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On the inclusion ideal graph of a poset

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Abstract. Let $(P, \leqslant)$ be an atomic partially ordered set (poset, briefly) with a minimum element 0 and $\mathcal{I}(P)$ the set of nontrivial ideals of P. The inclusion ideal graph of P, denoted by $\Omega(P)$, is an undirected and simple graph with the vertex set $\mathcal{I}(P)$ and two distinct vertices $I, J \in \mathcal{I}(P)$ are adjacent in $\Omega(P)$ if and only if $I \subset J$ or $J \subset I$. We study some connections between the graph theoretic properties of this graph and some algebraic properties of a poset. We prove that $\Omega(P)$ is not connected if and only if $P = \{0, a_1, a_2\}$, where a_1, a_2 are two atoms. Moreover, it is shown that if $\Omega(P)$ is connected, then $\mathrm{diam}(\Omega(P)) \leqslant 3$. Also, we show that if $\Omega(P)$ contains a cycle, then $\mathrm{girth}(\Omega(P)) \in \{3, 6\}$. Furthermore, all posets based on their diameters and girths of inclusion ideal graphs are characterized. Among other results, all posets whose inclusion ideal graphs are path, cycle and star are characterized.

1. Introduction

Recently, a major part of research in algebraic combinatorics has been devoted to the application of graph theory and combinatorics in abstract algebra. There are a lot of papers which apply combinatorial methods to obtain algebraic results in poset theory (for example see [3, 5, 6] and [7]).

Let $(P, \leqslant)$ be a poset with a least element 0. An element $m \in P$ is called *maximal*, if for every element $p \in P$ the relation $m \leqslant p$ implies

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that $m = p$. We denote the set of maximal elements of P by $Max(P)$. A *minimal element* of P is defined in a dual manner. We say x, y are *comparable* in P if either $x < y$ or $y < x$. A nonempty subset S of P is a *chain* in P, if every two elements of S are comparable. A finite chain with n elements can be written in the form $c_1 < c_2 < \cdots < c_n$. Such a chain said to have *length* $n - 1$. If $a < b$, then a chain from a to b in P is a chain in P whose minimum element is a and whose maximum element is b. An *order ideal* or a lower set (an *order filter* or an upper set) of P is a non-empty subset $I \subseteq P$ ($F \subseteq P$) such that for every $x, y \in P$, the relations $x \in I$ and $y \leqslant x$ imply that $y \in I$ (for every $x, y \in P$, the relations $x \in F$ and $x \leqslant y$ imply that $y \in F$). The set of all ideals of P is denoted by $\mathcal{J}(P)$ and $\mathcal{I}(P) = \mathcal{J}(P) \setminus \{\{0\}, P\}$. For every element x of P, the ideal $(x] := \{y \in P | y \leqslant x\}$(the filter $[x) := \{y \in P | x \leqslant y\}$) is called *principal* whose generator is x. Let $x, y \in P$. We say that y *covers* x in P if $x < y$ and no element in P lies strictly between x and y. If y covers x, then we write $x \sqsubset y$. Any cover of 0 in P is called an *atom*. The set of all atoms of P is denoted by $\text{Atom}(P)$. The poset P is called *atomic*, if for every non-zero element $x \in P$, we have $(x] \cap \text{Atom}(P) \neq \varnothing$. The set of all principal ideal generated by $\text{Atom}(P)$ and $P \setminus \text{Atom}(P)$ are denoted by $\mathcal{A}(P)$ and $\mathcal{B}(P)$, respectively. For any undefined notation or terminology in graph theory, we refer the reader to [8, 9].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. The *degree* of a vertex v in a graph G is the number of edges incident with v. The degree of a vertex v is denoted by $\deg(v)$. Let r be a non-negative integer. The graph G is said to be *r-regular*, if the degree of each vertex is r. The set of all adjacent vertices to $v \in V$ is denoted by $N(v)$. If u and v are two adjacent vertices of G, then we write $u — v$. We say that G is a *connected graph* if there is a path between each pair of distinct vertices of G. For two vertices x and y, let $d(x, y)$ denote their *distance*, that is, the length of the shortest path between x and y (we set $d(x, y) := \infty$ if there is no such path). The *diameter* of G is $\text{diam}(G) = sup\{d(x, y) \mid x$ and y are distinct vertices of $G\}$. The *girth* of G, denoted by $\text{girth}(G)$, is the length of the shortest cycle in G ($\text{girth}(G) = \infty$ if G has no cycles). A graph is called *complete*, if every pair of distinct vertices is joined by an edge. A complete graph of order n is denoted by K_n. A *bipartite* graph is one whose vertex-set can be partitioned into two (not necessarily nonempty) disjoint subsets in such a way that the two end vertices for each edge lie in distinct partitions. Among bipartite graphs, a *complete bipartite* graph is one in which each vertex is joined to every vertex that is not in the same partition. The complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$.

Graphs of the form $K_{1,n}$ are called *star graphs*. A graph G is k-partite if $V(G)$ can be expressed as the union of k (possibly empty) independent sets. A *cycle graph* is a graph that consists of a single cycle. For any undefined notation or terminology in graph theory, we refer the reader to [2, 10].

Throughout this paper P is a nontrivial atomic poset with a minimum element 0 and $\mathrm{Atom}(P) = \{a_1, a_2, \ldots, a_n, \ldots\}$ (at most countably infinite). The *inclusion ideal graph of* P, denoted by $\Omega(P)$, is an undirected and simple graph with the vertex set $\mathcal{I}(P)$ and two distinct vertices $I, J \in \mathcal{I}(P)$ are adjacent in $\Omega(P)$ if and only if $I \subset J$ or $J \subset I$. The idea of inclusion ideal graph associated with a ring was first introduced and studied in [1]. Motivated by [1], we define and study the inclusion ideal graph of a poset. In this paper, all posets whose inclusion ideal graphs are not connected are classified. Also, if $\Omega(P)$ is connected and contains a cycle, then upper bounds for the diameter and the girth of $\Omega(P)$ are given. Moreover, all posets based on their diameters and girths of inclusion ideal graphs are characterized.

2. Connectivity, diameter and girth of $\Omega(P)$

In this section, we study the connectivity, diameter and girth of an inclusion ideal graph associated with a poset. For instance, we characterize all posets whose inclusion ideal graphs are connected. Moreover, it is shown that the diameter of a connected inclusion ideal graph is at most three. Furthermore, we classify inclusion ideal graphs of posets in terms of their diameters and girthes.

We begin with the following lemma.

Lemma 2.1. Let $(P, \leqslant)$ be a poset. Then the following statements hold.

(i) $|V(\Omega(P))| = 0$ if and only if $|P| = 2$.

(ii) $|V(\Omega(P))| = 1$ if and only if P is a chain with $|P| = 3$.

Proof. (i) It is clear.

(ii) First suppose that $|V(\Omega_{UM}(P))| = 1$. If P is not a chain. Then P has at least two non-comparable elements say x, y. Obviously, $(x], (y] \in \mathcal{I}(P)$, which is impossible. Moreover, if $x, y \in P$ and $a_1 \sqsubset x$ and $x \sqsubset y$, then $\Omega_{UM}(P)$ has two vertices, a contradiction. Therefore $|P| = 3$.

The converse is clear. □

Two following lemmas have straightforward proofs and so their proofs are omitted.

Lemma 2.2. Let $(P, \leqslant)$ be a poset. Then the following statements hold.

(i) If P is a chain, then $|\mathcal{J}(P)| = |P|$.

(ii) For every $I \in \mathcal{I}(P)$, there exists $J \in \mathcal{A}(P)$ such that $J \subset I$.

(iii) Let $x, y \in P$. If $x < y$, then $(x] \subset (y]$.

Lemma 2.3. Let $(P, \leqslant)$ be a poset and I, J be two ideals of P. Then $I \cap J$ and $I \cup J$ are ideals of P.

By using Lemma 2.3, we characterize all posets whose inclusion ideal graphs are not connected.

Theorem 2.4. Let $(P, \leqslant)$ be a poset. Then $\Omega(P)$ is not connected if and only if $P = \{0, a_1, a_2\}$.

Proof. First, suppose that $\Omega(P)$ is not connected and $C_1, \ldots, C_k$ are components of $\Omega(P)$. Since $\Omega(P)$ is not connected, we deduce that $k \geqslant 2$. Let $I_i \in C_i$ and $I_j \in C_j$, for some $1 \leqslant i, j \leqslant k$. Obviously, I_i and I_j are not comparable. If $I_i \cap I_j \neq 0$, then by Lemma 2.3, one may find the path $I_i - I_i \cap I_j - I_j$, which is impossible. Thus $I_i \cap I_j = 0$. Therefore, there exist atoms $a_i \in I_i \setminus I_j$ and $a_j \in I_j \setminus I_i$. Consider the minimal ideals $I = \{0, a_i\}$ and $J = \{0, a_j\}$. It is clear that $I \in C_i$ and $J \in C_j$. If $I \cup J \neq P$, then by Lemma 2.3, one may find the path $I - I \cup J - J$ in $\Omega(P)$ and this contradicts the hypothesis. Hence $I \cup J = P = \{0, a_i, a_j\}$. This means that $k = 2$, as desired.

Conversely, suppose that $P = \{0, a_1, a_2\}$. Let $I_1 = \{0, a_1\}$ and $I_2 = \{0, a_2\}$. Then $\mathcal{I}(P) = \{I_1, I_2\}$. Therefore, two isolated vertices I_1, I_2 form $\Omega(P)$ and so the proof is complete. □

The next theorem states that the diameter of a connected inclusion ideal graph associated with a poset does not exceed three.

Theorem 2.5. If $\Omega(P)$ is connected, then $\operatorname{diam}(\Omega(P)) \leqslant 3$.

Proof. Let I_i and I_j be two non-adjacent vertices of $\Omega(P)$. If $I_i \cap I_j \neq 0$, then by Lemma 2.3, we have the path $I_i - I_i \cap I_j - I_j$. Hence suppose that $I_i \cap I_j = 0$. Therefore, there exist atoms $a_i \in I_i \setminus I_j$ and $a_j \in I_j \setminus I_i$. Since $\Omega(P)$ is connected, we deduce from Theorem 2.4 that $\{0, a_i, a_j\}$ is a vertex of $\Omega(P)$. Now, we continue the proof in the following cases:

Case 1. If $I_i = \{0, a_i\}$ and $I_j \neq \{0, a_j\}$, then consider the path $I_i - \{0, a_i, a_j\} - \{0, a_j\} - I_j$. The case $I_i \neq \{0, a_i\}$ and $I_j = \{0, a_j\}$ is similar.

Case 2. If $I_i = \{0, a_i\}$ and $I_j = \{0, a_j\}$, then we have the path $I_i - \{0, a_i, a_j\} - I_j$.

Case 3. Let $I_i \neq \{0, a_i\}$ and $I_j \neq \{0, a_j\}$. If $\{0, a_i\} \cup I_j \neq P$, then we have the path $I_i - \{0, a_i\} - \{0, a_i\} \cup I_j - I_j$. Hence, we may suppose that $\{0, a_i\} \cup I_j = P$. We claim that $\{0, a_j\} \cup I_i \neq P$. Assume to the contrary, $\{0, a_j\} \cup I_i = P$. Thus

$$0 = I_i \cap I_j = (P \setminus \{a_j\}) \cap (P \setminus \{a_i\}).$$

Thus $P = \{0, a_i, a_j\}$. By Theorem 2.4, $\Omega(P)$ is not connected, a contradiction and so the claim is proved. Therefore, I_i links to I_j through the path $I_i - \{0, a_j\} \cup I_i - \{0, a_j\} - I_j$.

The proof now is complete. □

To characterize inclusion ideal graphs in terms of their diameters, the following lemma is needed.

Lemma 2.6. Let $(P, \leqslant)$ be a poset and $I, J \in \mathcal{I}(P)$. Then the following statements hold:

(i) $d(I, J) = 1$ if and only if I and J are comparable.

(ii) $d(I, J) = 2$ if and only if I and J are not comparable and either $I \cap J \neq 0$ or $I \cup J \neq P$.

(iii) $d(I, J) \geqslant 3$ if and only if $I \cap J = 0$ and $I \cup J = P$.

Proof. (i) It is clear.

(ii) Let $d(I, J) = 2$. Then there exists $K \in \mathcal{I}(P)$ such that $d(I, K) = d(K, J) = 1$. It is obvious that I and J are not comparable. We show that either $I \cap J \neq 0$ or $I \cup J \neq P$. If $K \subset I$ and $K \subset J$, then $I \cap J \neq 0$. If $I \subset K$ and $J \subset K$, then $I \cup J \subseteq K$ and so $I \cup J \neq P$. Note that the case $I \subset K \subset J$ does not occur. The converse is clear.

(iii) It follows from parts (i) and (ii). □

Theorem 2.7. Let $(P, \leqslant)$ be a poset such that $\Omega(P)$ be connected. Then the following statements hold:

(i) $\operatorname{diam}(\Omega(P)) = 1$ if and only if P is a chain.

(ii) $\operatorname{diam}(\Omega(P)) = 2$ if and only if $|P| \geqslant 4$ and one of the following conditions holds:

(a) $|\operatorname{Atom}(P)| = 1$ and P is not a chain.

(b) $|\operatorname{Atom}(P)| \geqslant 2$ and $[a_i) \cap [a_j) \neq \varnothing$, for every $a_i, a_j \in \operatorname{Atom}(P)$.

(iii) $\operatorname{diam}(\Omega(P)) = 3$ if and only if $|\operatorname{Atom}(P)| \geqslant 2$, $|P| \geqslant 4$ and there exists at least one a_i in $\operatorname{Atom}(P)$ such that for every $a_j \in \operatorname{Atom}(P)$ with $j \neq i$, $[a_i) \cap [a_j) = \varnothing$.

Proof. (i) Suppose that $\Omega(P)$ is complete. We claim that P has a unique atom. If a_1, a_2 are two distinct atoms of P, then vertices $I_1 = \{0, a_1\}$

and $I_2 = \{0, a_2\}$ are not adjacent (as obviously they are not comparable), a contradiction and so the claim is proved. Similarly, P has a unique maximal element, say m_1. Let x, y be two non-comparable elements of P. Thus $x \neq a_1, m_1$ and $y \neq a_1, m_1$ and hence vertices $(x]$ and $(y]$ are not adjacent in $\Omega(P)$, which is impossible. Hence every pair of elements of P are comparable, as desired.

Conversely, assume that P is a chain, say $0 < a_1 < x_1 < x_2 < \cdots < x_{n-1} < x_n$, where $n+1$ is the length of chain. Then $I_1 = \{0, a_1\}, I_i = \{0, a_1, x_{i-1}\}(2 \leqslant i \leqslant n)$ are all ideals of P. It is clear that $I_1 \subset I_2 \subset \cdots \subset I_n$, i.e, $\Omega(P)$ is complete.

(ii) First, suppose that $\mathrm{diam}(\Omega(P)) = 2$. Then, there exist distinct vertices $I, J \in \mathcal{I}(P)$ such that $d(I,J) = 2$. Therefore, there is $K \in \mathcal{I}(P)$ such that $d(I,K) = d(K,J) = 1$. By Lemma 2.1 and Theorem 2.4, $|P| \geqslant 4$. Also, by part (i), P is not a chain. Obviously, $|\,\mathrm{Atom}(P)| \geqslant 1$. If $|\,\mathrm{Atom}(P)| \neq 1$, then we show that $[a_i) \cap [a_j) \neq \varnothing$, for every $a_i, a_j \in \mathrm{Atom}(P)$. Assume to the contrary, there exists $a_i \in \mathrm{Atom}(P)$ such that $[a_i) \cap [a_j) = \varnothing$, for every $a_j(\neq a_i) \in \mathrm{Atom}(P)$. Thus $[a_i) \cup \{0\} \in \mathcal{I}(P)$. Now, consider the path $[a_i) \cup \{0\} - \{0, a_i\} - \{0, a_i\} \cup (P \setminus [a_i)) - P \setminus [a_i)$. By part (iii) of Lemma 2.6, $d([a_i) \cup \{0\}, P \setminus [a_i)) = 3$. This contradicts the assumption $\mathrm{diam}(\Omega(P)) = 2$.

Conversely, assume that $|P| \geqslant 4$. If (a) holds, then $|\,\mathrm{Atom}(P)| = 1$ implies that $\mathrm{diam}(\Omega(P)) \leqslant 2$. Also, since P is not a chain, we deduce that $\mathrm{diam}(\Omega(P)) = 2$. Next, let (b) holds. Assume to the contrary, $\mathrm{diam}(\Omega(P)) \neq 2$. Then, by Theorem 2.5, $\mathrm{diam}(\Omega(P)) = 1$ or 3. If $\mathrm{diam}(\Omega(P)) = 1$, then by part (i), P is a chain, a contradiction. If $\mathrm{diam}(\Omega(P)) = 3$, then by part (iii) of Lemma 2.6, there exist $I, J \in V(\Omega(P))$ such that $I \cup J = P$ and $I \cap J = \{0\}$. Therefore, there are atoms $a_i \in I \setminus J$ and $a_j \in J \setminus I$. Let $y \in [a_i) \cap [a_j)$. With no loss of generality, we may assume that $y \in I \setminus J$. Thus $a_j \in I$, which is impossible and so $[a_i) \cap [a_j) = \varnothing$. This contradicts the assumption and hence $\mathrm{diam}(\Omega(P)) = 2$.

(iii) It is obvious by parts (i) and (ii). □

In the next result we study the girth of an inclusion ideal graph of a poset.

Theorem 2.8. Let $(P, \leqslant)$ be a poset. Then the following statements hold.

(i) $\mathrm{girth}(\Omega(P)) = 3$ if and only if $|P| \geqslant 5$.
(ii) $\mathrm{girth}(\Omega(P)) = 6$ if and only if $|P| = 4$ and $|\,\mathrm{Atom}(P)| = 3$.
(iii) $\mathrm{girth}(\Omega(P)) = \infty$ if and only if $|P| < 5$ and $|\,\mathrm{Atom}(P)| \neq 3$.

Proof. (i) Let $|P| \geqslant 5$. If $|\operatorname{Atom}(P)| \geqslant 3$ and $a_1, a_2, a_3 \in \operatorname{Atom}(P)$, then $\Omega(P)$ has the cycle $\{0, a_1\} - \{0, a_1, a_2\} - \{0, a_1, a_2, a_3\} - \{0, a_1\}$. To complete the proof, suppose that $|\operatorname{Atom}(P)| < 3$. First, let $|\operatorname{Atom}(P)| = 1$. Two following cases may occur:

Case 1. If P is a chain, then by part (i) of Theorem 2.7, $\Omega(P)$ is complete. Since $|P| \geqslant 5$, we deduce that $\operatorname{girth}(\Omega(P)) = 3$.

Case 2. If P is not a chain, then there exist at least two non-comparable elements in P, say x, y. Obviously, $a_1 \neq x$ and $a_1 \neq y$. If $(x] \cup (y] \neq P$, then by Lemma 2.3, the triangle $\{0, a_1\} - (x] - (x] \cup (y] - \{0, a_1\}$ in $\Omega(P)$ shows that $\operatorname{girth}(\Omega(P)) = 3$. Hence suppose that $(x] \cup (y] = P$. This means that x, y are only maximal elements of P. Since $|P| \geqslant 5$, with no loss of generality, one may assume that there exists a non atom $z \in P$ such that $z < x$. Now, by part (c) of Lemma 2.2, the cycle $\{0, a_1\} - (z] - (x] - \{0, a_1\}$ implies that $\operatorname{girth}(\Omega(P)) = 3$.

Finally, let $\operatorname{Atom}(P) = \{a_1, a_2\}$. Since $|P| \geqslant 5$, there exists an element $x \in P$ such that either $a_1 \sqsubset x$ or $a_2 \sqsubset x$. The cycle $\{0, a_1\} - \{0, a_1, a_2\} - \{0, a_1, a_2, x\} - \{0, a_1\}$ shows that $\operatorname{girth}(\Omega(P)) = 3$.

Conversely, suppose that $\operatorname{girth}(\Omega(P)) = 3$. Thus there exists a cycle of length three in $\Omega(P)$, say $I_1 - I_2 - I_3 - I_1$. With no loss of generality, we suppose that $I_1 \subset I_2 \subset I_3$. Since every ideals needs at least two elements and every I_i $(1 \leqslant i \leqslant 3)$ is non-trivial, we deduce that $|P| \geqslant 5$.

(ii) Let $P = \{0, a_1, a_2, a_3\}$. Then $\Omega(P)$ is the cycle $\{0, a_1\} - \{0, a_1, a_2\} - \{0, a_2\} - \{0, a_2, a_3\} - \{0, a_3\} - \{0, a_1, a_3\} - \{0, a_1\}$ and so $\operatorname{girth}(\Omega(P)) = 6$. Conversely, suppose that $\operatorname{girth}(\Omega(P)) = 6$. By part (i), $|P| < 5$. It follows from part (i) of Lemma 2.1 that $3 \leqslant |P| \leqslant 4$. If $|P| = 3$ and $|\operatorname{Atom}(P)| = 1$, then part (ii) of Lemma 2.1 implies that $\Omega(P) = K_1$, a contradiction. Also, if $P = \{0, a_1, a_2\}$, then by Theorem 2.4, $\Omega(P)$ is not connected, a contradiction. Hence we may suppose that $|P| = 4$. By part (i) of Theorem 2.7, P is not a chain and so $|\operatorname{Atom}(P)| \geqslant 2$. If $P = \{0, a_1, a_2, x\}$, where x is not an atom, then two following cases may occur:

Case 1. $a_1 \sqsubset x$ and $a_2 \sqsubset x$. It is not hard to check that $|\mathcal{I}(P)| = 3$, i.e, $\operatorname{girth}(\Omega(P)) \neq 6$, a contradiction.

Case 2. $a_1 \sqsubset x$ and a_2, x are not comparable. In this case, $|\mathcal{I}(P)| = 4$ and so $\operatorname{girth}(\Omega(P)) \neq 6$, a contradiction. The case "$a_2 \sqsubset x$ and a_1, x are not comparable" is similar.

Two above cases show that $|\operatorname{Atom}(P)| = 3$.

(iii) If $\operatorname{girth}(\Omega(P)) = \infty$, then by parts (i) and (ii), we deduce that $|P| < 5$ and $|\operatorname{Atom}(P)| \neq 3$. Conversely, suppose that $|P| < 5$ and $|\operatorname{Atom}(P)| \neq 3$. The following cases may happen:

Case 1. $|P| = 4$ and $|\operatorname{Atom}(P)| = 2$. By proof of part (ii), either $|\mathcal{I}(P)| = 3$ or $|\mathcal{I}(P)| = 4$. It is not hard to see that if $|\mathcal{I}(P)| = 3$, then

$\Omega(P)$ is a path of length two ($K_{1,2}$) and if $|\mathcal{I}(P)| = 4$, then $\Omega(P)$ is a path of length three.

Case 2. $|P| = 4$ and $|\operatorname{Atom}(P)| = 1$. If P is a chain, then by part (i) of Theorem 2.7, $\Omega(P)$ is K_2. If P is not a chain, then $\Omega(P)$ is $K_{1,2}$.

Case 3. $|P| = 3$ and $|\operatorname{Atom}(P)| = 2$. By Theorem 2.4, $\Omega(P)$ is not connected.

Case 4. $|P| = 3$ and $|\operatorname{Atom}(P)| = 1$. By part (ii) of Lemma 2.1, $\Omega(P)$ is K_1.

Case 5. $|P| = 2$. By part (i) of Lemma 2.1, $\Omega(P)$ is the empty graph.

In all of the above cases, $\operatorname{girth}(\Omega(P)) = \infty$, as desired. □

We close this section with the following corollary which is an immediate consequence of Theorem 2.8.

Corollary 2.9. Let $(P, \leqslant)$ be a poset. Then $\operatorname{girth}(\Omega(P)) \in \{3, 6, \infty\}$.

3. Some further classifications of $\Omega(P)$

In this section, some further classifications of $\Omega(P)$ are given. For instance, we classify all poset whose inclusion ideal graphs are regular. Moreover, it is shown that if $(P, \leqslant)$ is a finite poset with connected $\Omega(P)$, then $\Omega(P)$ is a $(|P| - 2)$-partite graph. Finally, posets whose inclusion ideal graphs are cycles or paths are characterized.

Theorem 3.1. Let $(P, \leqslant)$ be a finite poset. Then $\Omega(P)$ is regular if and only if P is one of the following posets.

a. P is a chain.

b. $P = \operatorname{Atom}(P) \cup \{0\}$ with $|P| \leqslant 4$.

Proof. Suppose that $\Omega(P)$ is a regular graph. If there exists a vertex of $V(\Omega(P))$ which is adjacent to every other vertex, then $\Omega(P)$ is complete. Thus by part (i) of Theorem 2.7, P is a chain. So assume that there is no vertex of $V(\Omega(P))$ adjacent to every other vertex. Hence $|\operatorname{Atom}(P)| \geqslant 2$. Consider the following cases:

Case 1. $P = \operatorname{Atom}(P) \cup \{0\}$. If $|\operatorname{Atom}(P)| = 2$, then by Theorem 2.4, $\Omega(P)$ is 0-regular. If $|\operatorname{Atom}(P)| = 3$, then by part (ii) of Theorem 2.8, $\Omega(P)$ is 2-regular. Let $\operatorname{Atom}(P) = \{a_1, \ldots, a_n\}$, where $n > 3$. If $I = \{0, a_1\}$ and $J = \{0, a_1, a_2\}$, then it is not hard to check that

$$\deg(I) = (n-1) + \binom{n-1}{2} + \binom{n-1}{3} + \cdots + \binom{n-1}{n-2},$$

whereas

$$\deg(J) = 2 + \binom{n-2}{1} + \binom{n-2}{2} + \cdots + \binom{n-2}{n-3}.$$

Since $n > 3$, we deduce that $\deg(I) > \deg(J)$, a contradiction.
Case 2. $P \neq \mathrm{Atom}(P) \cup \{0\}$. Thus there exists an element $x \in P \setminus \mathrm{Atom}(P)$. With no loss of generality, one may assume that $a_1 < x$ and there is no $y \in P$ such that $a_1 < y < x$. Let $I' = \{0, a_1\}$ and $J' = \{0, a_1, x\}$. Since $\Omega(P)$ is regular, we conclude that $P \neq \{0, a_1, a_2, x\}$ and so $|P| \geqslant 5$. It is not hard to check that $N(J') = \{I', \{L\}_{L \in B}\}$, where B consists of all ideals of P which contains J' and $N(I') \subseteq N(J') \cup \{\{0, a_1, a_2\}\}$. It means that $\deg(I') > \deg(J')$, which is impossible. This completes the proof. □

Next, we show that if $(P, \leqslant)$ is a finite poset with connected $\Omega(P)$, then $\Omega(P)$ is a $(|P|-2)$-partite graph. First, we state the following lemma.

Lemma 3.2. Let $(P, \leqslant)$ be a finite poset. Then the cardinal number of every minimal ideal is 2 and the cardinal number of every maximal ideal is $|P| - 1$.

Proof. It is not hard to check that every minimal ideal is of the form $\{0, a_i\}$, where $a_i \in \mathrm{Atom}(P)$, and every maximal ideal is of the form $P \setminus \{u\}$, where u is a maximal element of P. Thus the result follows. □

Theorem 3.3. Let $(P, \leqslant)$ be a finite poset. If $\Omega(P)$ is connected, then $\Omega(P)$ is a $(|P| - 2)$-partite graph.

Proof. Let I be an ideal of P. By Lemma 3.2, $2 \leqslant |I| \leqslant |P| - 1$. We claim that for every i, $2 \leqslant i \leqslant |P| - 1$, there exists at least one ideal I of P such that $|I| = i$. Assume that $Max(P) = \{m_1, \ldots, m_k\}$ and $M_1 = P - \{m_1\}$ is a maximal ideal of P. Set $M_i = M_{i-1} - \{m_i\}$, for every $2 \leqslant i \leqslant k$. Obviously, M_i is an ideal of order $|P| - i$, for every i, $1 \leqslant i \leqslant k$. Let $\{n_1, \ldots, n_s\}$ be maximal elements of M_k (with respect to elements contained in M_k). By repeating the above procedure, we obtain an ideal of order $|P| - k - j$, for every j, $1 \leqslant j \leqslant s$. By this method, our claim is proved. Define the relation $\sim$ on $V(\Omega(P))$ as follows: For ever $I, J \in V(\Omega(P))$ we write $I \sim J$ if and only if $|I| = |J|$. It is easily seen that $\sim$ is an equivalence relation on $V(\Omega(P))$. By $[I]$, we mean the equivalence class of I. It follows from the above argument that the number of equivalence classes is equal to $|P| - 2$. Now, suppose that $[I]$ and $[J]$ are two distinct arbitrary equivalence classes. It is easily seen that there is no adjacency between every pair of vertices contained $[I]$ and every vertex contained in $[I]$ is adjacent to at least one vertex contained in $[J]$. This completes the proof. □

Next we classify posets whose inclusion ideal graphs are cycles or paths.

Theorem 3.4. Let $(P, \leqslant)$ be a poset. Then $\Omega(P)$ is a cycle if and only if one of the following statements holds.

(i) P is a chain and $|P| = 5$.

(ii) $|P| = 4$ and $|\operatorname{Atom}(P)| = 3$.

Proof. Suppose that $\Omega(P)$ is a cycle of length n. By Theorem 2.8, either $n = 3$ or $n = 6$. If $n = 3$, then $\Omega(P)$ is a complete graph. Thus part (i) of Theorem 2.7 implies that P is a chain. Also, by part (i) of Theorem 2.8, $|P| \geqslant 5$. It is not hard to check that $|P| > 5$ implies that $|\mathcal{I}(P)| > 3$, which is impossible. Hence $|P| = 5$. If $n = 6$, then the result follows from part (ii) of Theorem 2.8. Conversely,

(i) Suppose that P is a chain with $P = \{0, a_1, x_1, x_2, x_3\}$. Then $\Omega(P)$ is the cycle $\{0, a_1\} - \{0, a_1, x_1\} - \{0, a_1, x_1, x_2\} - \{0, a_1\}$ of length 3.

(ii) If $P = \operatorname{Atom}(P) \cup \{0\}$ with $|P| = 5$, then by part (ii) of Theorem 2.8, $\Omega(P)$ is a cycle of length 6.

The proof is complete. □

Theorem 3.5. Let $(P, \leqslant)$ be a poset. Then $\Omega(P)$ is a path of positive length if and only if $|P| = 4$ and $|\operatorname{Atom}(P)| \neq 3$.

Proof. The result follows from Theorem 2.8. □

In the following result, we classify posets whose inclusion ideal graphs are star.

Theorem 3.6. Let $(P, \leqslant)$ be a poset. Then $\Omega(P)$ is a star graph if and only if $|P| = 4$ and one of the following statements holds.

(i) P is a chain ($\Omega(P) = K_2$).

(ii) P is not a chain and $|\mathcal{I}(P)| = 3$ ($\Omega(P) = K_{1,2}$).

Proof. Suppose that $\Omega(P)$ is star. By Theorem 2.8, $|P| < 5$. If $|P| < 4$, then $\Omega(P)$ is not star. Thus $|P| = 4$. Two following cases may occur.

(i) If P is a chain, then by part (i) of Theorem 2.7, $\Omega(P) = K_2$.

(ii) Suppose that P is not a chain. If $P = \{0, a_1, x, y\}$, then $\Omega(P)$ is the star graph $\{0, a_1, x\} - \{0, a_1\} - \{0, a_1, y\}$. Thus we may assume that $P = \{0, a_1, a_2, x\}$. If $a_1 < x$ and $a_2 < x$, then $\Omega(P)$ is the the star graph $\{0, a_1\} - \{0, a_1, a_2\} - \{0, a_2\}$. If $a_1 < x$ and $a_2 \nless x$, then $\Omega(P)$ is the the path $\{0, a_1, x\} - \{0, a_1\} - \{0, a_1, a_2\} - \{0, a_2\}$, which is impossible.

The case $a_2 < x$ and $a_1 \nless x$ leads to a similar contradiction. Also, if $P = \{0, a_1, a_2, a_3\}$, then by part (iii) of Theorem 2.8, $\Omega(P)$ is a cycle of length 6. Thus $|\mathcal{I}(P)| = 3$ and $\Omega(P) = K_{1,2}$.

The converse is clear. □

In light of Theorems 2.8 and 3.6, we have the following corollary.

Corollary 3.7. Let $(P, \leqslant)$ be a poset. Then the following statements are equivalent:

(i) $\Omega(P)$ is complete bipartite.

(ii) $\Omega(P)$ is star.

(iii) Either $\Omega(P) = K_2$ or $\Omega(P) = K_{1,2}$.

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