Algebra and Discrete Mathematics
Volume **21** (2016). Number 2, pp. 255–263 RESEARCH ARTICLE

Involution rings with unique minimal *-biideal

D. I. C. Mendes*

Communicated by V. M. Futorny

Abstract. The structure of certain involution rings which have exactly one minimal *-biideal is determined. In addition, involution rings with identity having a unique maximal biideal are characterized.

1. Introduction

In the category of involution rings, it is not plausible to use the concept of left (right) ideal, since a left (right) ideal which is closed under involution is an ideal. An appropriate generalization which has been efficient in playing the role of these in the case of involution rings is that of *-biideal, first used by Loi [9] for proving structure theorems for involution rings. For semiprime involution rings, Loi also investigated the interrelation between the existence of minimal *-biideals and minimal biideals and Aburawash [3] characterized minimal *-biideals by means of idempotent elements. In [12], the author described minimal *-biideals of an arbitrary involution ring. The structure and properties of certain classes of right subdirectly irreducible rings (that is, rings in which the intersection of all nonzero right ideals is nonzero) were determined by

*Research supported by FEDER and Portuguese funds through the Centre for Mathematics (University of Beira Interior) and the Portuguese Foundation for Science and Technology (FCT- Fundação para a Ciência e a Tecnologia), Project PEst-OE/MAT/UI0212/2013.

2010 MSC: Primary 16W10; Secondary 16D25, 16N20.

Key words and phrases: involution, biideal, nilpotent ring, local ring, subdirectly irreducible ring, Jacobson radical.

Desphande ([6] , [7]). It seems, therefore, pertinent to consider involution rings in which the intersection of all nonzero *-biideals is nonzero. In a broader setting, we shall determine the structure of involution rings, belonging to certain classes, having exactly one minimal *-biideal.

All rings considered are associative and do not necessarily have identity. Let us recall that an involution ring A is a ring with an additional unary operation $*$, called involution, such that $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in A$. An element of an involution ring A, which is either symmetric or skew-symmetric, shall be called a *-element. A biideal B of a ring A is a subring of A satisfying the inclusion $BAB \subseteq B$. An ideal (biideal) B of an involution ring A is called a *-ideal (*-biideal) of A if B is closed under involution; that is, $B^* = \{a^* \in A : a \in B\} \subseteq B$. An involution ring A is semiprime if and only if, for any *-ideal I of A, $I^2 = 0$ implies $I = 0$. An involution ring A is called *-subdirectly irreducible if the intersection of all nonzero *-ideals of A (called the *-heart of A) is nonzero.

2. Involution rings with unique minimal *-biideal

We begin by considering involution rings in which the intersection of nonzero *-biideals is nonzero, which are obviously *-subdirectly irreducible. These will be called *-bi-subdirectly irreducible rings. If p is a prime, then $Z(p)$ denotes the zero ring on the cyclic additive group of order p.

Proposition 1. *Let A be a *-bi-subdirectly irreducible (with unique minimal *-biideal B). Then one of the following holds:*

(i) *A is a division ring with involution;*

(ii) *$A \cong D \oplus D^{op}$, where D is a division ring and $D \oplus D^{op}$ is endowed with the exchange involution;*

(iii) *A is *-subdirectly irreducible involution ring with *-heart $B \cong Z(p)$ for some prime p;*

(iv) *A is a *-subdirectly irreducible involution ring with *-heart $H = K \oplus K^*$, where $K \cong Z(2) \cong K^*$ and $B = \{a + a^* : a \in K\} \cong Z(2)$.*

Proof. Since the intersection of the nonzero *-biideals of A is nonzero, B generates the *-heart H of A.

Case 1. $(H^2 \neq 0)$. Either H is a simple prime ring or $H = K \oplus K^*$, where the ideals K and K^* of A are simple prime rings [5].

The *-biideal B is contained in every nonzero *-biideal B_1 of H. Indeed, $0 \neq B_1HB_1$ is a *-biideal of A so that $B \subseteq B_1HB_1 \subseteq B_1$. Therefore,

H is a *-simple involution ring having a minimal *-biideal, namely B. If H is simple prime, then H has a minimal left ideal L and $L = He$ for some idempotent element e in H [1]. Then $0 \neq L^*L = e^*He$ is a minimal *-biideal of H. So $B = L^*L \subseteq L$. The *-ideal H does not contain other minimal left ideals besides L, for if L_1 is a minimal left ideal of H, then $B = L_1^*L_1 \subseteq L_1$. Now, $0 \neq B \subseteq L \cap L_1 \subseteq L_1$ and since L and L_1 are minimal left ideals, it follows that $L_1 = L$. Thus $H = L$ and H is a division ring. Since the *-essential *-ideal H has identity, we have, by ([11], Lemma 8) that $A = H$. Thus A is a division ring. If $H = K \oplus K^*$, then it is clear, from [1], that K and K^* have minimal left ideals. Moreover, it can be deduced that K and K^* have unique minimal left ideals and this implies that K and K^* are division rings. Consequently, $H = B$ and we have $A = H = K \oplus K^* \cong K \oplus K^{op}$ endowed with the exchange involution.

Case 2. $(H^2 = 0)$. In this case, the *-biideal $B \cong Z(p)$ for some prime p, according to ([12], Corollary 4(iii)). Moreover, every subgroup of H is a biideal of A. By ([8], Proposition 6.2), H^+, the additive group of H, is an elementary abelian p-group and hence is a direct sum of cyclic groups of order p. By our assumption on A, either $H \cong Z(p)$ or $H = K \oplus K^*$, where $K \cong Z(p) \cong K^*$. If $p \neq 2$, then the case $H = K \oplus K^*$ cannot occur, for then $\{a + a^* : a \in K\}$ and $\{a - a^* : a \in K\}$ would be two distinct minimal *-biideals of A. □

The following corollary is immediate:

Corollary 2. *An involution ring A is semiprime *-bi-subdirectly irreducible if and only if it is one of the following types:*

(i) *a division ring;*

(ii) *$D \oplus D^{op}$, where D is a division ring and $D \oplus D^{op}$ is endowed with the exchange involution.*

Next, we study certain classes of involution rings having exactly one atom in their lattice of *-biideals. In the sequel, $[a]$ and $\langle a \rangle$ denote, respectively, the subring of A and the biideal of A generated by $a \in A$. Furthermore, if B is a biideal of A with p elements (p prime), we let $A_B = \{a \in A : pa = 0 = a^2 \text{ and } a \notin B\}$.

Lemma 3. *Let A be a nilpotent involution p-ring (p prime). Then A has a unique minimal *-biideal if and only if A is *-bi-subdirectly irreducible.*

Proof. Let A have a unique minimal *-biideal B. Then $B^2 = 0$, B contains a minimal *-subring S of order p and $B = S + SAS$, the *-biideal generated by S. But SAS is a *-biideal of A and $SAS = sAs$ for some *-element $s \in S$. Hence, either $sAs = 0$ or $sAs = B$. The latter case cannot occur, because then we would have $0 \neq s = sas$ for some $a \in A$; a contradiction with the fact that A is nilpotent. Therefore $B = S \cong Z\,(p)$. Now we will show that S is contained in every nonzero *-biideal of A. Let B_1 be any nonzero *-biideal of A. There exists a nonzero *-element s_1 in B_1, of order p and such that $s_1^2 = 0$. If $s_1As_1 \neq 0$, then there exists a nonzero *-element s_2 in s_1As_1. Now $s_2As_2 \subseteq s_1As_1 \subseteq B_1$. Continuing in this way, we obtain a chain $\ldots \subseteq s_iAs_i \ldots \subseteq s_2As_2 \subseteq s_1As_1 \subseteq B_1$. Since A is nilpotent, eventually we must obtain $s_iAs_i = 0$ for some nonzero *-element $s_i \in B_1$. Hence $\langle s_i \rangle = [s_i] = S$ and so $S \subseteq B_1$.

The converse is clear. □

Proposition 4. *If A is a nilpotent involution p-ring ($p \neq 2$ and p prime), then the following conditions are equivalent:*

(i) *A has a unique minimal *-biideal B;*

(ii) *A is subdirectly irreducible with heart $B \cong Z\,(p)$ and, for each $a \in A_B$, at least one of the following holds: $aAa \neq 0$, $aAa^* \neq 0$, $a^*Aa \neq 0$, $a^*a \neq 0$, $aa^* \neq 0$.*

Proof. Suppose that (i) holds. From the Lemma 3, we know that B is contained in every nonzero *-biideal of A. By Proposition 1, A is *-subdirectly irreducible with *-heart $B \cong Z\,(p)$. Next, we show that A is, in fact, subdirectly irreducible. Let I be any nonzero ideal of A such that $I \neq I^*$. We claim that $I \cap I^* \neq 0$. Suppose, on the contrary, that $I \cap I^* = 0$. Since A is nilpotent, there exists a least positive integer $n \geqslant 2$ such that $I^n = 0$. If n is even, let $J = I^{\frac{n}{2}}$ and if n is odd, let $J = I^{\frac{n+1}{2}}$. Hence $J^2 = JJ^* = J^*J = 0$. Then, for $0 \neq j \in J$ such that $pj = 0$ and $K = [j]$, it is easy to see that $\{k + k^* : k \in K\}$ and $\{k - k^* : k \in K\}$ are two distinct *-biideals of A of order p, which is a contradiction with our assumption. Therefore $I \cap I^* \neq 0$ and $B \subseteq I \cap I^* \subseteq I$. Hence A is a subdirectly irreducible ring with heart B. Suppose that there exists $a \in A_B$ such that $aAa = aAa^* = a^*Aa = 0$ and $a^*a = aa^* = 0$. If a is a *-element, then $[a]$ is a minimal *-biideal of A, which is a contradiction with our assumption. If a is not a *-element, and $T = [a]$, then $\{a + a^* : a \in T\}$ and $\{a - a^* : a \in T\}$ are distinct minimal *-biideals of A, which is again a contradiction.

Suppose that (ii) holds and let C be a minimal *-biideal of A and $C \neq B$. Clearly there exists a *-element $a \in C \cap A_B$ and $CAC = 0$, whence $aAa = aAa^* = a^*Aa = 0$ and $a^*a = aa^* = 0$, contradicting (ii). □

Corollary 5. *If A is an involution p-ring ($p \neq 2$ and p prime) and $A^2 = 0$, then the following conditions are equivalent:*

(i) *A has a unique minimal *-biideal B;*
(ii) *A has a unique minimal *-subring B;*
(iii) *A has a unique minimal subring B;*
(iv) *A is subdirectly irreducible with heart $B \cong Z(p)$ and $A_B = \varnothing$.*

The following example illustrates that Corollary 5 is not true, in general, when $p = 2$.

Example 6. *The 2-ring $A = Z(2) \oplus Z(2)$, with the exchange involution, is such that $A^2 = 0$ and has a unique minimal *-biideal, $B = \{(0,0),(1,1)\}$. However, A is not subdirectly irreducible.*

As usual, a ring A with identity 1 is called a local ring if $A/\mathcal{J}(A)$ is a division ring, where $\mathcal{J}(A)$ denotes the Jacobson radical of A.

Proposition 7. *Let A be a local involution ring of characteristic p^n ($p \neq 2$, p prime and $n \geqslant 1$) and with nonzero nilpotent Jacobson radical $\mathcal{J}(A)$. Then*

(i) *if $\mathcal{J}(A)$ has a unique minimal *-biideal B, then B is the unique minimal *-biideal of A;*
(ii) *$B = \{a \in A : a\mathcal{J}(A) = a^*\mathcal{J}(A) = 0\}$;*
(iii) *for a fixed nonzero $b \in B$, $\mathcal{J}(A) = \{a \in A : ba = ba^* = 0\} = \{a \in A : aB = a^*B = 0\}$;*
(iv) *for any $b \in B$, $a \in \mathcal{J}(A) \backslash B$, there exist $a_1, a_2 \in \mathcal{J}(A) \backslash B$ such that either $b = aa_1 = a_2a$ (if a is a *-element) or $b = (a + a^*)a_1 = a_2(a + a^*)$ (if a is not a *-element).*

Proof. (i) Taking into account Proposition 1 and the fact that a local ring contains only the trivial idempotents, it is clear that any minimal *-biideal of A must be contained in the Jacobson radical $\mathcal{J}(A)$ of A. If $\mathcal{J}(A)$ has a unique minimal *-biideal B, then we know that $B \cong Z(p)$ (Proposition 4). Clearly, $BAB \subseteq \mathcal{J}(A)$ and so, if $BAB \neq 0$, then $B \subseteq BAB$. However, this is impossible since $\mathcal{J}(A)$ is nilpotent. Thus $BAB = 0$ and so B is a biideal of A. Since any minimal *-biideal C of A is contained in $\mathcal{J}(A)$, we must have $C = B$.

(ii) From Proposition 1, $B \cong Z(p)$ and B is a *-ideal of A. Hence, for any nonzero $b \in B$, $b\mathcal{J}(A) \subseteq B$ implies that $b\mathcal{J}(A) = 0$ or $b\mathcal{J}(A) = B$. However, the latter case cannot occur since $\mathcal{J}(A)$ is nilpotent. Similarly, $b^*\mathcal{J}(A) = 0$. Thus $B \subseteq \{a \in A : a\mathcal{J}(A) = a^*\mathcal{J}(A) = 0\}$. Now to prove the other inclusion, let $a \in A$ such that $a\mathcal{J}(A) = a^*\mathcal{J}(A) = 0$. Then

$a \in \mathcal{J}(A)$ and $a^2 = 0$. Moreover, we claim that $pa = 0$. Indeed, since $(p1)^n = p^n 1 = 0$, $p1$ is not invertible and hence $p1 \in \mathcal{J}(A)$ and $pa = a(p1) = 0$. Taking into account Proposition 4, it follows that $a \in B$.

(iii) Let b be a fixed nonzero element in B. If $x \in \mathcal{J}(A)$, then also $x^* \in \mathcal{J}(A)$ and it follows from (ii) that $bx = bx^* = 0$ and so $x \in \{a \in A : ba = ba^* = 0\}$. On the other hand, if $x \in A$ such that $bx = bx^* = 0$, then $x \in \mathcal{J}(A)$, since $\mathcal{J}(A)$ contains all the zero divisors of A. Since $Ab = B$, it is now clear that $\mathcal{J}(A) = \{a \in A : ba = ba^* = 0\} = \{a \in A : Ba = Ba^* = 0\} = \{a \in A : aB = a^*B = 0\}$.

(iv) Let $b \in B$ and $a \in \mathcal{J}(A) \backslash B$. If a is a *-element, then $b \in Aa \cap aA$. If, on the other hand, a is not a *-element, then $b \in A(a + a^*) \cap (a + a^*) A$. □

Lemma 8. *Let A be a direct sum of rings, $A = A_1 \oplus A_2 \oplus \ldots \oplus A_n$, and let B be a biideal of A. There exist biideals B_k of A_k, $k = 1, 2, \ldots, n$, such that $B \subseteq B_1 \oplus B_2 \oplus \ldots \oplus B_n$. In particular, if B is a minimal biideal of A, then there exist minimal biideals B_k of A_k, $k = 1, 2, \ldots, n$, such that $B \subseteq B_1 \oplus B_2 \oplus \ldots \oplus B_n$.*

Proof. For each $k = 1, 2, \ldots, n$, consider the epimorphism $\pi_k : A_1 \oplus A_2 \oplus \ldots \oplus A_n \to A_k$ given by $\pi_k((a_1, a_2, \ldots, a_n)) = a_k$ and let $\pi_k(B) = B_k$. Then B_k is a biideal of A_k. For $b = (b_1, b_2, \ldots, b_n) \in B$, $\pi_k(b) = b_k$ and hence $b \in B_1 \oplus B_2 \oplus \ldots \oplus B_n$. Therefore $B \subseteq B_1 \oplus B_2 \oplus \ldots \oplus B_n$. Clearly, if B is a minimal biideal of A, then $\pi_k(B) = B_k$ is a minimal biideal of A_k, $k = 1, 2, \ldots, n$. □

For any prime p, let A_p denote, as usual, the p-component of an involution ring A. In addition, an involution ring A is said to be a *CI*-involution ring if every idempotent in A is central. Now we are in a position to give the following classification theorem.

Theorem 9. *Let A be a CI-involution ring with descending chain condition on *-biideals. Then A is *-bi-subdirectly irreducible if and only if A is one of the following rings:*

(i) *A is a division ring with involution;*

(ii) *$A \cong D \oplus D^{op}$, where D is a division ring and $D \oplus D^{op}$ is endowed with the exchange involution;*

(iii) *A is a local involution ring of characteristic p^n (p prime and $n \geqslant 1$) with nonzero nilpotent Jacobson radical, having a unique minimal *-biideal;*

(iv) *$A \cong L \oplus L^{op}$ where each of the rings L and L^{op} is a local ring of characteristic 2^n $(n \geqslant 1)$ with nonzero nilpotent Jacobson radical having a unique minimal biideal and $L \oplus L^{op}$ is endowed with the exchange involution;*

(v) *A is a nilpotent involution p-ring (p prime) having a unique minimal *-biideal.*

Proof. First we prove the direct implication. It is well-known that an involution ring A has d.c.c. on *-biideals if and only if it is an artinian ring with artinian Jacobson radical $\mathcal{J}(A)$ and $\mathcal{J}(A)$ is nilpotent. Moreover, $A = F \oplus T$, where the *-ideal T is the maximal torsion ideal of A and F is a torsion-free *-ideal with identity and $\mathcal{J}(A) \subseteq T$ ([2], [4], [10]). Our assumption on A implies that the intersection of all nonzero *-biideals of A is a nonzero *-biideal and either $A = T = A_p$, for some prime p, or $A = F$. Suppose that $A = A_p$. Since A is artinian, either A_p has a nonzero idempotent or A_p is nilpotent. First, we consider the case when A_p has a nonzero idempotent. Then A_p has a nonzero idempotent e which is a *-element. Then e must be the identity of A_p. Indeed, if e is not the identity of A_p, then eA_p and $(1-e)A_p = \{a - ea : a \in A_p\}$ are nonzero *-biideals with zero intersection, contradicting our assumption. If e is the only nonzero idempotent in A_p, then, A_p, being artinian without nontrivial idempotents, is a local ring of characteristic p^n, for some integer $n \geqslant 1$, having a unique minimal *-biideal, and so (i) or (iii) holds.

If there is another nonzero idempotent element $f \neq e$ in A_p, then f is not a *-element and $ff^* = 0$. Indeed, if $ff^* \neq 0$, then $ff^* = 1$ and so $f = ff^*$, which is a contradiction with the fact that f is not a *-element. Likewise, $f^*f = 0$. Hence $f + f^*$ is the identity element of A_p. Furthermore, $A_p = fA_p \oplus f^*A_p$, where f and f^* are the only nonzero idempotents in fA_p and f^*A_p, respectively. Hence each of the ideals fA_p and f^*A_p is a local ring of characteristic p^n $(n \geqslant 1)$ with nilpotent Jacobson radical, having a unique minimal biideal. Thus (ii) or (iv) holds.

Notice that if $p \neq 2$ and S is the unique minimal biideal of fA_p, then $\{a + a^* : a \in S\}$ and $\{a - a^* : a \in S\}$ are two distinct minimal *-biideals of A_p. If A_p is nilpotent, then (v) holds. Suppose now that $A = F$. From Proposition 1 and the fact that A is torsion-free, it follows that A is either a division ring of characteristic zero or $A \cong D \oplus D^{op}$, where D is a division ring of characteristic zero and $D \oplus D^{op}$ is endowed with the exchange involution.

Conversely, it is clear that the involution rings in (i) and (ii) are *-bi-subdirectily irreducible (see [12]), and so are the involution rings in (iii) and (v). Taking into consideration Lemma 8, the involution rings in (iv) have a unique minimal *-biideal, so the descending chain condition on *biideals implies that these are *-bi-subdirectly irreducible. □

3. Involution rings with unique maximal biideal

The next proposition states that an involution ring with identity which has a unique maximal biideal B is a local involution ring with Jacobson radical B. The proof is an easy adaptation of the well-known result that if a ring A with identity has a unique maximal right ideal R, then R is in fact an ideal of A and $R = \mathcal{J}(A)$.

Proposition 10. *Let A be an involution ring with identity. If A has a unique maximal biideal B, then B is a *-ideal of A and $B = \mathcal{J}(A)$.*

Proof. Let $a \in A$. Then Ba is a biideal of A. If $Ba \neq A$, then Ba is contained in a maximal biideal of A. Indeed, it is easily deduced, using Zorn's Lemma, that every biideal is contained in a maximal biideal. Since B is the unique maximal biideal of A, $Ba \subseteq B$. On the other hand, if $Ba = A$, then $ba = 1$ and $b'a = a$ for certain $b, b' \in B$. Now $0 \neq ab = b'ab \in B$; hence $ab \neq 1$ and $1 - b'$ is not invertible and so $A(1 - b') \neq A$. But then $A(1 - b')$ is contained in a maximal biideal; that is, $1 - b' \in A(1 - b') \subseteq B$, whence $1 \in B$, which is a contradiction. Thus $Ba = A$ is impossible and so B is a right ideal of A. Since every right ideal is a biideal, we have that B is the unique maximal right ideal of A. As is well-known, B is therefore an ideal of A, it is also the unique maximal left ideal of A and $B = \mathcal{J}(A)$ is a *-ideal of A. □

Corollary 11. *A ring A with identity has a unique maximal biideal B if and only if it has a unique maximal right (left) ideal.*

Proof. The direct implication was proved in the previous proposition. Conversely, let A have a unique maximal right ideal R and let B_1 be a maximal biideal of A. Then $B_1 \subseteq B_1 A \subseteq R$ and, since a right ideal is also a biideal, the maximality of B_1 implies that $B_1 = R$. □

We now terminate with a result which permits us to conclude that an involution ring with identity having a unique maximal *-biideal may not be a local ring.

Proposition 12. *If B is a maximal *-biideal of an involution ring A with identity, then one of the following conditions holds:*

(i) *B is a maximal biideal of A;*
(ii) *there exist maximal biideals K and K^* of A such that $B = K \cap K^*$.*

Proof. Let B be a maximal *-biideal of A. If B is not a maximal biideal of A, then B is contained in a maximal biideal K of A. Since B is closed under involution, B is also contained in K^*. Now $B \subseteq K \cap K^*$, where $K \cap K^*$ is a *-biideal of A. The maximality of B now implies that $B = K \cap K^*$. $\square$

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Contact information

D. I. C. Mendes Department of Mathematics, University of Beira Interior, Covilhã, Portugal
E-Mail(s): `imendes@ubi.pt`

Received by the editors: 31.10.2012
and in final form 04.07.2015.