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RESEARCH ARTICLE

Positive posets with linear Hasse diagrams

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Abstract. This paper studies positive (i.e., with positive Tits quadratic form) posets whose Hasse diadrams are linear. A criterion for the existence of such a nonserial poset of an arbitrary fixed order and height (width) is provided. We do not use the previously obtained by us complete list of nonserial positive posets, which contains 108 elements up to isomorphism and duality.

Introduction

In the paper [1], for a finite quiver $Q = (Q_0, G_1)$ with the set of vertices Q_0 and the set of arrows Q_1, P. Gabriel introduced the quadratic form

$$q_Q(z) := \sum_{i \in Q_0} z_i^2 - \sum_{i \to j} z_i z_j$$

($i \to j$ runs through Q_1) called by him the *Tits quadratic form* of the quiver Q. He proved that a connected quiver is of finite representation type over a field if and only if its underlying non-oriented graph is one of the (simply faced) Dynkin diagrams. In the language of the quadratic forms this means that the quivers of finite representation type coincide with the quivers whose Tits quadratic form is positive. This Gabriel's work laid the foundations of a new direction in the theory of algebra dealing with the investigation of relationships between properties

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of representations of various objects and properties of quadratic forms associated with these objects.

The above quadratic form is naturally generalized to a finite poset (partially ordered set) $S \not\ni 0$ [2]:

$$q_S(z) := z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i.$$

In [2], Yu. A. Drozd showed that a poset S is of finite representation type if and only if its Tits quadratic form is weakly positive, i.e., takes a positive value on any nonzero vector with non-negative coordinates (representations of posets over a field were introduced by L. A. Nazarova and A. V. Roiter [3][1]).

In contrast to quivers, the sets of posets with weakly positive and with positive Tits quadratic form do not coincide. Therefore, the investigations of posets with positive Tits form seems to be quite natural: they are analogs of the Dynkin diagrams. Posets of such type were first classified in [11, 12]. The paper [12] also classified the minimal posets with nonpositive Tits quadratic form.

In solving these problems the minimax equivalence method, introduced by the first author [13,14], was first applied. It has been subsequently used to solve many other classification problems (see, e.g., [15–19]).

This paper continues to investigate posets with positive Tits quadratic form, which are now called *positive*. Namely, we study positive posets whose Hasse diagrams are linear. We provide a criterion for the existence of such a nonserial poset of an arbitrary fixed order and height (width). Crucially, we do not rely on the complete list of 108 nonserial positive posets previously obtained by us up to isomorphism and duality.

Such a strategy for studying large lists of objects allows for a deeper understanding of the underlying causes. It reveals the core structural properties of these posets.

1. Positive posets

Throughout this paper, all posets are assumed to be nonempty and finite (without the symbol 0), with all subposets being induced. Sometime (in definitions or statements), we admit empty posets, which are or may later become subposets of some original posets.

[1] See also [4–10].

By the Hasse diagram of a poset S, we mean an undirected graph with the vertices $x \in S$ and the edges (x, y), where y covers x (i.e., $x < y$ and there is no element z satisfying $x < z < y$). In drawings, the edges always go upward from x to y.

A poset S is called positive if so is its Tits quadratic form

$$q_S(z) = z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i.$$

Each positive poset is of one of two types: serial or nonserial. A positive poset S is called *serial* if for any natural m, there is a positive poset $S(m) \supset S$ such that $|S(m) \setminus S| = m$, and *nonserial* otherwise.

This section is devoted to the classification of serial and nonserial positive posets obtained by the authors (see Introduction).

1.1. Classification of serial positive posets. First let us give some definitions.

A poset S is called a *sum* of subposets A and B and is written $S = A + B$, if $S = A \cup B$ and $A \cap B = \varnothing$. If any two elements of different summands are incomparable, the sum is called *direct*.

A sum $S = A + B$ with $A, B \neq \varnothing$ is said to be *left* (resp. *right*) if $a < b$ (resp. $b < a$) for some $a \in A$, $b \in B$ and there are no $a' \in A$, $b' \in B$ satisfying $a' > b'$ (resp. $b' > a'$). Finally, a sum $S = A + B$ is called *minimax* if $x < y$ with x and y belonging to different summands implies that x is minimal and y is maximal in S.

Now we move on to the formulation of the main theorem [12].

A linearly ordered set of order n is called a *chain of length* n. A poset of order $m \geq 2$ with exactly one pair of incomparable elements is called an *almost chain of length* $m - 1$ (such a poset is obtained from a chain of length $m - 1$ by bifurcating one of its elements).

Theorem 1. *A poset T is serial positive if and only if it is isomorphic to one of the following posets S:*

(1) *S is a direct sum of a chain of length $k \geq 0$ and a chain of length $s \geq 1$, where $k \leq s$;*

(2) *S is a left minimax sum of two chains of lengths $k \geq 1$ and $s \geq 1$, where $k + s > 3$;*

(3) *S is a direct sum of an almost chain of length $k \geq 1$ and a chain of length $s \geq 0$, where $k + s > 1$.*

Moreover, all these posets are pairwise nonisomorphic.

Theorem 2. *Any positive poset of order $n \neq 5, 6, 7$ is serial.*

1.2. Classification of nonserial positive posets. In the case when the elements of a poset are numbered by integers, the partial order relation is denoted by $\preceq$ (and one always assumes that $i \prec j$ implies $i < j$). A poset T is called *dual* to a poset S and is denoted by S^{op} if $T = S$ as ordinary sets and $x < y$ in T if and only if $x > y$ in S. For posets X, Y, we denote by $X \sqcup Y$ their direct sum (i.e., such union that elements from the different posets are incomparable) From Dilworth's theorem it follows that any poset can be represented in the form $\bigsqcup_{i=1}^{m} X_i$ with X_i being chains and additional relations $y < z$ for y and z belonging to different components. Below A_s, B_s, C_s denote, respectively, the chains $a_1 < \ldots < a_s$, $b_1 < \ldots < b_s$, $c_1 < \ldots < c_s$.

The positive nonserial posets were classified by the authors in [12] in terms of their Hasse diagrams. In [20, Section 4], we provided these diagrams, sorting them by the order of the posets (which can only be equal to 5, 6 or 7; see Theorem 2 above).

The next three theorems from [20, Section 2] give a set-theoretic reformulation of our classification; a number m in parentheses indicates the corresponding number from [12], and m^{op} means that one must take the poset dual to the poset with number m).

Theorem 3. *The nonserial posets of order 5 are exhausted, up to isomorphism and duality, by the following 10 posets:*

$NSP5.1(3)$ $A_2 \sqcup B_3,\ a_1 \prec b_2;$
$NSP5.2(4)$ $A_2 \sqcup B_3,\ a_2 \prec b_3;$
$NSP5.3(5)$ $A_2 \sqcup B_3,\ a_1 \prec b_2,\ a_2 \prec b_3;$
$NSP5.4(1)$ $A_1 \sqcup B_4,\ a_1 \prec b_3;$
$NSP5.5(2)$ $A_2 \sqcup B_3,\ a_1 \prec b_1,\ a_2 \prec b_3;$
$NSP5.6(46)$ $A_1 \sqcup B_2 \sqcup C_2;$
$NSP5.7(48)$ $A_1 \sqcup B_2 \sqcup C_2,\ b_1 \prec c_2;$
$NSP5.8(49)$ $A_1 \sqcup B_2 \sqcup C_2,\ a_1 \prec b_2,\ b_1 \prec c_2;$
$NSP5.9(47)$ $A_1 \sqcup B_1 \sqcup C_3,\ b_1 \prec c_3;$
$NSP5.10(50)$ $A_1 \sqcup B_3 \sqcup C_1,\ a_1 \prec b_3,\ b_1 \prec c_1.$

Theorem 4. *The nonserial posets of order 6 are exhausted, up to isomorphism and duality, by the following 32 posets:*

$NSP6.1(12)$ $A_3 \sqcup B_3,\ a_1 \prec b_2;$
$NSP6.2(20)$ $A_3 \sqcup B_3,\ a_1 \prec b_2,\ a_2 \prec b_3;$
$NSP6.3(10)$ $A_2 \sqcup B_4,\ a_1 \prec b_2;$
$NSP6.4(11)$ $A_2 \sqcup B_4,\ a_1 \prec b_3;$

$NSP6.5(13)$ $A_2 \sqcup B_4,\ a_2 \prec b_4$;
$NSP6.6(14)$ $A_2 \sqcup B_4,\ a_1 \prec b_2,\ a_2 \prec b_3$;
$NSP6.7(16)$ $A_2 \sqcup B_4,\ a_1 \prec b_2,\ a_2 \prec b_4$;
$NSP6.8(18)$ $A_2 \sqcup B_4,\ a_1 \prec b_3,\ a_2 \prec b_4$;
$NSP6.9(19^{op})$ $A_3 \sqcup B_3,\ a_1 \prec b_2,\ a_3 \prec b_3$;
$NSP6.10(14^{op})$ $A_3 \sqcup B_3,\ a_2 \prec b_2,\ a_3 \prec b_3$;
$NSP6.11(15)$ $A_3 \sqcup B_3,\ a_1 \prec b_1,\ a_2 \prec b_2,\ a_3 \prec b_3$;
$NSP6.12(6)$ $A_1 \sqcup B_5,\ a_1 \prec b_3$;
$NSP6.13(8)$ $A_1 \sqcup B_5,\ a_1 \prec b_4$;
$NSP6.14(7)$ $A_2 \sqcup B_4,\ a_1 \prec b_1,\ a_2 \prec b_3$;
$NSP6.15(9)$ $A_2 \sqcup B_4,\ a_1 \prec b_1,\ a_2 \prec b_4$;
$NSP6.16(58)$ $A_2 \sqcup B_2 \sqcup C_2,\ b_1 \prec c_2$;
$NSP6.17(66)$ $A_2 \sqcup B_2 \sqcup C_2,\ a_1 \prec b_2,\ b_1 \prec c_2$;
$NSP6.18(51)$ $A_1 \sqcup B_2 \sqcup C_3$;
$NSP6.19(55)$ $A_1 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_2$;
$NSP6.20(56)$ $A_1 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_3$;
$NSP6.21(57)$ $A_2 \sqcup B_1 \sqcup C_3,\ b_1 \prec c_3$;
$NSP6.22(60)$ $A_1 \sqcup B_2 \sqcup C_3,\ a_1 \prec b_2,\ b_1 \prec c_2$;
$NSP6.23(61)$ $A_1 \sqcup B_2 \sqcup C_3,\ a_1 \prec b_2,\ b_1 \prec c_3$;
$NSP6.24(62)$ $A_1 \sqcup B_3 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_1$;
$NSP6.25(63)$ $A_1 \sqcup B_3 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_2$;
$NSP6.26(59)$ $A_1 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_2,\ b_2 \prec c_3$;
$NSP6.27(67)$ $A_1 \sqcup B_3 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_1,\ b_2 \prec c_2$;
$NSP6.28(52)$ $A_1 \sqcup B_1 \sqcup C_4,\ b_1 \prec c_3$;
$NSP6.29(54)$ $A_1 \sqcup B_1 \sqcup C_4,\ b_1 \prec c_4$;
$NSP6.30(64)$ $A_1 \sqcup B_4 \sqcup C_1,\ a_1 \prec b_3,\ b_1 \prec c_1$;
$NSP6.31(65)$ $A_1 \sqcup B_4 \sqcup C_1,\ a_1 \prec b_4,\ b_1 \prec c_1$;
$NSP6.32(53)$ $A_1 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_1,\ b_2 \prec c_3$.

Theorem 5. *The nonserial posets of order 7 are exhausted, up to isomorphism and duality, by the following 66 posets:*

$NSP7.1(29)$ $A_3 \sqcup B_4,\ a_1 \prec b_3$;
$NSP7.2(30)$ $A_3 \sqcup B_4,\ a_2 \prec b_4$;
$NSP7.3(42)$ $A_3 \sqcup B_4,\ a_1 \prec b_2,\ a_2 \prec b_4$;
$NSP7.4(43)$ $A_3 \sqcup B_4,\ a_1 \prec b_3,\ a_2 \prec b_4$;
$NSP7.5(44)$ $A_3 \sqcup B_4,\ a_1 \prec b_3,\ a_3 \prec b_4$;
$NSP7.6(45)$ $A_3 \sqcup B_4,\ a_1 \prec b_2,\ a_2 \prec b_3,\ a_3 \prec b_4$;
$NSP7.7(26)$ $A_2 \sqcup B_5,\ a_1 \prec b_2$;
$NSP7.8(27)$ $A_2 \sqcup B_5,\ a_1 \prec b_4$;

$NSP7.9(28)$ $A_2 \sqcup B_5,\ a_2 \prec b_5$;
$NSP7.10(31)$ $A_2 \sqcup B_5,\ a_1 \prec b_2,\ a_2 \prec b_3$;
$NSP7.11(33)$ $A_2 \sqcup B_5,\ a_1 \prec b_2,\ a_2 \prec b_4$;
$NSP7.12(36)$ $A_2 \sqcup B_5,\ a_1 \prec b_2,\ a_2 \prec b_5$;
$NSP7.13(38)$ $A_2 \sqcup B_5,\ a_1 \prec b_3,\ a_2 \prec b_5$;
$NSP7.14(40)$ $A_2 \sqcup B_5,\ a_1 \prec b_4,\ a_2 \prec b_5$;
$NSP7.15(35^{op})$ $A_3 \sqcup B_4,\ a_2 \prec b_2,\ a_3 \prec b_3$;
$NSP7.16(41^{op})$ $A_4 \sqcup B_3,\ a_1 \prec b_2,\ a_4 \prec b_3$;
$NSP7.17(39^{op})$ $A_4 \sqcup B_3,\ a_2 \prec b_2,\ a_4 \prec b_3$;
$NSP7.18(37^{op})$ $A_4 \sqcup B_3,\ a_3 \prec b_2,\ a_4 \prec b_3$;
$NSP7.19(32)$ $A_3 \sqcup B_4,\ a_1 \prec b_1,\ a_2 \prec b_2,\ a_3 \prec b_3$;
$NSP7.20(34)$ $A_3 \sqcup B_4,\ a_1 \prec b_1,\ a_2 \prec b_2,\ a_3 \prec b_4$;
$NSP7.21(21)$ $A_1 \sqcup B_6,\ a_1 \prec b_3$;
$NSP7.22(24)$ $A_1 \sqcup B_6,\ a_1 \prec b_5$;
$NSP7.23(22)$ $A_2 \sqcup B_5,\ a_1 \prec b_1,\ a_2 \prec b_3$;
$NSP7.24(25)$ $A_2 \sqcup B_5,\ a_1 \prec b_1,\ a_2 \prec b_5$;
$NSP7.25(23)$ $A_3 \sqcup B_4,\ a_2 \prec b_1,\ a_3 \prec b_3$;
$NSP7.26(75)$ $A_1 \sqcup B_3 \sqcup C_3,\ b_1 \prec c_3$;
$NSP7.27(78)$ $A_2 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_2$;
$NSP7.28(79)$ $A_3 \sqcup B_1 \sqcup C_3,\ b_1 \prec c_3$;
$NSP7.29(80)$ $A_3 \sqcup B_2 \sqcup C_2,\ b_1 \prec c_2$;
$NSP7.30(89)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_2,\ b_1 \prec c_3$;
$NSP7.31(91)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_3,\ b_1 \prec c_2$;
$NSP7.32(92)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_3,\ b_1 \prec c_3$;
$NSP7.33(99)$ $A_2 \sqcup B_2 \sqcup C_3,\ a_1 \prec b_2,\ b_1 \prec c_2$;
$NSP7.34(100)$ $A_2 \sqcup B_2 \sqcup C_3,\ a_1 \prec b_2,\ b_1 \prec c_3$;
$NSP7.35(101)$ $A_2 \sqcup B_3 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_2$;
$NSP7.36(102)$ $A_2 \sqcup B_3 \sqcup C_2,\ a_2 \prec b_3,\ b_1 \prec c_1$;
$NSP7.37(85)$ $A_1 \sqcup B_3 \sqcup C_3,\ b_1 \prec c_2,\ b_2 \prec c_3$;
$NSP7.38(86)$ $A_2 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_2,\ b_2 \prec c_3$;
$NSP7.39(108)$ $A_2 \sqcup B_3 \sqcup C_2,\ a_2 \prec b_3,\ b_1 \prec c_1,\ b_2 \prec c_2$;
$NSP7.40(108^{op})$ $A_2 \sqcup B_3 \sqcup C_2,\ a_1 \prec b_2,\ a_2 \prec b_3,\ b_1 \prec c_2$;
$NSP7.41(68)$ $A_1 \sqcup B_2 \sqcup C_4$;
$NSP7.42(72)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_2$;
$NSP7.43(73)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_3$;
$NSP7.44(74)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_4$;
$NSP7.45(76)$ $A_2 \sqcup B_1 \sqcup C_4,\ b_1 \prec c_3$;
$NSP7.46(87)$ $A_1 \sqcup B_2 \sqcup C_4,\ a_1 \prec b_2,\ b_1 \prec c_2$;
$NSP7.47(88)$ $A_1 \sqcup B_2 \sqcup C_4,\ a_1 \prec b_2,\ b_1 \prec c_4$;

$NSP7.48(90)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_3,\ b_1 \prec c_1$;
$NSP7.49(93)$ $A_1 \sqcup B_4 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_1$;
$NSP7.50(94)$ $A_1 \sqcup B_4 \sqcup C_2,\ a_1 \prec b_3,\ b_1 \prec c_2$;
$NSP7.51(95)$ $A_1 \sqcup B_4 \sqcup C_2,\ a_1 \prec b_4,\ b_1 \prec c_2$;
$NSP7.52(81)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_2,\ b_2 \prec c_3$;
$NSP7.53(83)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_2,\ b_2 \prec c_4$;
$NSP7.54(84^{op})$ $A_1 \sqcup B_3 \sqcup C_3,\ b_2 \prec c_2,\ b_3 \prec c_3$;
$NSP7.55(77)$ $A_2 \sqcup B_2 \sqcup C_3,\ b_1 \prec c_1,\ b_2 \prec c_3$;
$NSP7.56(103)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_3,\ b_1 \prec c_1,\ b_2 \prec c_2$;
$NSP7.57(104)$ $A_1 \sqcup B_3 \sqcup C_3,\ a_1 \prec b_3,\ b_1 \prec c_1,\ b_2 \prec c_3$;
$NSP7.58(105)$ $A_1 \sqcup B_4 \sqcup C_2,\ a_1 \prec b_4,\ b_1 \prec c_1,\ b_2 \prec c_2$;
$NSP7.59(106)$ $A_1 \sqcup B_4 \sqcup C_2,\ a_1 \prec b_4,\ b_2 \prec c_1,\ b_3 \prec c_2$;
$NSP7.60(82)$ $A_1 \sqcup B_3 \sqcup C_3,\ b_1 \prec c_1,\ b_2 \prec c_2,\ b_3 \prec c_3$;
$NSP7.61(69)$ $A_1 \sqcup B_1 \sqcup C_5,\ b_1 \prec c_3$;
$NSP7.62(71)$ $A_1 \sqcup B_1 \sqcup C_5,\ b_1 \prec c_5$;
$NSP7.63(96)$ $A_1 \sqcup B_5 \sqcup C_1,\ a_1 \prec b_3,\ b_1 \prec c_1$;
$NSP7.64(97)$ $A_1 \sqcup B_5 \sqcup C_1,\ a_1 \prec b_4,\ b_1 \prec c_1$;
$NSP7.65(98)$ $A_1 \sqcup B_5 \sqcup C_1,\ a_1 \prec b_5,\ b_1 \prec c_1$;
$NSP7.66(70)$ $A_1 \sqcup B_2 \sqcup C_4,\ b_1 \prec c_1,\ b_2 \prec c_3$.

2. Main results

2.1. Formulation of the main theorem. An undirected graph is said to be *linear* or a *chain* if its vertices can be labeled from 1 to n such that its edges are precisely $(i, i+1)$ for $i = 1, \dots, n-1$.

Recall that the *height* (resp. *width*) of a poset S is defined as the maximum number of its pairwise comparable (resp. incomparable) elements and is denoted by $h(S)$ (resp. $w(S)$).

For any natural numbers x and y, let $\mathcal{H}(x, y)$ (resp. $\mathcal{W}(x, y)$) denote the class of all posets of order x and height (resp. weight) y. Furthermore, we introduce $LP_{ns}^{+}\mathcal{H}(x, y)$ (resp. $LP_{ns}^{+}\mathcal{W}(x, y)$) as the subclass of $\mathcal{H}(x, y)$ (resp. $\mathcal{W}(x, y)$) consisting of all nonserial positive posets whose Hasse diagrams are linear graphs.

Theorem 6. *$LP_{ns}^{+}\mathcal{H}(x, y) \neq \varnothing$ if and only if one of the following condition holds:*

1) *$x = 5$ and $y = 2, 3$;*
2) *$x = 6$ and $y = 2, 3, 4$;*
3) *$x = 7$ and $y = 3, 4, 5$.*

Theorem 7. *$LP^+_{ns}\mathcal{W}(x,y) \neq \varnothing$ if and only if $x = 5,6,7$ and $y = 2,3$.*

Theorems 6 and 7 follow from the complete classification of the positive posets previously obtained by us (see Subsections 1.2 and 1.3). The main goal of our paper is to prove these theorems without using the list of nonserial positive posets (which contain 108 elements up to isomorphism and duality).

2.2. Proof of Theorem 6

Sufficiency. Let A and B be chains. We write $A \nearrow B$ if we add the relation $a < b$ to their direct sum. Here, a is the least element of A and b is the greatest element of B. The notation $A \longrightarrow B$ means that we add a relation between the greatest elements of A and B. Obviously, the posets $A \nearrow B$ and $A \longrightarrow B$ have linear Hasse diagrams.

Put

$H_{52} = \{1 \prec 3, 2 \prec 3, 2 \prec 5, 4 \prec 5\}$,
$H_{53} = \{1 \prec 2 \prec 5, 3 \prec 4 \prec 5\}$,
$H_{62} = \{1 \prec 2, 1 \prec 4, 3 \prec 4, 3 \prec 6, 5 \prec 6\}$,
$H_{63} = \{1 \prec 3, 2 \prec 3, 2 \prec 6, 4 \prec 5 \prec 6\}$,
$H_{64} = \{1 \prec 2 \prec 6, 3 \prec 4 \prec 5 \prec 6\}$,
$H_{73} = \{1 \prec 4, 2 \prec 3 \prec 4, 2 \prec 7, 5 \prec 6 \prec 7\}$,
$H_{74} = \{1 \prec 3, 2 \prec 3, 2 \prec 7, 4 \prec 5 \prec 6 \prec 7\}$,
$H_{75} = \{1 \prec 2 \prec 7, 3 \prec 4 \prec 5 \prec 6 \prec 7\}$.

For $(x,y) \in \{(5,2),(5,3),(6,2),(6,3),(6,4),(7,3),(7,4),(7,5)\}$, the poset H_{xy} belongs, obviously, to the set $\mathcal{H}(x,y)$.

Futher, the posets H_{xy} have linear Hasse diagrams because

$H_{52} = \{1\} \nearrow \{2 \prec 3\} \nearrow \{4 \prec 5\}$,
$H_{53} = \{1 \prec 2\} \longrightarrow \{3 \prec 4 \prec 5\}$,
$H_{62} = \{1 \prec 2\} \nearrow \{3 \prec 4\} \nearrow \{5 \prec 6\}$,
$H_{63} = \{1\} \nearrow \{2 \prec 3\} \nearrow \{4 \prec 5 \prec 6\}$,
$H_{64} = \{1 \prec 2\} \longrightarrow \{3 \prec 4 \prec 5 \prec 6\}$,
$H_{73} = \{1\} \nearrow \{2 \prec 3 \prec 4\} \nearrow \{5 \prec 6 \prec 7\}$,
$H_{74} = \{1\} \nearrow \{2 \prec 3\} \nearrow \{4 \prec 5 \prec 6 \prec 7\}$,
$H_{75} = \{1 \prec 2\} \longrightarrow \{3 \prec 4 \prec 5 \prec 6 \prec 7\}$.

Finally, these posets are positive since for their Tits quadratic forms $q_{ij}(z) := q_{H_{ij}}(z)$ we have the following equations:

$8q_{52} = 2(2z_1 + z_3 - z_0)^2 + 2(2z_2 + z_3 + z_5 - z_0)^2 + (2z_3 - z_5)^2 + 2(2z_4 + z_5 - z_0)^2 + 3z_5^2 + 2z_0^2$,

$12q_{53} = 3(2z_1 + z_2 + z_5 - z_0)^2 + (3z_2 + z_5 - z_0)^2 + 3(2z_3 + z_4 + z_5 - z_0)^2 + (3z_4 + z_5 - z_0)^2 + (2z_5 + z_0)^2 + 3z_0^2$,

$420q_{62} = 105(2z_1+z_2+z_4-z_0)^2+35(3z_2-z_4-z_0)^2+105(2z_3+z_4+z_6-z_0)^2+7(5z_4-3z_6-z_0)^2+105(2z_5+z_6-z_0)^2+3(7z_6-z_0)^2+60z_0^2,$

$168q_{63} = 42(2z_1+z_3-z_0)^2+42(2z_2+z_3+z_6-z_0)^2+21(2z_3-z_6)^2+42(2z_4+z_5+z_6-z_0)^2+14(3z_5+z_6-z_0)^2+(7z_6+2z_0)^2+24z_0^2,$

$168q_{64} = 42(2z_1+z_2+z_6-z_0)^2+14(3z_2+z_6-z_0)^2+42(2z_3+z_4+z_5+z_6-z_0)^2+14(3z_4+z_5+z_6-z_0)^2+7(4z_5+z_6-z_0)^2+(7z_6+5z_0)^2+24z_0^2,$

$240q_{73} = 60(2z_1+z_4-z_0)^2+60(2z_2+z_3+z_4+z_7-z_0)^2+20(3z_3+z_4-z_7-z_0)^2+4(5z_4-2z_7+z_0)^2+60(2z_5+z_6+z_7-z_0)^2+20(3z_6+z_7-z_0)^2+(8z_7+z_0)^2+15z_0^2,$

$48q_{74} = 12(2z_1+z_3-z_0)^2+12(2z_2+z_3+z_7-z_0)^2+6(2z_3-z_7)^2+12(2z_4+z_5+z_6+z_7-z_0)^2+4(3z_5+z_6+z_7-z_0)^2+2(4z_6+z_7-z_0)^2+3(2z_7+z_0)^2+3z_0^2,$

$240q_{75} = 60(2z_1+z_2+z_7-z_0)^2+20(3z_2+z_7-z_0)^2+60(2z_3+z_4+z_5+z_6+z_7-z_0)^2+20(3z_4+z_5+z_6+z_7-z_0)^2+10(4z_5+z_6+z_7-z_0)^2+6(5z_6+z_7-z_0)^2+(8z_6+7z_0)^2+15z_0^2.$

Necessity. Let $(x,y) \neq (5,2),(5,3),(6,2),(6,3),(6,4),(7,3),(7,4),(7,5)$. By Theorem 2, the sets $LP^+_{ns}\mathcal{H}(x,y)$ are empty for $x \neq 5,6,7$. In the cases $(x,y) = (5,1),(6,1),(7,1),(7,2)$ the sets $LP^+_{ns}\mathcal{H}(x,y)$ are empty since $n \leq h(S)w(S)$ for any poset of order n, and a poset consisting of four pairwise incomparable elements is not positive[2]. Finally, any element of the set $\mathcal{H}(x,x)$ is a chain, and of the set $\mathcal{H}(x,x-1)$ is isomorphic to a poset $A \nearrow B$, where A or B is of order 1 ($x = 5,6,7$). By Theorem 1, in both the cases we have only serial posets.

2.3. Proof of Theorem 7. The sets $LP^+_{ns}\mathcal{W}(x,y)$ are not empty for $x = 5,6,7$ and $y = 2,3$ since as elements $W_{xy} \in LP^+_{ns}\mathcal{W}(x,y)$ one can take the following poset (see previous subsection): $W_{52} = H_{53}$, $W_{53} = H_{52}$, $W_{62} = H_{64}$, $W_{63} = H_{63}$, $W_{72} = H_{75}$, $W_{73} = H_{73}$.

The necessity of the condition follows from Theorems 2 and Footnote 2.

2.4. Existence tables. In the final part of the paper, we formulate the main results in terms of Hasse diagrams. Theorems 6 and 7 correspond to the first and second tables, respectively.

[2]More precisely, the Tits quadratic form of the poset with elements 1,2,3,4 vanishes at $z_0 = 2, z_1 = z_2 = z_3 = z_4 = 1$.

$n\backslash h$	2	3	4	5
5				
6				
7				

$n\backslash w$	2	3
5		
6		
7		

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