

On some non-nilpotent Leibniz algebras of dimension 3 and their automorphism groups

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ABSTRACT. Let L be an algebra over a field F with the binary operations $+$ and $[\cdot]$. Then L is called a (left) Leibniz algebra if it satisfies the (left) Leibniz identity: $[a, [b, c]] = [[a, b], c] + [b, [a, c]]$ for all $a, b, c \in L$. A linear transformation f of L is called an endomorphism of L if $f([a, b]) = [f(a), f(b)]$ for all $a, b \in L$. A bijective endomorphism of L is called an automorphism of L . The main goal of this article is to describe the structure of automorphism groups of certain types of non-nilpotent three-dimensional Leibniz algebras over an arbitrary field F .

Introduction

Let L be an algebra over a field F with binary operations $+$ and $[\cdot]$. We say that L is a (left) *Leibniz algebra* if it satisfies the (left) Leibniz identity

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]]$$

for all elements $a, b, c \in L$ [2]. The term “Leibniz algebra” first appeared in the book [14] and in the article [15]. The modern theory of Leibniz algebras includes numerous results concerning their structure, classification, and properties; these results can be found, for instance, in the books [1] and [10].

We recall that Leibniz algebras form a broad generalization of Lie algebras. On the other hand, if L is a Leibniz algebra such that $[a, a] = 0$

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for every element $a \in L$, then L is a Lie algebra. Equivalently, Lie algebras can be characterized as anticommutative Leibniz algebras. Although many results for Lie algebras have analogues in the theory of Leibniz algebras, there are also substantial differences between these two classes of non-associative algebras; see, for example, the survey articles [3, 4, 12, 17].

Let f be a linear transformation of a Leibniz algebra L . The map f is called an *endomorphism* of L if $f([a, b]) = [f(a), f(b)]$ for all elements $a, b \in L$. A bijective endomorphism of L is called an *automorphism* of L . It is well known that the set $Aut_{[\cdot, \cdot]}(L)$ of all automorphisms of L forms a group under composition; see, for example, [9].

Describing the automorphism groups of algebraic structures is a fundamental and important problem in algebra, and Leibniz algebras are no exception. It is natural to begin with Leibniz algebras whose structure is already well understood. The first such class consists of one-generated Leibniz algebras. The automorphism groups of infinite-dimensional one-generated Leibniz algebras were described in [13], while the finite-dimensional case was considered in [9].

The next natural step is to consider low-dimensional Leibniz algebras. The case of dimension 2 is rather simple. Up to isomorphism, there are only two non-Lie Leibniz algebras of dimension 2 (see [4]); only non-zero products are displayed:

- $Lei_1(2, F) = Fa_1 \oplus Fa_2, \quad [a_1, a_1] = a_2;$
- $Lei_2(2, F) = Fa_1 \oplus Fa_2, \quad [a_1, a_1] = [a_1, a_2] = a_2.$

The automorphism groups of these Leibniz algebras were described in the paper [11].

The situation for three-dimensional Leibniz algebras is considerably more complicated. The most comprehensive description of such algebras over arbitrary fields can be found in [8]. Below we list the three-dimensional Leibniz algebras of the form $Fa_1 \oplus Fa_2 \oplus Fa_3$ whose automorphism groups have already been determined. Only non-zero products are displayed:

- $Lei_1(3, F) : \quad [a_1, a_1] = a_2, \quad [a_1, a_2] = a_3 \quad [11];$
- $Lei_2(3, F) : \quad [a_1, a_1] = a_3 \quad [11];$
- $Lei_3(3, F) : \quad [a_1, a_1] = [a_1, a_2] = a_3 \quad [6];$
- $Lei_4(3, F) : \quad [a_1, a_1] = a_3, \quad [a_2, a_2] = \lambda a_3, \quad \lambda \neq 0 \quad [7];$
- $Lei_5(3, F) : \quad [a_1, a_1] = a_3, \quad [a_1, a_2] = a_2 + \lambda a_3 \quad [5];$

- $Leib_6(3, F) : [a_1, a_1] = [a_2, a_1] = a_3$ [16].

In this paper, we extend our investigation to several further types of three-dimensional Leibniz algebras.

1. Preliminaries and remarks

We recall some necessary definitions.

Let L be a Leibniz algebra over a field F . The algebra L is called *abelian* if $[a, b] = 0$ for all $a, b \in L$. If A and B are subspaces of L , then $[A, B]$ denotes the subspace generated by all products $[a, b]$, where $a \in A$ and $b \in B$.

A subspace S of L is called a *subalgebra* of L if $[a, b] \in S$ for all $a, b \in S$.

A subalgebra I of L is called a *left ideal* of L if $[b, a] \in I$ for all $a \in I$ and $b \in L$. Similarly, I is called a *right ideal* of L if $[a, b] \in I$ for all $a \in I$ and $b \in L$. A subalgebra I of L is called an *ideal* of L if it is both a left and a right ideal.

Denote by $Leib(L)$ the subspace generated by all elements $[a, a]$, where $a \in L$. It is easy to verify that $Leib(L)$ is an ideal of L . This ideal is called the *Leibniz kernel* of L .

The *left center* $\zeta^{\text{left}}(L)$ and the *right center* $\zeta^{\text{right}}(L)$ of L are defined as follows:

$$\begin{aligned} \zeta^{\text{left}}(L) &= \{a \in L \mid [a, b] = 0 \text{ for each } b \in L\}, \\ \zeta^{\text{right}}(L) &= \{a \in L \mid [b, a] = 0 \text{ for each } b \in L\}. \end{aligned}$$

We observe that the left center of L is an ideal, whereas this is not necessarily true for the right center. The right center is a subalgebra of L ; in general, the left and right centers are distinct.

The *center* $\zeta(L)$ of L is defined as follows:

$$\zeta(L) = \{a \in L \mid [a, b] = 0 = [b, a] \text{ for each } b \in L\}.$$

The center is an ideal of L .

The *lower central series* of L is defined by

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \dots \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \dots \gamma_\tau(L),$$

where $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α , and

$$\gamma_\lambda(L) = \bigcap_{\mu < \lambda} \gamma_\mu(L)$$

for every limit ordinal λ . We say that L is *nilpotent* if there exists a positive integer k such that $\gamma_k(L) = \langle 0 \rangle$.

Let L be a Leibniz algebra over a field F , let A be a subalgebra of L , and let $G = \text{Aut}_{[\cdot, \cdot]}(L)$. We put

$$C_G(A) = \{ \alpha \in G \mid \alpha(x) = x \text{ for every } x \in A \}.$$

Let L be a non-nilpotent three-dimensional Leibniz algebra. As usual, we assume that L is not a Lie algebra, so that $\text{Leib}(L)$ is nonzero. Thus, there are two possible cases: $\dim_F(\text{Leib}(L)) = 2$ or $\dim_F(\text{Leib}(L)) = 1$.

Consider the first case. Let $K = \text{Leib}(L)$. Since L is not a Lie algebra, there exists an element a_1 such that $[a_1, a_1] = a_3 \neq 0$. By the structure of two-dimensional Leibniz algebras, either $[a_1, a_3] = [a_3, a_1] = 0$, or one can choose an element a_1 such that $[a_3, a_1] = 0$ and $[a_1, a_3] = a_3$. Put $A_1 = Fa_1 \oplus Fa_3$. The case in which the subalgebra A_1 is nilpotent was studied in [5]. We now suppose that the subalgebra A_1 is non-nilpotent, that is, $[a_3, a_1] = 0$ and $[a_1, a_3] = a_3$.

Since the subalgebra K is abelian of dimension 2, we have $K = Fa_3 \oplus Fb$ for some element b . It is not hard to see that the subalgebra $\langle a_3 \rangle$ is an ideal of K . Since an abelian Leibniz algebra is a Lie algebra, the assumption that the factor algebra L/Fa_3 is abelian leads to a contradiction with the equality $\text{Leib}(L) = Fa_3 \oplus Fb$. Hence, the factor algebra L/Fa_3 is non-abelian. Therefore, L/Fa_3 has a coset $c + Fa_3$ such that $\langle c, Fa_3 \rangle = \text{Leib}(L)$ and

$$[a_1 + Fa_3, c + Fa_3] = c + Fa_3.$$

Put $c = a_2$. Then $[a_1, a_2] = a_2 + \lambda a_3$ for some scalar $\lambda \in F$. Since $a_2 \in \text{Leib}(L)$, we have $[a_2, a_1] = 0$.

If $\lambda = 0$, then the subalgebra Fa_2 is an ideal of L , and we obtain the following type of Leibniz algebras:

$$\text{Leib}_7(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3, \text{ where}$$

$$[a_1, a_1] = [a_1, a_3] = a_3, [a_1, a_2] = a_2,$$

$$[a_2, a_1] = [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

In other words, $\text{Leib}_7(3, F) = L$ is the sum of the ideal $A_2 = Fa_2$ and the non-nilpotent one-generated Leibniz algebra $A_1 = Fa_1 \oplus Fa_3$ of dimension 2. Moreover,

$$[A_1, A_2] = A_2, \quad \text{Leib}(L) = Fa_2 \oplus Fa_3 = [L, L] = \zeta^{\text{left}}(L), \\ \zeta^{\text{right}}(L) = \zeta(L) = \langle 0 \rangle.$$

If $\lambda \neq 0$, then we obtain the following type of Leibniz algebras:

$$\begin{aligned} \text{Leis}(3, F) &= Fa_1 \oplus Fa_2 \oplus Fa_3, \text{ where} \\ [a_1, a_1] &= [a_1, a_3] = a_3, [a_1, a_2] = a_2 + \lambda a_3, \quad 0 \neq \lambda \in F, \\ [a_2, a_1] &= [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0. \end{aligned}$$

In other words, $\text{Leis}(3, F) = L$ is the sum of the ideal $\text{Leib}(L) = Fa_2 \oplus Fa_3$ and the non-nilpotent one-generated Leibniz algebra $A_1 = Fa_1 \oplus Fa_3$ of dimension 2. Moreover,

$$\begin{aligned} [A_1, \text{Leib}(L)] &= \text{Leib}(L), \quad \text{Leib}(L) = [L, L] = \zeta^{\text{left}}(L), \\ \zeta^{\text{right}}(L) &= \zeta(L) = \langle 0 \rangle. \end{aligned}$$

In the next section, we discuss the structure of the automorphism groups of these Leibniz algebras.

2. The automorphism groups of $\text{Lei}_7(3, F)$ and $\text{Lei}_8(3, F)$

Denote by Ξ the canonical monomorphism of $\text{Aut}_{[1]}(L)$ in $M_3(F)$.

Theorem 1. *Let G be an automorphism group of the Leibniz algebra $\text{Lei}_7(3, F)$. Then G is isomorphic to a subgroup of $GL_3(F)$ that consists of matrices of the form:*

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & 1 + \alpha_3 \end{pmatrix},$$

$\alpha_2, \alpha_3, \beta_2, \beta_3 \in F$. Furthermore, G is isomorphic to the group $GL_2(F)$. Moreover, it is a product of two subgroups $C_1 = C_G(a_1)$, $C_2 = C_G(a_2)$, and $\Xi(C_1)$ is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & \beta_3 & 1 \end{pmatrix},$$

$\beta_2, \beta_3 \in F$, and $\Xi(C_2)$ is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 & \alpha_2 \\ \alpha_3 & 0 & 1 + \alpha_3 \end{pmatrix},$$

$\alpha_2, \alpha_3 \in F$. A subgroup C_1 is a semidirect product of a normal subgroup C_3 , which is isomorphic to the additive group of a field F , and a subgroup C_4 , which is isomorphic to the multiplicative group of a field F ; a subgroup C_2 is a semidirect product of a normal subgroup, which is isomorphic to the additive group of a field F , and a subgroup that is isomorphic to the multiplicative group of a field F .

Proof. Let $L = Lei_7(3, F)$ and let $f \in Aut_{[\cdot]}(L)$. By [9, Lemma 2.1], $f([L, L]) = [L, L]$, so that

$$\begin{aligned} f(a_1) &= \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \\ f(a_2) &= \beta_2 a_2 + \beta_3 a_3, \\ f(a_3) &= \gamma_2 a_2 + \gamma_3 a_3. \end{aligned}$$

Then

$$\begin{aligned} f(a_3) &= f([a_1, a_1]) = [f(a_1), f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1^2 [a_1, a_1] + \alpha_1 \alpha_2 [a_1, a_2] + \alpha_1 \alpha_3 [a_1, a_3] \\ &= \alpha_1^2 a_3 + \alpha_1 \alpha_2 a_2 + \alpha_1 \alpha_3 a_3 \\ &= \alpha_1 \alpha_2 a_2 + (\alpha_1^2 + \alpha_1 \alpha_3) a_3, \\ f(a_3) &= f([a_1, a_3]) = [f(a_1), f(a_3)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \gamma_2 a_2 + \gamma_3 a_3] \\ &= \alpha_1 \gamma_2 [a_1, a_2] + \alpha_1 \gamma_3 [a_1, a_3] \\ &= \alpha_1 \gamma_2 a_2 + \alpha_1 \gamma_3 a_3, \\ f(a_2) &= f([a_1, a_2]) = [f(a_1), f(a_2)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \beta_2 a_2 + \beta_3 a_3] \\ &= \alpha_1 \beta_2 [a_1, a_2] + \alpha_1 \beta_3 [a_1, a_3] \\ &= \alpha_1 \beta_2 a_2 + \alpha_1 \beta_3 a_3. \end{aligned}$$

Thus, we can see that

$$\begin{aligned} \alpha_1 \alpha_2 a_2 + (\alpha_1^2 + \alpha_1 \alpha_3) a_3 &= \alpha_1 \gamma_2 a_2 + \alpha_1 \gamma_3 a_3 = \gamma_2 a_2 + \gamma_3 a_3, \\ \alpha_1 \beta_2 a_2 + \alpha_1 \beta_3 a_3 &= \beta_2 a_2 + \beta_3 a_3. \end{aligned}$$

It follows that

$$\alpha_1 \alpha_2 = \alpha_1 \gamma_2 = \gamma_2, \quad \alpha_1^2 + \alpha_1 \alpha_3 = \alpha_1 \gamma_3 = \gamma_3, \quad \alpha_1 \beta_2 = \beta_2, \quad \alpha_1 \beta_3 = \beta_3.$$

If we suppose that $\alpha_1 = 0$, then $f(a_1) \in Fa_2 \oplus Fa_3$. Since we have $f(a_2), f(a_3) \in Fa_2 \oplus Fa_3$, then $f(L) \leq Fa_2 \oplus Fa_3$ – in particular, $f(L) \neq L$ – and we obtain a contradiction with the fact that f is an automorphism of L . Hence, $\alpha_1 \neq 0$. It follows that $\alpha_2 = \gamma_2$.

If we suppose that both $\beta_2 = \gamma_2 = 0$, then $f(a_2) \in Fa_3$ and $f(a_3) \in Fa_3$, so that $f([L, L]) \leq Fa_3$. On the other hand, $\dim_F([L, L]) = 2$, and we, again, obtain a contradiction with the fact that f is an automorphism of L . Hence, $(\beta_2, \gamma_2) \neq (0, 0)$. It follows that $\alpha_1 = 1$ and then $1 + \alpha_3 = \gamma_3$.

Denote by Ξ the canonical monomorphism of $\text{Aut}_{[1]}(L)$ in $M_3(F)$. Then $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & 1 + \alpha_3 \end{pmatrix},$$

$\alpha_2, \alpha_3, \beta_2, \beta_3 \in F$.

Conversely, let f be a linear transformation of L , having in a basis $\{a_1, a_2, a_3\}$ the matrix above. Let x, y be the arbitrary elements of L , $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, $y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. Then

$$\begin{aligned} [x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\ &= \xi_1 \eta_1 [a_1, a_1] + \xi_1 \eta_2 [a_1, a_2] + \xi_1 \eta_3 [a_1, a_3] \\ &= \xi_1 \eta_1 a_3 + \xi_1 \eta_2 a_2 + \xi_1 \eta_3 a_3 \\ &= \xi_1 \eta_2 a_2 + (\xi_1 \eta_1 + \xi_1 \eta_3) a_3 = \xi_1 \eta_2 a_2 + \xi_1 (\eta_1 + \eta_3) a_3, \\ f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\ &= \xi_1 (a_1 + \alpha_2 a_2 + \alpha_3 a_3) + \xi_2 (\beta_2 a_2 + \beta_3 a_3) + \xi_3 (\alpha_2 a_2 + (1 + \alpha_3) a_3) \\ &= \xi_1 a_1 + (\xi_1 \alpha_2 + \xi_2 \beta_2 + \xi_3 \alpha_2) a_2 + (\xi_1 \alpha_3 + \xi_2 \beta_3 + \xi_3 + \xi_3 \alpha_3) a_3, \\ f(y) &= \eta_1 a_1 + (\eta_1 \alpha_2 + \eta_2 \beta_2 + \eta_3 \alpha_2) a_2 + (\eta_1 \alpha_3 + \eta_2 \beta_3 + \eta_3 + \eta_3 \alpha_3) a_3. \end{aligned}$$

Therefore,

$$\begin{aligned} f([x, y]) &= f(\xi_1 \eta_2 a_2 + \xi_1 (\eta_1 + \eta_3) a_3) = \xi_1 \eta_2 f(a_2) + \xi_1 (\eta_1 + \eta_3) f(a_3) \\ &= \xi_1 \eta_2 (\beta_2 a_2 + \beta_3 a_3) + \xi_1 (\eta_1 + \eta_3) (\alpha_2 a_2 + (1 + \alpha_3) a_3) \\ &= (\xi_1 \eta_2 \beta_2 + \xi_1 (\eta_1 + \eta_3) \alpha_2) a_2 + (\xi_1 \eta_2 \beta_3 + \xi_1 (\eta_1 + \eta_3) (1 + \alpha_3)) a_3, \\ &\qquad [f(x), f(y)] \\ &= [\xi_1 a_1 + (\xi_1 \alpha_2 + \xi_2 \beta_2 + \xi_3 \alpha_2) a_2 + (\xi_1 \alpha_3 + \xi_2 \beta_3 + \xi_3 + \xi_3 \alpha_3) a_3, \\ &\quad \eta_1 a_1 + (\eta_1 \alpha_2 + \eta_2 \beta_2 + \eta_3 \alpha_2) a_2 + (\eta_1 \alpha_3 + \eta_2 \beta_3 + \eta_3 + \eta_3 \alpha_3) a_3] \end{aligned}$$

$$\begin{aligned}
 &= \xi_1 \eta_1 [a_1, a_1] + \xi_1 (\eta_1 \alpha_2 + \eta_2 \beta_2 + \eta_3 \alpha_2) [a_1, a_2] \\
 &\quad + \xi_1 (\eta_1 \alpha_3 + \eta_2 \beta_3 + \eta_3 + \eta_3 \alpha_3) [a_1, a_3] \\
 &= \xi_1 \eta_1 a_3 + \xi_1 (\eta_1 \alpha_2 + \eta_2 \beta_2 + \eta_3 \alpha_2) a_2 + \xi_1 (\eta_1 \alpha_3 + \eta_2 \beta_3 + \eta_3 + \eta_3 \alpha_3) a_3 \\
 &= \xi_1 (\eta_1 \alpha_2 + \eta_2 \beta_2 + \eta_3 \alpha_2) a_2 + \xi_1 (\eta_1 + \eta_1 \alpha_3 + \eta_2 \beta_3 + \eta_3 + \eta_3 \alpha_3) a_3.
 \end{aligned}$$

We can see that $f([x, y]) = [f(x), f(y)]$. It follows that a linear transformation f having in a basis $\{a_1, a_2, a_3\}$ the above matrix is an automorphism of a Leibniz algebra.

Denote by Φ the mapping of $\Xi(G)$ in $GL_2(F)$ defined by the rule:

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & 1 + \alpha_3 \end{pmatrix} \rightarrow \begin{pmatrix} \beta_2 & \alpha_2 \\ \beta_3 & 1 + \alpha_3 \end{pmatrix}.$$

We have:

$$\begin{aligned}
 &\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & 1 + \alpha_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \gamma_2 & \kappa_2 & \gamma_2 \\ \gamma_3 & \kappa_3 & 1 + \gamma_3 \end{pmatrix} = \\
 &\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 + \beta_2 \gamma_2 + \alpha_2 \gamma_3 & \beta_2 \kappa_2 + \alpha_2 \kappa_3 & \beta_2 \gamma_2 + \alpha_2 (1 + \gamma_3) \\ \alpha_3 + \beta_3 \gamma_2 + (1 + \alpha_3) \gamma_3 & \beta_3 \kappa_2 + (1 + \alpha_3) \kappa_3 & \beta_3 \gamma_2 + (1 + \alpha_3) (1 + \gamma_3) \end{pmatrix}
 \end{aligned}$$

and

$$\begin{aligned}
 &\begin{pmatrix} \beta_2 & \alpha_2 \\ \beta_3 & 1 + \alpha_3 \end{pmatrix} \begin{pmatrix} \kappa_2 & \gamma_2 \\ \kappa_3 & 1 + \gamma_3 \end{pmatrix} \\
 &= \begin{pmatrix} \beta_2 \kappa_2 + \alpha_2 \kappa_3 & \beta_2 \gamma_2 + \alpha_2 (1 + \gamma_3) \\ \beta_3 \kappa_2 + (1 + \alpha_3) \kappa_3 & \beta_3 \gamma_2 + (1 + \alpha_3) (1 + \gamma_3) \end{pmatrix}.
 \end{aligned}$$

These equalities show that a mapping Φ is a homomorphism. Clearly, $Ker(\Phi)$ consists only of the identity matrix. Finally, let

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

be the arbitrary element of $GL_2(F)$. Consider the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ \sigma_{12} & \sigma_{11} & \sigma_{12} \\ \sigma_{22} - 1 & \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

This matrix is a preimage of the matrix above by the mapping Φ . Moreover, both these matrices have the same determinants. It follows that the last matrix belongs to $\Xi(G)$, so that a mapping Φ is an isomorphism.

Let $C_1 = C_G(a_1)$, $C_2 = C_G(a_2)$. Then $\Xi(C_1)$ is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & \beta_3 & 1 \end{pmatrix},$$

$\beta_2, \beta_3 \in F$, and $\Xi(C_2)$ is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 & \alpha_2 \\ \alpha_3 & 0 & 1 + \alpha_3 \end{pmatrix},$$

$\alpha_2, \alpha_3 \in F$. We have:

$$\begin{pmatrix} 1 & 0 & 0 \\ \sigma_2 & 1 & \sigma_2 \\ \sigma_3 & 0 & 1 + \sigma_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & \kappa_3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \sigma_2 & \kappa_2 + \sigma_2 \kappa_3 & \sigma_2 \\ \sigma_3 & (1 + \sigma_3) \kappa_3 & 1 + \sigma_3 \end{pmatrix}.$$

If

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & 1 + \alpha_3 \end{pmatrix}$$

is an arbitrary matrix of $\Xi(G)$, then we obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & \beta_2 & \alpha_2 \\ \alpha_3 & \beta_3 & 1 + \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \sigma_2 & 1 & \sigma_2 \\ \sigma_3 & 0 & 1 + \sigma_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & \kappa_3 & 1 \end{pmatrix},$$

where $\sigma_2 = \alpha_2$, $\sigma_3 = \alpha_3$, $\kappa_3 = \beta_3(1 + \alpha_3)^{-1}$, $\kappa_2 = \beta_2 - \alpha_2 \beta_3(1 + \alpha_3)^{-1}$. Thus, we can see that a group G is a product of the subgroups C_1 and C_2 .

Furthermore, a subgroup C_1 has a normal subgroup C_3 such that $\Xi(C_3)$ is the subset of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta_3 & 1 \end{pmatrix},$$

which is isomorphic to the additive group of a field F , and a subgroup C_4 such that $\Xi(C_4)$ is the subset of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is isomorphic to the multiplicative group of a field F . Moreover, C_1 is clearly a semidirect product of C_3 and C_4 .

Now, consider the group $\Xi(C_2)$. It is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 & \alpha_2 \\ \alpha_3 & 0 & 1 + \alpha_3 \end{pmatrix},$$

$\alpha_2, \alpha_3 \in F$. Denote by Φ the mapping of $\Xi(C_2)$ in $GL_2(F)$ defined by the rule:

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 & \alpha_2 \\ \alpha_3 & 0 & 1 + \alpha_3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 + \alpha_3 \end{pmatrix}.$$

We have:

$$\begin{aligned} & \begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 & 1 & \alpha_2 \\ \alpha_3 & 0 & 1 + \alpha_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \gamma_2 & 1 & \gamma_2 \\ \gamma_3 & 0 & 1 + \gamma_3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ \alpha_2 + \gamma_2 + \alpha_2\gamma_3 & 1 & \gamma_2 + \alpha_2(1 + \gamma_3) \\ \alpha_3 + (1 + \alpha_3)\gamma_3 & 0 & (1 + \alpha_3)(1 + \gamma_3) \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 + \alpha_3 \end{pmatrix} \begin{pmatrix} 1 & \gamma_2 \\ 0 & 1 + \gamma_3 \end{pmatrix} = \begin{pmatrix} 1 & \gamma_2 + \alpha_2(1 + \gamma_3) \\ 0 & (1 + \alpha_3)(1 + \gamma_3) \end{pmatrix}.$$

These equalities show that a mapping Φ is a homomorphism. Clearly, $Ker(\Phi)$ consists only of identity matrix. Finally, let

$$\begin{pmatrix} 1 & \sigma_{12} \\ 0 & \sigma_{22} \end{pmatrix}$$

be the arbitrary element of $GL_2(F)$. Consider the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ \sigma_{12} & 1 & \sigma_{12} \\ \sigma_{22} - 1 & 0 & \sigma_{22} \end{pmatrix}.$$

This matrix is a preimage of the matrix above by Φ . Moreover, both these matrices have the same determinants. The last matrix belongs to $\Xi(G)$, so a mapping Φ is an isomorphism.

In turn, clearly $\Xi(C_2)$ is a semidirect product of a normal subgroup that consists of matrices of the form

$$\begin{pmatrix} 1 & \sigma_{12} \\ 0 & 1 \end{pmatrix},$$

(i.e., a subgroup $UT_2(F)$) that is isomorphic to the additive group of a field F and a subgroup that consists of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \sigma_{22} \end{pmatrix},$$

which is isomorphic to the multiplicative group of a field F . □

Theorem 2. *Let G be an automorphism group of the Leibniz algebra $Leis_8(3, F)$. Then G is isomorphic to a subgroup of $GL_3(F)$ that consists of matrices of the form*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ \beta_1 - 1 & \beta_2 & \beta_1 \end{pmatrix},$$

$\beta_1, \beta_2 \in F$. Furthermore, G is abelian, $G = C \times A$ such that $\Xi(C)$ is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sigma & 1 \end{pmatrix},$$

$\sigma \in F$, and $\Xi(A)$ is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ \beta - 1 & 0 & \beta \end{pmatrix},$$

$\beta \in F$. A subgroup C is isomorphic to the additive group of a field F , and a subgroup A is isomorphic to the multiplicative group of a field F .

Proof. Let $L = Leis_8(3, F)$ and let $f \in Aut_{[1]}(L)$. By [9, Lemma 2.1], $f([L, L]) = [L, L]$, so that

$$\begin{aligned} f(a_1) &= \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \\ f(a_2) &= \beta_2 a_2 + \beta_3 a_3, \\ f(a_3) &= \gamma_2 a_2 + \gamma_3 a_3. \end{aligned}$$

Then

$$\begin{aligned}
f(a_3) &= f([a_1, a_1]) = [f(a_1), f(a_1)] \\
&= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\
&= \alpha_1^2 [a_1, a_1] + \alpha_1 \alpha_2 [a_1, a_2] + \alpha_1 \alpha_3 [a_1, a_3] \\
&= \alpha_1^2 a_3 + \alpha_1 \alpha_2 (a_2 + \lambda a_3) + \alpha_1 \alpha_3 a_3 \\
&= \alpha_1 \alpha_2 a_2 + (\alpha_1^2 + \lambda \alpha_1 \alpha_2 + \alpha_1 \alpha_3) a_3, \\
f(a_3) &= f([a_1, a_3]) = [f(a_1), f(a_3)] \\
&= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \gamma_2 a_2 + \gamma_3 a_3] \\
&= \alpha_1 \gamma_2 [a_1, a_2] + \alpha_1 \gamma_3 [a_1, a_3] \\
&= \alpha_1 \gamma_2 (a_2 + \lambda a_3) + \alpha_1 \gamma_3 a_3 \\
&= \alpha_1 \gamma_2 a_2 + (\lambda \alpha_1 \gamma_2 + \alpha_1 \gamma_3) a_3, \\
f([a_1, a_2]) &= [f(a_1), f(a_2)] = [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, \beta_2 a_2 + \beta_3 a_3] \\
&= \alpha_1 \beta_2 [a_1, a_2] + \alpha_1 \beta_3 [a_1, a_3] \\
&= \alpha_1 \beta_2 (a_2 + \lambda a_3) + \alpha_1 \beta_3 a_3 \\
&= \alpha_1 \beta_2 a_2 + (\lambda \alpha_1 \beta_2 + \alpha_1 \beta_3) a_3, \\
f([a_1, a_2]) &= f(a_2 + \lambda a_3) = f(a_2) + \lambda f(a_3) \\
&= \beta_2 a_2 + \beta_3 a_3 + \lambda (\gamma_2 a_2 + \gamma_3 a_3) \\
&= (\beta_2 + \lambda \gamma_2) a_2 + (\beta_3 + \lambda \gamma_3) a_3.
\end{aligned}$$

Thus, we can see that

$$\begin{aligned}
\alpha_1 \alpha_2 a_2 + (\alpha_1^2 + \lambda \alpha_1 \alpha_2 + \alpha_1 \alpha_3) a_3 &= \\
\alpha_1 \gamma_2 a_2 + (\lambda \alpha_1 \gamma_2 + \alpha_1 \gamma_3) a_3 &= \gamma_2 a_2 + \gamma_3 a_3, \\
\alpha_1 \beta_2 a_2 + (\lambda \alpha_1 \beta_2 + \alpha_1 \beta_3) a_3 &= (\beta_2 + \lambda \gamma_2) a_2 + (\beta_3 + \lambda \gamma_3) a_3.
\end{aligned}$$

It follows that

$$\begin{aligned}
\alpha_1 \alpha_2 = \alpha_1 \gamma_2 = \gamma_2, \quad \alpha_1^2 + \lambda \alpha_1 \alpha_2 + \alpha_1 \alpha_3 = \lambda \alpha_1 \gamma_2 + \alpha_1 \gamma_3 = \gamma_3, \\
\alpha_1 \beta_2 = \beta_2 + \lambda \gamma_2, \quad \lambda \alpha_1 \beta_2 + \alpha_1 \beta_3 = \beta_3 + \lambda \gamma_3.
\end{aligned}$$

If we suppose that $\alpha_1 = 0$, then $f(a_1) \in Fa_2 \oplus Fa_3$. Since we have $f(a_2), f(a_3) \in Fa_2 \oplus Fa_3$, then $f(L) \leq Fa_2 \oplus Fa_3$ – in particular, $f(L) \neq L$ – and we obtain a contradiction with the fact that f is an automorphism of L . Hence, $\alpha_1 \neq 0$. It follows that $\alpha_2 = \gamma_2$.

If we suppose that both $\beta_2 = \gamma_2 = 0$, then $f(a_2) \in Fa_3$, and $f(a_3) \in Fa_3$. We obtain that $f([L, L]) \leq Fa_3$. On the other hand,

$\dim_F([L, L]) = 2$, and we, again, obtain a contradiction with the fact that f is an automorphism of L . Hence, $(\beta_2, \gamma_2) \neq (0, 0)$. Then, if we suppose that $\beta_2 = 0$, we will see that $\lambda\gamma_2 = 0$. Since $\lambda \neq 0$, we obtain that $\gamma_2 = 0$, which is impossible. It follows that $\beta_2 \neq 0$. Suppose that $\gamma_2 = 0$. Since $\beta_2 \neq 0$, we obtain that $\alpha_1 = 1$. If $\gamma_2 \neq 0$, then from $\alpha_1\gamma_2 = \gamma_2$, we obtain that $\alpha_1 = 1$. Thus, in every case, $\alpha_1 = 1$. Then we obtain an equation $\beta_2 = \beta_2 + \lambda\gamma_2$ or $\lambda\gamma_2 = 0$. Since $\lambda \neq 0$, we obtain that $\gamma_2 = 0$. Then $\alpha_2 = 0$, and we obtain $1 + \alpha_3 = \gamma_3$, $\lambda\beta_2 + \beta_3 = \beta_3 + \lambda\gamma_3$. The last equation implies that $\lambda\beta_2 = \lambda\gamma_3$. Since $\lambda \neq 0$, we obtain that $\beta_2 = \gamma_3$.

Denote by Ξ the canonical monomorphism of $\text{Aut}_{[\cdot]}(L)$ in $M_3(F)$. Then $\Xi(f)$ is the following matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ \beta_2 - 1 & \beta_3 & \beta_2 \end{pmatrix},$$

$\beta_2, \beta_3 \in F$. Since this matrix must be non-degenerate, $\beta_2 \neq 0$.

Conversely, let f be a linear transformation of L , having in a basis $\{a_1, a_2, a_3\}$ the matrix above. Let x, y be the arbitrary elements of L , $x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3$, $y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3$, where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3 \in F$. Then

$$\begin{aligned} [x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\ &= \xi_1 \eta_1 [a_1, a_1] + \xi_1 \eta_2 [a_1, a_2] + \xi_1 \eta_3 [a_1, a_3] \\ &= \xi_1 \eta_1 a_3 + \xi_1 \eta_2 (a_2 + \lambda a_3) + \xi_1 \eta_3 a_3 \\ &= \xi_1 \eta_2 a_2 + (\xi_1 \eta_1 + \lambda \xi_1 \eta_2 + \xi_1 \eta_3) a_3 \end{aligned}$$

and

$$\begin{aligned} f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\ &= \xi_1 (a_1 + (\beta_2 - 1)a_3) + \xi_2 (\beta_2 a_2 + \beta_3 a_3) + \xi_3 (\beta_2 a_3) \\ &= \xi_1 a_1 + \xi_2 \beta_2 a_2 + (\xi_1 (\beta_2 - 1) + \xi_2 \beta_3 + \xi_3 \beta_2) a_3, \\ f(y) &= \eta_1 a_1 + \eta_2 \beta_2 a_2 + (\eta_1 (\beta_2 - 1) + \eta_2 \beta_3 + \eta_3 \beta_2) a_3. \end{aligned}$$

Therefore,

$$\begin{aligned} f([x, y]) &= f(\xi_1 \eta_2 a_2 + (\xi_1 \eta_1 + \lambda \xi_1 \eta_2 + \xi_1 \eta_3) a_3) \\ &= \xi_1 \eta_2 f(a_2) + (\xi_1 \eta_1 + \lambda \xi_1 \eta_2 + \xi_1 \eta_3) f(a_3) \\ &= \xi_1 \eta_2 (\beta_2 a_2 + \beta_3 a_3) + (\xi_1 \eta_1 + \lambda \xi_1 \eta_2 + \xi_1 \eta_3) \beta_2 a_3 \end{aligned}$$

$$\begin{aligned}
 &= \xi_1\eta_2\beta_2a_2 + (\xi_1\eta_2\beta_3 + \beta_2\xi_1\eta_1 + \lambda\xi_1\eta_2\beta_2 + \xi_1\eta_3\beta_2)a_3, \\
 [f(x), f(y)] &= [\xi_1a_1 + \xi_2\beta_2a_2 + (\xi_1(\beta_2 - 1) + \xi_2\beta_3 + \xi_3\beta_2)a_3, \\
 &\quad \eta_1a_1 + \eta_2\beta_2a_2 + (\eta_1(\beta_2 - 1) + \eta_2\beta_3 + \eta_3\beta_2)a_3] \\
 &= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2\beta_2[a_1, a_2] + \xi_1(\eta_1(\beta_2 - 1) + \eta_2\beta_3 + \eta_3\beta_2)[a_1, a_3] \\
 &= \xi_1\eta_1a_3 + \xi_1\eta_2\beta_2(a_2 + \lambda a_3) + \xi_1(\eta_1(\beta_2 - 1) + \eta_2\beta_3 + \eta_3\beta_2)a_3 \\
 &= \xi_1\eta_2\beta_2a_2 + (\xi_1\eta_1 + \xi_1\eta_2\beta_2\lambda + \xi_1\eta_1\beta_2 - \xi_1\eta_1 + \xi_1\eta_2\beta_3 + \xi_1\eta_3\beta_2)a_3 \\
 &= \xi_1\eta_2\beta_2a_2 + (\xi_1\eta_2\beta_2\lambda + \xi_1\eta_1\beta_2 + \xi_1\eta_2\beta_3 + \xi_1\eta_3\beta_2)a_3.
 \end{aligned}$$

We can see that $f([x, y]) = [f(x), f(y)]$. It follows that a linear transformation f having in a basis $\{a_1, a_2, a_3\}$ the matrix above, is an automorphism of a Leibniz algebra L .

The equality

$$\begin{aligned}
 &\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ \beta - 1 & \gamma & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ \sigma - 1 & \kappa & \sigma \end{pmatrix} \\
 = &\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta\sigma & 0 \\ \beta - 1 + \beta(\sigma - 1) & \gamma\sigma + \beta\kappa & \beta\sigma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta\sigma & 0 \\ \beta\sigma - 1 & \gamma\sigma + \beta\kappa & \beta\sigma \end{pmatrix}
 \end{aligned}$$

shows that a group G is abelian.

According to the proof above, a subspace Fa_3 is G -invariant. Let $C = C_G(Fa_3)$. Then we can see that $\Xi(C)$ is the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix},$$

$\beta \in F$. It is not hard to see that C is isomorphic to the additive group of a field F .

Denote by A the preimage by Ξ of the set of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ \beta - 1 & 0 & \beta \end{pmatrix},$$

$\beta \in F$. We have:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ \beta - 1 & 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sigma & 0 \\ \sigma - 1 & 0 & \sigma \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta\sigma & 0 \\ \beta - 1 + \beta(\sigma - 1) & 0 & \beta\sigma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta\sigma & 0 \\ \beta\sigma - 1 & 0 & \beta\sigma \end{pmatrix}.$$

We can now see that $\Xi(A)$ is a subgroup of $GL_3(F)$, so that A is a subgroup of G .

Moreover, the mapping, defined by the rule

$$\beta \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ \beta - 1 & 0 & \beta \end{pmatrix},$$

$\beta \in F$, is an isomorphism of the multiplicative group of F on $\Xi(A)$. Thus, we obtain that A is isomorphic to the multiplicative group of a field F .

Furthermore, the equality

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ \beta - 1 & \sigma & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ \beta - 1 & 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta^{-1}\sigma & 1 \end{pmatrix}$$

shows that G is a product of the subgroups C and A . Moreover, this product is direct. □

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