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RESEARCH ARTICLE

On tensor products of indecomposable representations of the cyclic group of prime order p over the ring $\mathbb{Z}_p[[x]]$

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Abstract. Let C_p be a cyclic group of prime order p, let $R = \mathbb{Z}_p[[x]]$ be the ring of formal power series in one indeterminate x with coefficients in the ring $\mathbb{Z}_p$ of p-adic integers, and let RC_p be the corresponding group ring. We determine the decompositions into indecomposable summands of the tensor products over R of all indecomposable RC_p-modules that are free as R-modules. The resulting formulas cover tensor products involving the regular and cyclotomic modules, as well as two infinite families of mutually contragredient indecomposable modules of R-rank p, including tensor squares and mixed products. For odd p, the proof is based on exact sequences, suitable reductions of modules, and decompositions of symmetric and exterior squares. The case $p = 2$ is covered by earlier direct computations.

Introduction

Tensor products of representations provide a standard construction in the representation theory of finite groups, and their decomposition into indecomposable summands is a natural problem. Over splitting fields of characteristic zero, this problem is closely related to character theory

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and the structure of representation rings. Over non-field coefficient rings, however, the situation is considerably more complicated, since representations need not be completely reducible and the decomposition problem depends essentially on the arithmetic structure of the coefficient ring.

The systematic study of such decomposition problems in the modular case is naturally formulated in terms of the Green ring, or modular representation algebra, whose addition and multiplication are induced by direct sums and tensor products, respectively. This approach was developed by J. A. Green in [1] and became a standard framework for the study of tensor products in modular representation theory, including representations of cyclic p-groups.

In the integral p-adic setting, tensor products of representations of finite groups were investigated in a series of works by P. M. Gudivok, V. P. Rud'ko, and their collaborators. In particular, they obtained decomposition formulas for tensor products of certain indecomposable representations of cyclic p-groups over the ring of p-adic integers. An early contribution in this direction is due to V. P. Rud'ko [2], who studied the rational tensor algebra of modular representations of cyclic p-groups. Further results concerning tensor products of integral p-adic representations of finite groups were obtained by P. M. Gudivok, S. F. Goncharova and V. P. Rud'ko [3]. The methods and results developed in this line of research were subsequently presented in the monograph by P. M. Gudivok and V. P. Rud'ko [4] devoted to tensor products of representations of finite groups. These works provide an important background for decomposition problems for tensor products over p-adic and related local coefficient rings.

The present paper extends this line of research to representations of the cyclic group of prime order p over the formal power series ring $R = \mathbb{Z}_p[[x]]$, where $\mathbb{Z}_p$ is the ring of p-adic integers. We determine the decompositions into indecomposable summands of the tensor products of all such indecomposable representations that are free as R-modules. The first three decomposition formulas in Theorem 1 follow from the results of Rud'ko [2]. The remaining formulas, involving the two infinite families of mutually contragredient indecomposable representations of R-rank p, are established in the present paper and include tensor squares and mixed tensor products.

The proof proceeds in several steps. First, suitable exact sequences relating the indecomposable modules are established and used, together with Schanuel's lemma and the Krull–Schmidt cancellation theorem, to

determine several products involving the trivial, cyclotomic, and regular modules. Reduction modulo powers of x is then used to treat tensor products of modules with distinct indices. For odd p, the tensor squares are obtained by decomposing them into their symmetric and exterior squares and determining these summands separately. The remaining mixed products with equal indices are then derived from the exact sequences, while the case $p = 2$ follows from the direct computations in [5].

This paper is dedicated to the memory of V. P. Rud'ko. The ideas developed here originated in his memorable talks delivered over the years at the algebra seminar of Uzhhorod University and in numerous discussions of the problems he proposed.

1. Preliminaries and main result

Throughout the paper, let p be a prime, let $R = \mathbb{Z}_p[[x]]$, let $C_p = \langle a \rangle$ be the cyclic group of order p, and let RC_p be its group ring over R. As usual, we identify the identity element of C_p and the identity element of R with the identity element 1 of the group ring RC_p. Let $\Gamma = RC_p$ denote the regular RC_p-module. For each positive integer i, let Γ_i be the RC_p-module that is free as an R-module with R-basis $u_1, \ldots, u_p$, with respect to which the corresponding matrix representation of C_p is given by

$$a \mapsto \begin{pmatrix} \widetilde{\varepsilon} & x^i e_1^{\mathsf{T}} \\ 0 & 1 \end{pmatrix}, \tag{1}$$

where $\widetilde{\varepsilon}$ is the companion matrix of the p-th cyclotomic polynomial

$$\Phi_p(y) = y^{p-1} + \cdots + y + 1, \quad e_1 = (1, 0, \ldots, 0) \in R^{p-1}.$$

For $i = 0$, the module arising from this construction is isomorphic to the regular RC_p-module; more precisely, $\Gamma_0 \cong \Gamma$. With respect to the above basis, the action of a is given by

$$\begin{aligned} au_j &= u_{j+1}, \qquad j = 1, \ldots, p-2, \\ au_{p-1} &= -u_1 - \cdots - u_{p-1}, \\ au_p &= x^i u_1 + u_p. \end{aligned} \tag{2}$$

In what follows, let Γ_i^* denote the RC_p-module contragredient to Γ_i, let Δ_0 denote the RC_p-module affording the trivial R-representation, and let $\Delta_1 = \langle u_1, \ldots, u_{p-1} \rangle$ denote the RC_p-submodule of Γ_i whose corresponding R-representation is given by $a \mapsto \widetilde{\varepsilon}$.

It was shown in [6] that the RC_p-modules Δ_0, Δ_1, Γ, Γ_i, and Γ_i^* ($i \in \mathbb{N}$) form a complete list of pairwise non-isomorphic indecomposable RC_p-modules affording R-representations of the group C_p.

For an arbitrary RC_p-module M and any positive integer n, we shall use nM throughout to denote the external direct sum

$$\underbrace{M \oplus \cdots \oplus M}_{n \text{ copies}}.$$

The notation $0M$ will indicate the absence of such a summand in a decomposition.

Since Δ_0 is the trivial RC_p-module of R-rank one, for every RC_p-module M that is free as an R-module, we have

$$\Delta_0 \otimes_R M \cong M.$$

Theorem 1. *For arbitrary positive integers i and j with $i \leq j$, the following decompositions of tensor products of indecomposable RC_p-modules that are free as R-modules hold:*

$$\begin{gathered}
\Delta_1 \otimes_R \Delta_1 \cong \Delta_0 \oplus (p-2)\Gamma, \\
\Delta_1 \otimes_R \Gamma \cong (p-1)\Gamma, \\
\Gamma \otimes_R \Gamma \cong p\Gamma, \\
\Delta_1 \otimes_R \Gamma_i \cong \Gamma_i^* \oplus (p-2)\Gamma, \\
\Delta_1 \otimes_R \Gamma_i^* \cong \Gamma_i \oplus (p-2)\Gamma, \\
\Gamma \otimes_R \Gamma_i \cong \Gamma \otimes_R \Gamma_i^* \cong p\Gamma, \\
\Gamma_i \otimes_R \Gamma_j \cong \Gamma_i \oplus \Gamma_i^* \oplus (p-2)\Gamma, \\
\Gamma_i \otimes_R \Gamma_j^* \cong \Gamma_i^* \otimes_R \Gamma_j \cong \Gamma_i^* \otimes_R \Gamma_j^* \cong \Gamma_i \otimes_R \Gamma_j.
\end{gathered}$$

Remark 1. Since the tensor product over R is symmetric up to RC_p-module isomorphism, products that differ only by the order of their factors are not listed separately. The first three formulas are included for completeness and follow from the results of Rud'ko [2].

Example 1. Let $p = 3$. Then, for arbitrary positive integers i and j with $i \leq j$, the decomposition formulas in Theorem 1 involving the modules Γ_i and Γ_i^* take the form

$$\Delta_1 \otimes_R \Gamma_i \cong \Gamma_i^* \oplus \Gamma, \qquad \Delta_1 \otimes_R \Gamma_i^* \cong \Gamma_i \oplus \Gamma,$$

$$\Gamma \otimes_R \Gamma_i \cong \Gamma \otimes_R \Gamma_i^* \cong 3\Gamma,$$

and

$$\Gamma_i \otimes_R \Gamma_j \cong \Gamma_i \otimes_R \Gamma_j^* \cong \Gamma_i^* \otimes_R \Gamma_j \cong \Gamma_i^* \otimes_R \Gamma_j^* \cong \Gamma_i \oplus \Gamma_i^* \oplus \Gamma.$$

2. Proof of the main result

Lemma 1. *There exist exact sequences of RC_p-modules*

$$0 \longrightarrow \Delta_0 \longrightarrow \Gamma \longrightarrow \Delta_1 \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow \Delta_1 \longrightarrow \Gamma \longrightarrow \Delta_0 \longrightarrow 0. \tag{4}$$

Moreover, for every positive integer i, there exist exact sequences of RC_p-modules

$$0 \longrightarrow \Gamma_i \longrightarrow \Gamma \oplus \Gamma \longrightarrow \Gamma_i^* \longrightarrow 0, \tag{5}$$

$$0 \longrightarrow \Gamma_i^* \longrightarrow \Gamma \oplus \Gamma \longrightarrow \Gamma_i \longrightarrow 0. \tag{6}$$

Proof. To prove the existence of the exact sequences (3) and (4), consider the R-submodules Δ_0' and Δ_1' of the regular RC_p-module Γ with R-bases

$$1 + a + a^2 + \cdots + a^{p-1}$$

and

$$a - 1, \quad a^2 - a, \quad \ldots, \quad a^{p-1} - a^{p-2},$$

respectively. A direct verification shows that Δ_0' and Δ_1' are RC_p-submodules of Γ and that the following isomorphisms hold:

$$\Delta_0' \cong \Delta_0, \qquad \Delta_1' \cong \Delta_1,$$

$$\Gamma/\Delta_0' \cong \Delta_1, \qquad \Gamma/\Delta_1' \cong \Delta_0.$$

This proves (3) and (4).

To prove the existence of the exact sequence (5), consider, for an arbitrary fixed positive integer i, the RC_p-submodule Γ_i' of the free RC_p-module $\Gamma \oplus \Gamma$ with RC_p-basis $e_1' = (1, 0)$, $e_2' = (0, 1)$, generated by the elements $(a-1)e_1'$ and $x^i e_1' + (1 + a + \cdots + a^{p-1})e_2'$. The RC_p-submodule Γ_i' is a free R-module with R-basis

$$u_1' = (a-1)e_1', \quad u_2' = (a^2 - a)e_1', \quad \ldots, \quad u_{p-1}' = (a^{p-1} - a^{p-2})e_1',$$
$$u_p' = x^i e_1' + (1 + a + \cdots + a^{p-1})e_2'.$$

It is easy to verify that, with respect to the R-basis $u_1', \ldots, u_p'$, the matrix of the action of a on Γ_i' coincides with the matrix in (1). Therefore, the correspondence $u_k \mapsto u_k'$ for $k = 1, \ldots, p$ defines an RC_p-module isomorphism $\Gamma_i \cong \Gamma_i'$.

In the quotient module $(\Gamma \oplus \Gamma)/\Gamma'_i$, the elements

$$u''_1 = -e'_1 + \Gamma'_i, \quad u''_2 = e'_2 + \Gamma'_i, \quad \ldots, \quad u''_p = a^{p-2}e'_2 + \Gamma'_i$$

form an R-basis with respect to which the matrix R-representation of C_p is given by

$$a \mapsto \begin{pmatrix} 1 & x^i e_{p-1} \\ 0 & \widetilde{\varepsilon} \end{pmatrix}, \tag{7}$$

where $e_{p-1} = (0, \ldots, 0, 1) \in R^{p-1}$. On the other hand, the contragredient module Γ_i^* admits an R-basis with respect to which the action of a is given by the same matrix. Hence, $(\Gamma \oplus \Gamma)/\Gamma'_i \cong \Gamma_i^*$. This completes the proof of the existence of the exact sequence (5).

Finally, to prove the existence of the exact sequence (6), we proceed similarly. Let Θ_i be the RC_p-submodule of $\Gamma \oplus \Gamma$ generated by the elements $(1 + a + \cdots + a^{p-1})e'_1$ and $(a-1)e'_2 + x^i e'_1$. Indeed, the elements $v'_1 = (1 + a + \cdots + a^{p-1})e'_1$ and $v'_k = (a^{k-1} - a^{k-2})e'_2 + x^i a^{k-2} e'_1$ for $k = 2, \ldots, p$ form an R-basis of Θ_i, with respect to which the matrix R-representation of C_p coincides with (7). Hence, $\Theta_i \cong \Gamma_i^*$.

Moreover, in the quotient module $(\Gamma \oplus \Gamma)/\Theta_i$, the elements

$$e'_1 + \Theta_i, \quad ae'_1 + \Theta_i, \quad \ldots, \quad a^{p-2}e'_1 + \Theta_i, \quad -e'_2 + \Theta_i$$

form an R-basis with respect to which the matrix R-representation of C_p coincides with (1). Therefore, $(\Gamma \oplus \Gamma)/\Theta_i \cong \Gamma_i$. This completes the proof of the existence of the exact sequence (6). □

Lemma 2. *Let M be an RC_p-module that is free as an R-module of rank n. Then*

$$\Gamma \otimes_R M \cong n\Gamma.$$

Proof. The assertion follows directly from the proof of [4, Lemma 3.1]. □

Corollary 1. *Let i be a positive integer. Then*

$$\Gamma \otimes_R \Gamma_i \cong \Gamma \otimes_R \Gamma_i^* \cong p\Gamma.$$

Proof. Since the RC_p-modules Γ_i and Γ_i^* are free R-modules of rank p, the result follows immediately from Lemma 2. □

Lemma 3. *For every positive integer i, one has*

$$\Delta_1 \otimes_R \Gamma_i \cong \Gamma_i^* \oplus (p-2)\Gamma \tag{8}$$

and

$$\Delta_1 \otimes_R \Gamma_i^* \cong \Gamma_i \oplus (p-2)\Gamma. \tag{9}$$

Proof. Tensoring the exact sequence (4) over R with Γ_i, we obtain an exact sequence

$$0 \longrightarrow \Delta_1 \otimes_R \Gamma_i \longrightarrow \Gamma \otimes_R \Gamma_i \longrightarrow \Delta_0 \otimes_R \Gamma_i \longrightarrow 0.$$

Since $\Delta_0 \otimes_R \Gamma_i \cong \Gamma_i$ and $\Gamma \otimes_R \Gamma_i \cong p\Gamma$, this gives

$$0 \longrightarrow \Delta_1 \otimes_R \Gamma_i \longrightarrow p\Gamma \longrightarrow \Gamma_i \longrightarrow 0. \tag{10}$$

On the other hand, taking the direct sum of (6) with the split exact sequence

$$0 \longrightarrow (p-2)\Gamma \longrightarrow (p-2)\Gamma \longrightarrow 0 \longrightarrow 0,$$

we obtain

$$0 \longrightarrow \Gamma_i^* \oplus (p-2)\Gamma \longrightarrow p\Gamma \longrightarrow \Gamma_i \longrightarrow 0. \tag{11}$$

Since $p\Gamma$ is a free, hence projective, RC_p-module, Schanuel's lemma applied to the exact sequences (10) and (11) gives

$$(\Delta_1 \otimes_R \Gamma_i) \oplus p\Gamma \cong \big(\Gamma_i^* \oplus (p-2)\Gamma\big) \oplus p\Gamma.$$

Therefore, by the Krull–Schmidt cancellation theorem, we obtain (8).

The second isomorphism is obtained in the same way, replacing Γ_i by Γ_i^* and using the exact sequence (5) instead of (6). Alternatively, it follows by dualizing the first isomorphism and using the standard identification $(M \otimes_R N)^* \cong M^* \otimes_R N^*$ for R-free modules of finite rank, together with $\Delta_1^* \cong \Delta_1$, $\Gamma^* \cong \Gamma$, and $(\Gamma_i^*)^* \cong \Gamma_i$. □

Lemma 4. *Let i and j be positive integers such that $i < j$. Then*

$$\Gamma_i \otimes_R \Gamma_j \cong \Gamma_i \oplus \Gamma_i^* \oplus (p-2)\Gamma. \tag{12}$$

Proof. We shall use the following standard fact. Let M be an RC_p-module which is free as an R-module of finite rank, and let N be a projective RC_p-submodule of M. If an R-basis of N can be extended to an R-basis of M, then the inclusion $N \hookrightarrow M$ is R-split. Hence, since projective RC_p-modules are relatively injective with respect to R-split exact sequences, N is an RC_p-direct summand of M.

For each positive integer k, the module Γ_k contains Δ_1 as an RC_p-submodule whose R-basis can be extended to an R-basis of Γ_k. Therefore, the tensor product $\Gamma_i \otimes_R \Gamma_j$ contains the RC_p-submodule $\Delta_1 \otimes_R \Delta_1$, whose R-basis can be extended to an R-basis of $\Gamma_i \otimes_R \Gamma_j$.

Since $\Delta_1 \otimes_R \Delta_1 \cong \Delta_0 \oplus (p-2)\Gamma$, the module $\Gamma_i \otimes_R \Gamma_j$ contains a projective RC_p-submodule isomorphic to $(p-2)\Gamma$, with an R-basis extendable to an R-basis of $\Gamma_i \otimes_R \Gamma_j$. Hence, by the standard fact recalled above, $(p-2)\Gamma$ is an RC_p-direct summand of $\Gamma_i \otimes_R \Gamma_j$.

After splitting off the direct summand $(p-2)\Gamma$, it follows from the classification of R-free indecomposable RC_p-modules that each remaining indecomposable direct summand of $\Gamma_i \otimes_R \Gamma_j$ is isomorphic to one of the modules Δ_0, Δ_1, Γ, Γ_k, Γ_k^* ($k \in \mathbb{N}$).

For a positive integer j, put $R_j = R/Rx^j$. If M is an RC_p-module, then by $M^{(j)}$ we denote the R_jC_p-module $M^{(j)} = M/x^jM$. Clearly, for every positive integer $k \geq j$, one has

$$\Gamma_k^{(j)} \cong \Delta_1^{(j)} \oplus \Delta_0^{(j)} \cong \Delta_1^{(j)} \oplus \Delta_0. \tag{13}$$

Reducing the isomorphism (8) modulo x^j, we obtain, for every positive integer i,

$$\big(\Delta_1 \otimes_R \Gamma_i\big)^{(j)} \cong (\Gamma_i^*)^{(j)} \oplus (p-2)R_jC_p. \tag{14}$$

Combining the isomorphisms (13) and (14), and using the natural isomorphism

$$(\Gamma_i \otimes_R \Gamma_j)^{(j)} \cong (\Gamma_i)^{(j)} \otimes_{R_j} (\Gamma_j)^{(j)},$$

we obtain

$$\begin{aligned}(\Gamma_i \otimes_R \Gamma_j)^{(j)} &\cong (\Gamma_i)^{(j)} \otimes_{R_j} \big(\Delta_1^{(j)} \oplus \Delta_0\big)\\ &\cong (\Gamma_i)^{(j)} \oplus (\Gamma_i^*)^{(j)} \oplus (p-2)R_jC_p.\end{aligned}$$

Comparing this reduction with the possible reductions of the remaining indecomposable summands listed above, and using the Krull–Schmidt theorem, we conclude that these summands are Γ_i and Γ_i^*. Thus we obtain (12). □

Corollary 2. *Under the same assumptions on i and j, one has*

$$\Gamma_i \otimes_R \Gamma_j^* \cong \Gamma_i^* \otimes_R \Gamma_j \cong \Gamma_i^* \otimes_R \Gamma_j^* \cong \Gamma_i \otimes_R \Gamma_j.$$

Proof. It remains to prove the mixed case. Repeating the reduction argument used in the proof of (12), we note that

$$(\Gamma_j^*)^{(j)} \cong \Delta_0 \oplus \Delta_1^{(j)}.$$

Hence

$$\begin{aligned}(\Gamma_i \otimes_R \Gamma_j^*)^{(j)} &\cong (\Gamma_i)^{(j)} \otimes_{R_j} (\Gamma_j^*)^{(j)} \\ &\cong (\Gamma_i)^{(j)} \otimes_{R_j} \left(\Delta_0 \oplus \Delta_1^{(j)}\right) \\ &\cong (\Gamma_i)^{(j)} \oplus (\Gamma_i^*)^{(j)} \oplus (p-2)R_jC_p.\end{aligned}$$

As in the proof of (12), comparison with the possible reductions of the remaining indecomposable direct summands gives

$$\Gamma_i \otimes_R \Gamma_j^* \cong \Gamma_i \oplus \Gamma_i^* \oplus (p-2)\Gamma.$$

Dualizing the two decompositions obtained above and using the standard identifications recalled earlier, we get

$$\Gamma_i^* \otimes_R \Gamma_j^* \cong \Gamma_i \otimes_R \Gamma_j, \qquad \Gamma_i^* \otimes_R \Gamma_j \cong \Gamma_i \otimes_R \Gamma_j^*.$$

Thus all four tensor products are isomorphic. □

To study the tensor square $\Gamma_i \otimes_R \Gamma_i$, we consider the symmetric square $\mathrm{S}^2_R(\Gamma_i)$ and the exterior square $\bigwedge^2_R(\Gamma_i)$ of the R-module Γ_i. We recall the definitions of these notions for an arbitrary R-module M; see [7].

Let M be an R-module. The symmetric square of M is defined by

$$\mathrm{S}^2_R(M) = (M \otimes_R M) \big/ \langle \{u \otimes v - v \otimes u \mid u, v \in M\} \rangle_R .$$

The image of $u \otimes v$ in $\mathrm{S}^2_R(M)$ is denoted by $u \odot v$.

The exterior square of M is defined by

$$\textstyle\bigwedge^2_R(M) = (M \otimes_R M) \big/ \langle \{u \otimes u \mid u \in M\} \rangle_R .$$

The image of $u \otimes v$ in $\bigwedge^2_R(M)$ is denoted by $u \wedge v$.

Assume from now on that p is odd. Since $2 \in R^*$, the following well-known isomorphism holds:

$$M \otimes_R M \cong \mathrm{S}^2_R(M) \oplus \textstyle\bigwedge^2_R(M).$$

The generator a of the group C_p acts on $\mathrm{S}^2_R(M)$ and $\bigwedge^2_R(M)$ according to the formulas

$$a(u \odot v) = (au) \odot (av), \qquad a(u \wedge v) = (au) \wedge (av),$$

where $u, v \in M$. Thus, $\mathrm{S}^2_R(M)$ and $\bigwedge^2_R(M)$ acquire natural RC_p-module structures.

Lemma 5. *Let p be an odd prime. Then*

$$\mathrm{S}^2_R(\Delta_1) \cong \tfrac{p-1}{2}\Gamma, \qquad \bigwedge\nolimits^2_R(\Delta_1) \cong \Delta_0 \oplus \tfrac{p-3}{2}\Gamma.$$

Proof. Let P be the ideal of R generated by p and x. Then $R/P \cong \mathbb{F}_p$, where $\mathbb{F}_p$ is the finite field with p elements. For each k with $1 \leq k \leq p$, let V_k denote the indecomposable $\mathbb{F}_pC_p$-module of dimension k over $\mathbb{F}_p$. For an RC_p-module M, denote by $\overline{M}$ the quotient $\mathbb{F}_pC_p$-module $\overline{M} = M/PM$.

It is straightforward to verify that

$$\overline{\Delta_0} \cong V_1, \qquad \overline{\Delta_1} \cong V_{p-1}, \qquad \overline{\Gamma} \cong V_p, \qquad \overline{\Gamma_i} \cong \overline{\Gamma_i^*} \cong V_1 \oplus V_{p-1}$$

for every positive integer i.

Since

$$\overline{\mathrm{S}^2_R(\Delta_1)} \cong \mathrm{S}^2_{\mathbb{F}_p}\big(\overline{\Delta_1}\big) \cong \mathrm{S}^2_{\mathbb{F}_p}(V_{p-1}),$$

and, by [8],

$$\mathrm{S}^2_{\mathbb{F}_p}(V_{p-1}) \cong \tfrac{p-1}{2}V_p,$$

it follows that

$$\overline{\mathrm{S}^2_R(\Delta_1)} \cong \tfrac{p-1}{2}V_p.$$

By the classification of indecomposable R-free RC_p-modules recalled above and the Krull–Schmidt theorem, this implies $\mathrm{S}^2_R(\Delta_1) \cong \frac{p-1}{2}\Gamma$. Hence, we obtain the first isomorphism in the statement of the lemma.

Finally, since, on the one hand, $\Delta_1 \otimes_R \Delta_1 \cong \Delta_0 \oplus (p-2)\Gamma$, and, on the other hand,

$$\Delta_1 \otimes_R \Delta_1 \cong \mathrm{S}^2_R(\Delta_1) \oplus \bigwedge\nolimits^2_R(\Delta_1),$$

the Krull–Schmidt theorem, together with the first isomorphism, yields

$$\bigwedge\nolimits^2_R(\Delta_1) \cong \Delta_0 \oplus \tfrac{p-3}{2}\Gamma.$$

Thus, the second isomorphism in the statement of the lemma also follows. □

Lemma 6. *Let p be an odd prime. Then, for every positive integer i,*

$$\mathrm{S}^2_R(\Gamma_i) \cong \Gamma_i \oplus \tfrac{p-1}{2}\Gamma.$$

Proof. It follows immediately from the construction of the RC_p-module Δ_1 that

$$\mathrm{S}^2_R(\Delta_1) \subset \mathrm{S}^2_R(\Gamma_i),$$

and that the elements $u_1 \odot u_p, \dots, u_p \odot u_p$ extend the R-basis

$$u_1 \odot u_1,\ u_1 \odot u_2,\ u_2 \odot u_2,\ \dots,\ u_1 \odot u_{p-1},\ \dots,\ u_{p-1} \odot u_{p-1}$$

of the RC_p-module $\mathrm{S}^2_R(\Delta_1)$ to an R-basis of the RC_p-module $\mathrm{S}^2_R(\Gamma_i)$.

Consider the quotient module

$$\Lambda = \mathrm{S}^2_R(\Gamma_i)/\mathrm{S}^2_R(\Delta_1),$$

and denote its elements of the form $u \odot v + \mathrm{S}^2_R(\Delta_1)$ by $\overline{u \odot v}$. Then, for $k = 1, \dots, p-2$, we have

$$a\overline{u_k \odot u_p} = \overline{au_k \odot au_p} = \overline{u_{k+1} \odot (u_p + x^i u_1)} = \overline{u_{k+1} \odot u_p},$$

since $x^i u_{k+1} \odot u_1 \in \mathrm{S}^2_R(\Delta_1)$. For $k = p-1$, we obtain

$$\begin{aligned} a\overline{u_{p-1} \odot u_p} &= \overline{au_{p-1} \odot au_p} \\ &= \overline{(-u_1 - \dots - u_{p-1}) \odot (u_p + x^i u_1)} \\ &= -\overline{u_1 \odot u_p} - \overline{u_2 \odot u_p} - \dots - \overline{u_{p-1} \odot u_p}. \end{aligned}$$

Finally, for $k = p$, we have

$$\begin{aligned} a\overline{u_p \odot u_p} &= \overline{au_p \odot au_p} \\ &= \overline{(u_p + x^i u_1) \odot (u_p + x^i u_1)} \\ &= \overline{u_p \odot u_p} + 2x^i \overline{u_1 \odot u_p}. \end{aligned}$$

The last equalities follow from the fact that both $x^i(u_1 + \dots + u_{p-1}) \odot u_1$ and $x^{2i} u_1 \odot u_1$ belong to $\mathrm{S}^2_R(\Delta_1)$.

It follows from the above computations that, with respect to the R-basis $\overline{u_1 \odot u_p}, \dots, \overline{u_p \odot u_p}$ of Λ, the action of a is represented by the matrix

$$\begin{pmatrix} \widetilde{\varepsilon} & 2x^i e_1^{\mathsf{T}} \\ 0 & 1 \end{pmatrix}.$$

By the assumptions of the lemma, $2 \in R^*$. Therefore, replacing the last basis element $\overline{u_p \odot u_p}$ by $2^{-1}\overline{u_p \odot u_p}$ yields the matrix representation (1). Hence, $\Lambda \cong \Gamma_i$. Since the R-basis of $\mathrm{S}^2_R(\Delta_1)$ extends to an R-basis of $\mathrm{S}^2_R(\Gamma_i)$, the short exact sequence

$$0 \longrightarrow \mathrm{S}^2_R(\Delta_1) \longrightarrow \mathrm{S}^2_R(\Gamma_i) \longrightarrow \Lambda \longrightarrow 0$$

splits over R. Moreover, $\mathrm{S}^2_R(\Delta_1) \cong \frac{p-1}{2}\Gamma$ is relatively injective with respect to R-split exact sequences. Hence

$$\mathrm{S}^2_R(\Gamma_i) \cong \mathrm{S}^2_R(\Delta_1) \oplus \Gamma_i,$$

which, together with Lemma 5, completes the proof of the lemma. □

Lemma 7. *Let p be an odd prime. Then, for every positive integer i,*

$$\bigwedge\nolimits^2_R(\Gamma_i) \cong \Gamma_i^* \oplus \tfrac{p-3}{2}\Gamma.$$

Proof. As in the proof of the preceding lemma, for every positive integer i, we have

$$\bigwedge\nolimits^2_R(\Delta_1) \subseteq \bigwedge\nolimits^2_R(\Gamma_i).$$

Moreover, adjoining the elements $u_1 \wedge u_p, \ldots, u_{p-1} \wedge u_p$ to the R-basis

$$u_1 \wedge u_2,\ u_1 \wedge u_3,\ u_2 \wedge u_3,\ \ldots,\ u_1 \wedge u_{p-1},\ \ldots,\ u_{p-2} \wedge u_{p-1}$$

of the RC_p-module $\bigwedge^2_R(\Delta_1)$ yields an R-basis of the RC_p-module $\bigwedge^2_R(\Gamma_i)$.

For a fixed integer j satisfying $1 < j < p$, we compute the orbit of the element $u_1 \wedge u_j$ under the action of the group C_p. For $k = 0, 1, \ldots, p-1$, we have

$$a^k(u_1 \wedge u_j) = \begin{cases} u_{k+1} \wedge u_{k+j}, & 0 \le k \le p-j-1, \\ -u_{p-j+1} \wedge s, & k = p-j, \\ u_{k+1} \wedge u_{k+j-p}, & p-j+1 \le k \le p-2, \\ -s \wedge u_{j-1}, & k = p-1, \end{cases} \tag{15}$$

where $s = u_1 + \cdots + u_{p-1}$. Now suppose that $p > 3$. It is straightforward to verify that the elements of the orbit in (15) are linearly independent over R. For $j = 2, \ldots, \frac{p-1}{2}$, set $v_j = u_1 \wedge u_j$. Since the orbit $C_p v_j$ is linearly independent over R, it forms an R-basis of the RC_p-submodule $RC_p v_j$. Therefore, the RC_p-homomorphism

$$\varphi_j \colon \Gamma \longrightarrow RC_p v_j, \qquad \alpha \longmapsto \alpha v_j,$$

is an isomorphism. Thus $RC_p v_j \cong \Gamma$. Next, let

$$M = RC_p v_2 + \cdots + RC_p v_{\frac{p-1}{2}}.$$

It follows from (15) that $u_k \wedge u_l \in M$ for all integers $1 \le k < l \le p-1$ such that $l - k \ne \frac{p-1}{2}$ and $l - k \ne \frac{p+1}{2}$. Let N denote the R-submodule of M whose R-basis consists of all such elements. One verifies directly that adjoining the elements

$$u_1 \wedge s,\ \ldots,\ u_{\frac{p-3}{2}} \wedge s,\ s \wedge u_{\frac{p+3}{2}},\ \ldots,\ s \wedge u_{p-1}$$

to the R-basis of N yields an R-basis of M.

The above complementary set can be replaced by one adapted to the action of C_p. Put

$$w_1 = u_1 \wedge u_{\frac{p+1}{2}} + u_1 \wedge u_{\frac{p+3}{2}}, \qquad w_2 = u_1 \wedge u_{\frac{p+3}{2}} + u_2 \wedge u_{\frac{p+3}{2}}.$$

By the definition of N, for $r = 0, 1, \ldots, \frac{p-5}{2}$, we have

$$a^r w_1 \equiv u_{r+1} \wedge s \pmod{N}, \qquad a^r w_2 \equiv s \wedge u_{\frac{p+3}{2}+r} \pmod{N}.$$

Consequently, adjoining the elements

$$w_1,\ aw_1,\ \ldots,\ a^{\frac{p-5}{2}} w_1,\ w_2,\ aw_2,\ \ldots,\ a^{\frac{p-5}{2}} w_2$$

to the R-basis of N yields an R-basis of M.

Since $\operatorname{rank}_R(RC_p v_j) = p$ for each $j = 2, \ldots, \frac{p-1}{2}$, and

$$\operatorname{rank}_R M = \sum_{j=2}^{\frac{p-1}{2}} \operatorname{rank}_R(RC_p v_j),$$

the natural epimorphism $\bigoplus_{j=2}^{\frac{p-1}{2}} RC_p v_j \longrightarrow M$ is an isomorphism, since its domain and codomain are free R-modules of the same finite rank. Hence

$$M = RC_p v_2 \oplus \cdots \oplus RC_p v_{\frac{p-1}{2}} \cong \tfrac{p-3}{2}\Gamma.$$

As in the proof of the preceding lemma, consider the quotient RC_p-module

$$\Lambda = \textstyle\bigwedge_R^2(\Gamma_i)/M,$$

and denote the coset $u \wedge v + M$ by $\overline{u \wedge v}$, for $u, v \in \Gamma_i$.

In the RC_p-module Λ, consider the following set of elements:

$$\overline{u_1 \wedge u_{\frac{p+1}{2}}},\ \overline{u_1 \wedge u_p},\ a\overline{u_1 \wedge u_p},\ \ldots,\ a^{p-2}\overline{u_1 \wedge u_p}. \tag{16}$$

Put $z = \overline{u_1 \wedge u_{\frac{p+1}{2}}}$, $y = \overline{u_1 \wedge u_p}$. Since $w_1, w_2 \in M$, we have

$$\begin{aligned} az &= \overline{u_2 \wedge u_{\frac{p+3}{2}}} = \overline{w_2 - u_1 \wedge u_{\frac{p+3}{2}}} \\ &= \overline{w_2 - w_1 + u_1 \wedge u_{\frac{p+1}{2}}} = \overline{u_1 \wedge u_{\frac{p+1}{2}}} = z. \end{aligned}$$

Since

$$a^k u_p = u_p + x^i(u_1 + \cdots + u_k), \qquad 0 \le k \le p-2,$$

we obtain

$$a^k y = \overline{u_{k+1} \wedge u_p} - x^i \sum_{r=1}^{k} \overline{u_r \wedge u_{k+1}}.$$

By the definition of N, all summands in the latter sum belong to $N \subset M$, except possibly when

$$k + 1 - r \in \left\{ \frac{p-1}{2}, \frac{p+1}{2} \right\}.$$

For $k > \frac{p-1}{2}$, the sum of the two remaining terms is $a^{k-\frac{p+1}{2}} w_2 \in M$. Hence

$$a^k y = \begin{cases} \overline{u_{k+1} \wedge u_p}, & 0 \leq k \leq \frac{p-3}{2}, \\ \overline{u_{\frac{p+1}{2}} \wedge u_p} - x^i z, & k = \frac{p-1}{2}, \\ \overline{u_{k+1} \wedge u_p}, & \frac{p+1}{2} \leq k \leq p-2. \end{cases}$$

Summing these equalities, we obtain

$$a^{p-1} y = -\overline{s \wedge u_p} = -\sum_{k=0}^{p-2} a^k y - x^i z.$$

The representatives of the elements in (16), together with the R-basis of M described above, form an R-basis of $\bigwedge_R^2(\Gamma_i)$. Therefore, the system (16) is an R-basis of Λ. With respect to this basis, the matrix representation of Λ is obtained from the matrix in (7) by replacing its upper-right block by its negative. Replacing z by $-z$, we obtain precisely the matrix in (7). Hence $\Lambda \cong \Gamma_i^*$. Moreover, the chosen representatives of the basis elements in (16) determine an R-linear section of the canonical projection

$$\bigwedge_R^2(\Gamma_i) \longrightarrow \Lambda.$$

Therefore, the short exact sequence

$$0 \longrightarrow M \longrightarrow \bigwedge_R^2(\Gamma_i) \longrightarrow \Lambda \longrightarrow 0$$

splits as a sequence of R-modules. Since RC_p is a symmetric R-algebra, the regular module $\Gamma = RC_p$ is relatively injective with respect to R-split exact sequences. Therefore, $M \cong \frac{p-3}{2}\Gamma$ is relatively injective, and the above sequence splits as a sequence of RC_p-modules. Consequently, for $p > 3$,

$$\bigwedge_R^2(\Gamma_i) \cong M \oplus \Lambda \cong \Gamma_i^* \oplus \tfrac{p-3}{2}\Gamma.$$

Finally, suppose that $p = 3$. Put

$$z = u_1 \wedge u_2, \qquad y = u_1 \wedge u_3.$$

Then

$$az = z, \qquad ay = u_2 \wedge u_3 - x^i z,$$

so that z, y, ay form an R-basis of $\bigwedge^2_R(\Gamma_i)$. Moreover,

$$a(ay) = -y - ay - x^i z.$$

After replacing z by $-z$, these relations coincide with those determining the matrix representation in (7) for $p = 3$. Therefore,

$$\bigwedge^2_R(\Gamma_i) \cong \Gamma_i^*.$$

This completes the proof of the lemma. □

We now complete the proof of Theorem 1 for odd p. Combining Lemmas 6 and 7, we obtain

$$\Gamma_i \otimes_R \Gamma_i \cong \mathrm{S}^2_R(\Gamma_i) \oplus \bigwedge^2_R(\Gamma_i) \cong \Gamma_i \oplus \Gamma_i^* \oplus (p-2)\Gamma.$$

It remains to consider the mixed product with equal indices. Tensoring the exact sequence (6) over R with Γ_i, and using Corollary 1, we obtain

$$0 \longrightarrow \Gamma_i \otimes_R \Gamma_i^* \longrightarrow 2p\Gamma \longrightarrow \Gamma_i \otimes_R \Gamma_i \longrightarrow 0.$$

On the other hand, taking the direct sum of the exact sequences (5) and (6) with the split exact sequence

$$0 \longrightarrow 0 \longrightarrow (p-2)\Gamma \longrightarrow (p-2)\Gamma \longrightarrow 0,$$

we obtain

$$0 \longrightarrow \Gamma_i \oplus \Gamma_i^* \longrightarrow (p+2)\Gamma \longrightarrow \Gamma_i \otimes_R \Gamma_i \longrightarrow 0.$$

Schanuel's lemma and the Krull–Schmidt cancellation theorem yield

$$\Gamma_i \otimes_R \Gamma_i^* \cong \Gamma_i \oplus \Gamma_i^* \oplus (p-2)\Gamma.$$

By symmetry and duality, the same decomposition holds for $\Gamma_i^* \otimes_R \Gamma_i$ and $\Gamma_i^* \otimes_R \Gamma_i^*$.

Thus, Theorem 1 holds for every odd prime p. The case $p = 2$ is verified by direct computation of the tensor products of the corresponding

matrix representations, as shown in [5]. This completes the proof of the theorem.

The possibility of obtaining a complete tensor product table in the present setting relies essentially on the classification of indecomposable R-free representations of the group C_p. As shown in [6], the representation problem for a finite p-group H over $\mathbb{Z}_p[[x]]$ is wild whenever $|H| > p$. Moreover, it is already wild for every nontrivial finite p-group over $\mathbb{Z}_p[[x_1, \ldots, x_m]]$ whenever $m > 1$. Thus, the construction of analogous complete tensor product tables for all indecomposable representations encounters the difficulties inherent in wild representation type. Possible extensions would therefore have to focus on particular explicitly described classes of modules.

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