add a middle column showing the equivalent conditions in the intermediate GZ scheme object. See Table 2.

Hive	**GZ scheme**	**LR tableau**
RC(1)	IC(1)	trivial
RC(2)	IC(2)	Semistandard condition
RC(3)	Exponents	Yamanouchi

TABLE 2. Equivalent LR tableau, GZ scheme and hive conditions in Corollary 1

5. LR Tableaux and GT Patterns II

In this section, we show that the semistandard and Yamanouchi conditions for tableaux are equivalent to, respectively, the exponent and semistandard conditions for their *companion tableaux*. This correspondence

[1]The conditions RC(1), RC(2) and RC(3) are referred to as R2, R3 and R1 respectively by [18].

is obtained by taking the transpose of matrices describing tableaux. As a result, we show that the companion tableaux of LR tableaux are GZ schemes under the tableau-pattern bijection.

5.1. For a (skew) semistandard tableau Y, as in (5), we let $a_{i,j}(Y)$ denote the number of i's in the jth row.

Definition 5. For a (skew) semistandard tableau Y, its *companion tableau* Y^c is defined as a non-skew tableau whose entries are weakly increasing along each row and whose number of i's in the jth row is equal to $a_{j,i}(Y)$; that is, for $1 \leqslant i, j \leqslant n$,

$$a_{i,j}(Y^c) = a_{j,i}(Y). \tag{14}$$

Example 3. For the LR tableau Y from Example 2, the associated matrix is

$$a_{i,j}(Y) = \begin{bmatrix} 6 & 1 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Then, from its transpose, we have the following companion tableau Y^c.

1	1	1	1	1	1	2
2	2	2	3	4		
3	3	3				
4	4					

Note that Y is of shape $(11, 7, 5, 3)/(5, 3, 1, 0)$ and content $(7, 5, 3, 2)$ while its companion tableau Y^c is of shape $(7, 5, 3, 2)$ and content $(6, 4, 4, 3)$, which is $(11, 7, 5, 3) - (5, 3, 1, 0)$. The GT pattern T_{Y^c} corresponding to Y^c is

$$\begin{array}{ccccccc} 7 & & 5 & & 3 & & 2 \\ & 7 & & 4 & & 3 & \\ & & 7 & & 3 & & \\ & & & 6 & & & \end{array}.$$

We want to show that this correspondence $Y \mapsto T_{Y^c}$ gives another bijection from the set of LR tableaux to the set of GZ schemes.

In [29], van Leeuwen replaced the Yamanouchi condition in LR tableaux with the semistandard condition in their companion tableaux. Here, we show that the semistandard condition in LR tableaux has a counterpart in the companion tableaux as well, and then we identify the companion tableaux as an independent object equivalent to GZ schemes.

Theorem 4. *For a LR tableau Y, we let Y^c denote its companion tableau and let T_{Y^c} denote the GT pattern corresponding to Y^c. The map $\psi(Y) = T_{Y^c}$ gives a bijection from $LR(\lambda/\mu, \nu)$ to $GZ(\nu, \lambda-\mu, \mu)$. In particular, the Yamanouchi and semistandard conditions in Y are equivalent to, respectively, the interlacing condition IC(2) and the exponent condition in T_{Y^c}.*

Proof. From (14), Y is a tableau of shape λ/μ if and only if the content of Y^c is equal to $\lambda-\mu$. The content of Y is equal to the shape of Y^c. The type and weight of T_{Y^c} are therefore ν and $\lambda-\mu$, respectively.

Recall the Yamanouchi condition in Y (11): for $0 \leqslant i < n$ and $1 \leqslant j < n$,

$$\sum_{k=1}^{i} a_{j,k}(Y) \geqslant \sum_{k=1}^{i+1} a_{j+1,k}(Y). \tag{15}$$

Since $a_{i,j}(Y) = a_{j,i}(Y^c)$ for all i and j, this inequality, in terms of the entries in Y^c, is saying that the number of entries less than or equal to $i+1$ in the $(j+1)$th row is not more than the number of entries less than or equal to i in the jth row. It is the semistandard condition for Y^c and therefore the interlacing condition for T_{Y^c}.

To show this, consider expressing the elements of the GT pattern $T_{Y^c} = (t_j^{(i)})$ in terms of $a_{i,j}(Y^c)$. From the standard bijection between semistandard tableaux and GT patterns, (7), we have

$$t_j^{(i)} = \sum_{k=1}^{i} a_{k,j}(Y^c)$$

where $a_{i,j}(Y^c)$ is the number of i entries in the jth row of Y^c.

Consider the interlacing condition IC(2): $t_j^{(i)} \geqslant t_{j+1}^{(i+1)}$ where $0 \leqslant i < n$ and $1 \leqslant j < n$. Writing this with the above relation gives

$$\sum_{k=1}^{i} a_{k,j}(Y^c) \geqslant \sum_{k=1}^{i+1} a_{k,j+1}(Y^c) \quad \Leftrightarrow \quad \sum_{k=1}^{i} a_{j,k}(Y) \geqslant \sum_{k=1}^{i+1} a_{j+1,k}(Y)$$

which is exactly the expression for the Yamanouchi condition (15). It can be similarly shown that, as mentioned in Remark 2, IC(1) is equivalent to $a_{i,j}(Y) \geqslant 0$.

Using (10), the semistandard condition for Y says we have, for all $1 \leqslant \ell \leqslant n$ and $1 \leqslant m < n$,

$$\left(\sum_{k=1}^{\ell} a_{k,m+1}(Y) - \sum_{k=1}^{\ell-1} a_{k,m}(Y) \right) \leqslant (\mu_m - \mu_{m+1}) \tag{16}$$

or

$$\sum_{k=1}^{\ell-1}\left(a_{m+1,k}(Y^c)-a_{m,k}(Y^c)\right)+a_{m+1,\ell}(Y^c)\leqslant(\mu_m-\mu_{m+1}).$$

To finish our proof, it is enough to show that the left hand side of the above inequality is the exponent $\varepsilon_\ell^{(m)}(T_{Y^c})$. This can be easily seen, by using (8), as

$$\begin{aligned}\varepsilon_\ell^{(m)}(T_{Y^c}) &= \sum_{1\leqslant h<\ell}\left(t_h^{(m+1)}-2t_h^{(m)}+t_h^{(m-1)}\right)+\left(t_\ell^{(m+1)}-t_\ell^{(m)}\right)\\ &= \sum_{1\leqslant h<\ell}\left(\sum_{k=h}^{m+1}a_{k,h}(Y^c)-2\sum_{k=h}^{m}a_{k,h}(Y^c)+\sum_{k=h}^{m-1}a_{k,h}(Y^c)\right)\\ &\quad+\left(\sum_{k=\ell}^{m+1}a_{k,\ell}(Y^c)-\sum_{k=\ell}^{m}a_{k,\ell}(Y^c)\right)\\ &= \sum_{1\leqslant k<\ell}\left(a_{m+1,k}(Y^c)-a_{m,k}(Y^c)\right)+a_{m+1,\ell}(Y^c). \qquad\square\end{aligned}$$

We now have an alternative proof of Corollary 1.

Corollary 2. *There is a bijection between $LR(\lambda/\mu,\nu)$ and $\mathcal{H}^\circ(\mu,\nu,\lambda)$.*

Proof. We can map any $Y\in LR(\lambda/\mu,\nu)$ to $T_{Y^c}\in GZ(\nu,\lambda-\mu,\mu)$ via the bijection in Theorem 4. From Theorem 2 there is a bijection between $\mathcal{H}^\circ(\mu,\nu,\lambda)$ and $GZ(\nu,\lambda-\mu,\mu)$ through the derived t-array T_2 of a hive. The composition of the first bijection with the inverse of the second then gives a bijection which assigns $Y\in LR(\lambda/\mu,\nu)$ to $H\in\mathcal{H}^\circ(\mu,\nu,\lambda)$ if and only if $T_2(H)=T_{Y^c}$. $\square$

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