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# On combinatorial properties of minimal posets with nonnegative Tits quadratic form

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Abstract. In this paper, we study combinatorial properties of finite posets connected with the negativity of their Tits quadratic form. We calculate the coefficients of transitivity for all minimal posets with nonnegative Tits quadratic form (such posets are called $NP$-critical and their number is 115 up to isomorphism and duality). Some relationships between these coefficients and the heights of posets are established.

## 1. Introduction

When studying the representations of quivers, P. Gabriel [1] introduced a quadratic form $q_Q(z) = q_Q(z_1, \dots, z_n)$ for any finite quiver $Q = (Q_0, Q_1)$ with the set of vertices $Q_0$ and the set of arrows $Q_1$:

$$q_Q(z) := \sum_{i \in Q_0} z_i^2 - \sum_{i \to j} z_i z_j,$$

where $n = |Q_0|$ and $i \to j$ runs through $Q_1$. This form was called the *Tits quadratic form* of the quiver $Q$. P. Gabriel proved that a connected quiver is of finite representation type over a field if and only if the corresponding non-oriented graph is one of the Dynkin diagrams, and if and only if its Tits quadratic form is positive. This Gabriel's work laid the

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foundations of a new direction in the representation theory. The direction deals with the investigation of the relationships between properties of representations of various objects and properties of quadratic forms associated with these objects.

In [2], Yu. A. Drozd showed that a finite poset $S$ is of finite representation type if and only if its Tits quadratic form

$$q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j;\, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$$

is weakly positive, i.e., positive on the nonzero vectors with nonnegative coordinates (matrix representations of posets were introduced in [3]; see also [4]–[10]). In contrast to quivers, the posets with weakly positive and with positive Tits quadratic forms do not coincide. Therefore investigations related to posets with positive Tits form are natural. In [11], the author together with V. M. Bondarenko classified all posets having positive Tits quadratic form and the minimal posets with nonpositive Tits quadratic form (they are called respectively *positive* and *P-critical*).

We have a similar situation for quivers and posets of tame type. According the papers [12, 13], a connected quiver is of tame infinite type if and only if the corresponding non-oriented graph is an extended Dynkin diagram. On the other hand, the connected quivers with nonnegative, but not positive, Tits form coincide with the quivers, the corresponding graphs of which are extended Dynkin diagrams [14]. A poset $S$ is of tame type if and only if its quadratic Tits form is weakly nonnegative [15]. Since (in contrast to quivers) the classes of posets with weakly nonnegative and with nonnegative Tits forms do not coincide, the investigations related to posets with nonnegative Tits form are also natural. In [16], the author together with V. M. Bondarenko classified minimal posets with nonnegative Tits quadratic form, which were called *NP-critical*.

The present paper, which is a natural continuation of the papers [17, 18] on positive and $P$-critical posets, is devoted to the investigation of combinatorial properties of $NP$-critical posets.

## 2. Main result

Let $S$ be a poset and $S^2_{\prec} := \{(x, y) \,|\, x, y \in S, x \prec y\}$. Elements $x$ and $y$ are called *neighboring* if $(x, y) \in S^2_{<}$ and there no $z$ satisfying $x \prec z \prec y$. Denote by $n_w = n_w(S)$ the order of the set $S^2_{\prec}$ and by $n_e = n_e(S)$ the number of pairs $(x, y)$ of neighboring elements of $S$. On the language of the Hasse diagram $H(S)$ of $S$ (that represents $S$ in the plane), $n_e$ is equal to the number of all its edges and $n_w$ to the number of all its ways going bottom-up, up to parallelism (i.e. with the same start and terminate vertices). The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and $n_w$ is called the *coefficient of transitivity* (if $n_w = 0$, one assumes that $k_t = 0$); see, e.g., [18].

The main result of this paper is the following theorem ($h$ denotes the height of a poset, i.e. the maximum length of its subchain).

**Theorem 1.** *Let $S$ and $T$ be $NP$-critical posets. Then*
(1) $k_t(T) > k_t(S)$ *if* $h(T) > h(S) + 2$*;*
(2) $k_t(T) > k_t(S) - \frac{1}{5}$ *if* $h(T) = h(S) + 1$*;*
(3) $k_t(T) > k_t(S) - \frac{1}{50}$ *if* $h(T) = h(S) + 2$*.*

Since dual posets have the same coefficient of transitivity, under the proving of the theorem we can use the classification of $NP$-critical poset not only up to isomorphism, but simultaneously also up to duality.

## 3. Classification of $NP$-critical posets

For subsets $X, Y$ of a poset $S$, we denote by $X \sqcup Y$ their direct sum (i.e. $X \cup Y$, where $x \in X$ and $y \in Y$ are always incomparable). From Dilworth's theorem it follows that any poset can be represented in the form $\sqcup_{i=1}^{m} X_i$, where all $X_i$ are chains and there is allowed additional relations $y < z$ for the elements belonging to different components (which it is natural to write up to transitivity). By $A_s, B_s, C_s$ we denote, respectively, the chains $a_1 < \ldots < a_s$, $b_1 < \ldots < b_s$, $c_1 < \ldots < c_s$.

The $NP$-critical posets were classified in [16]. We formulate the corresponding theorem with another numbering of the posets.

**Theorem 2.** *The $NP$-critical posets are exhausted, up to isomorphism and duality, by the posets*
***of order 5***
$NP_{1.1} = A_1 \sqcup B_2 \sqcup C_2, a_1 < b_2, a_1 < c_2, b_1 < c_2, c_1 < b_2$*;*
$NP_{1.2} = A_1 \sqcup B_1 \sqcup C_1 \sqcup D_2, a_1 < d_2, b_1 < d_2, c_1 < d_2$*;*
$NP_{1.3} = A_1 \sqcup B_1 \sqcup C_1 \sqcup D_1 \sqcup E_1$*;* $NP_{2.1} = A_2 \sqcup B_3, a_1 < b_2, b_1 < a_2$*;*
$NP_{2.2} = A_1 \sqcup B_2 \sqcup C_2, a_1 < b_2, b_1 < c_2, c_1 < b_2$*;*
$NP_{2.3} = A_1 \sqcup B_3 \sqcup C_1, a_1, c_1 < b_2$*;* $NP_{2.4} = A_1 \sqcup B_3 \sqcup C_1, a_1, c_1 < b_3$*;*
$NP_{2.5} = A_1 \sqcup B_3 \sqcup C_1, a_1 < b_3, b_1 < a_1, b_1 < c_1, c_1 < b_3$*;*
$NP_{2.6} = A_1 \sqcup B_1 \sqcup C_1 \sqcup D_2$*;* $NP_{2.7} = A_1 \sqcup B_1 \sqcup C_2 \sqcup D_1, b_1 < c_2, d_1 < c_2$*;*
***of order 7***

$NP_{3.1} = A_3 \sqcup B_4, a_2 < b_3$; $NP_{3.2} = A_2 \sqcup B_5, a_2 < b_3$;
$NP_{3.3} = A_2 \sqcup B_5, a_2 < b_4$; $NP_{3.4} = A_3 \sqcup B_4, a_2 < b_2$;
$NP_{3.5} = A_3 \sqcup B_4, a_1 < b_1, a_3 < b_3$; $NP_{3.6} = A_3 \sqcup B_4, a_1 < b_1, a_3 < b_4$;
$NP_{3.7} = A_2 \sqcup B_2 \sqcup C_3$; $NP_{3.8} = A_2 \sqcup B_2 \sqcup C_3, b_2 < c_3$;
$NP_{3.9} = A_1 \sqcup B_3 \sqcup C_3, a_1 < b_3, b_2 < c_3$;
$NP_{3.10} = A_2 \sqcup B_2 \sqcup C_3, a_1 < b_2, b_1 < c_2, c_1 < a_2$;
$NP_{3.11} = A_1 \sqcup B_2 \sqcup C_4, b_2 < c_3$; $NP_{3.12} = A_1 \sqcup B_2 \sqcup C_4, b_2 < c_4$;
$NP_{3.13} = A_1 \sqcup B_3 \sqcup C_3, b_1 < c_1, b_3 < c_3$;
$NP_{3.14} = A_1 \sqcup B_3 \sqcup C_3, a_1 < b_3, b_2 < c_2$;
$NP_{3.15} = A_1 \sqcup B_4 \sqcup C_2, a_1 < b_4, b_3 < c_2$;

***of order 8***

$NP_{4.1} = A_4 \sqcup B_4, a_1 < b_2$; $NP_{4.2} = A_3 \sqcup B_5, a_1 < b_2$;
$NP_{4.3} = A_3 \sqcup B_5, a_1 < b_3$; $NP_{4.4} = A_3 \sqcup B_5, a_3 < b_5$;
$NP_{4.5} = A_2 \sqcup B_6, a_1 < b_3$; $NP_{4.6} = A_2 \sqcup B_6, a_1 < b_4$;
$NP_{4.7} = A_1 \sqcup B_7, a_1 < b_4$; $NP_{4.8} = A_1 \sqcup B_7, a_1 < b_5$;
$NP_{4.9} = A_2 \sqcup B_6, a_1 < b_1, a_2 < b_4$; $NP_{4.10} = A_2 \sqcup B_6, a_1 < b_1, a_2 < b_5$;
$NP_{4.11} = A_3 \sqcup B_5, a_2 < b_1, a_3 < b_4$; $NP_{4.12} = A_1 \sqcup B_3 \sqcup C_4$;
$NP_{4.13} = A_3 \sqcup B_1 \sqcup C_4, b_1 < c_4$; $NP_{4.14} = A_1 \sqcup B_3 \sqcup C_4, a_1 < b_2, b_1 < c_2$;
$NP_{4.15} = A_1 \sqcup B_3 \sqcup C_4, a_1 < b_2, b_1 < c_3$;
$NP_{4.16} = A_2 \sqcup B_2 \sqcup C_4, a_2 < b_2, b_1 < c_3$;
$NP_{4.17} = A_2 \sqcup B_2 \sqcup C_4, a_2 < b_2, b_1 < c_4$; $NP_{4.18} = A_2 \sqcup B_1 \sqcup C_5, b_1 < c_4$;
$NP_{4.19} = A_2 \sqcup B_1 \sqcup C_5, b_1 < c_5$; $NP_{4.20} = A_2 \sqcup B_2 \sqcup C_4, b_1 < c_1, b_2 < c_4$;
$NP_{4.21} = A_1 \sqcup B_2 \sqcup C_5, a_1 < b_2, b_1 < c_3$;
$NP_{4.22} = A_1 \sqcup B_2 \sqcup C_5, a_1 < b_2, b_1 < c_4$; $NP_{4.23} = A_1 \sqcup B_1 \sqcup C_6, b_1 < c_4$;
$NP_{4.24} = A_1 \sqcup B_1 \sqcup C_6, b_1 < c_5$; $NP_{4.25} = A_1 \sqcup B_2 \sqcup C_5, b_1 < c_1, b_2 < c_4$;
$NP_{4.26} = A_1 \sqcup B_2 \sqcup C_5, b_1 < c_1, b_2 < c_5$;

***of order 9 (part 1)***

$NP_{5.1} = A_4 \sqcup B_5, a_1 < b_4$; $NP_{5.2} = A_4 \sqcup B_5, a_2 < b_5$;
$NP_{5.3} = A_3 \sqcup B_6, a_1 < b_5$; $NP_{5.4} = A_3 \sqcup B_6, a_2 < b_6$;
$NP_{5.5} = A_2 \sqcup B_7, a_1 < b_2$; $NP_{5.6} = A_2 \sqcup B_7, a_1 < b_6$;
$NP_{5.7} = A_2 \sqcup B_7, a_2 < b_7$; $NP_{5.8} = A_1 \sqcup B_8, a_1 < b_3$;
$NP_{5.9} = A_1 \sqcup B_8, a_1 < b_7$; $NP_{5.10} = A_2 \sqcup B_7, a_1 < b_1, a_2 < b_3$;
$NP_{5.11} = A_2 \sqcup B_7, a_1 < b_1, a_2 < b_7$;
$NP_{5.12} = A_3 \sqcup B_6, a_2 < b_1, a_3 < b_3$;
$NP_{5.13} = A_4 \sqcup B_5, a_3 < b_1, a_4 < b_3$; $NP_{5.14} = A_4 \sqcup B_1 \sqcup C_4, b_1 < c_3$;
$NP_{5.15} = A_4 \sqcup B_2 \sqcup C_3, b_1 < c_1, b_2 < c_3$;
$NP_{5.16} = A_1 \sqcup B_4 \sqcup C_4, a_1 < b_2, b_1 < c_4$;
$NP_{5.17} = A_3 \sqcup B_1 \sqcup C_5, b_1 < c_3$; $NP_{5.18} = A_5 \sqcup B_1 \sqcup C_3, b_1 < c_3$;
$NP_{5.19} = A_3 \sqcup B_2 \sqcup C_4, b_1 < c_1, b_2 < c_3$;

$NP_{5.20} = A_1 \sqcup B_3 \sqcup C_5, a_1 < b_2, b_1 < c_5$;
$NP_{5.21} = A_1 \sqcup B_5 \sqcup C_3, a_1 < b_2, b_1 < c_3$; $NP_{5.22} = A_1 \sqcup B_2 \sqcup C_6$;
$NP_{5.23} = A_2 \sqcup B_1 \sqcup C_6, b_1 < c_3$; $NP_{5.24} = A_2 \sqcup B_2 \sqcup C_5, b_1 < c_1, b_2 < c_3$;
$NP_{5.25} = A_2 \sqcup B_3 \sqcup C_4, b_2 < c_1, b_2 < c_3$;
$NP_{5.26} = A_1 \sqcup B_2 \sqcup C_6, a_1 < b_2, b_1 < c_2$;
$NP_{5.27} = A_1 \sqcup B_2 \sqcup C_6, a_1 < b_2, b_1 < c_6$; $NP_{5.28} = A_1 \sqcup B_1 \sqcup C_7, b_1 < c_3$;
$NP_{5.29} = A_1 \sqcup B_1 \sqcup C_7, b_1 < c_7$; $NP_{5.30} = A_1 \sqcup B_2 \sqcup C_6, b_1 < c_1, b_2 < c_3$;
$NP_{5.31} = A_1 \sqcup B_3 \sqcup C_5, b_2 < c_1, b_2 < c_3$;

***of order 9 (part 2)***

$NP_{6.1} = A_4 \sqcup B_5, a_1 < b_4, a_2 < b_5$; $NP_{6.2} = A_3 \sqcup B_6, a_1 < b_5, a_2 < b_6$;
$NP_{6.3} = A_2 \sqcup B_7, a_1 < b_2, a_2 < b_3$; $NP_{6.4} = A_2 \sqcup B_7, a_1 < b_2, a_2 < b_7$;
$NP_{6.5} = A_2 \sqcup B_7, a_1 < b_6, a_2 < b_7$; $NP_{6.6} = A_6 \sqcup B_3, a_1 < b_2, a_6 < b_3$;
$NP_{6.7} = A_6 \sqcup B_3, a_5 < b_2, a_6 < b_3$;
$NP_{6.8} = A_3 \sqcup B_6, a_1 < b_1, a_2 < b_2, a_3 < b_3$;
$NP_{6.9} = A_4 \sqcup B_5, a_2 < b_1, a_3 < b_2, a_4 < b_3$;
$NP_{6.10} = A_3 \sqcup B_2 \sqcup C_4, b_1 < c_2, b_2 < c_3$;
$NP_{6.11} = A_4 \sqcup B_2 \sqcup C_3, b_1 < c_2, b_2 < c_3$;
$NP_{6.12} = A_1 \sqcup B_4 \sqcup C_4, a_1 < b_3, b_1 < c_4$;
$NP_{6.13} = A_3 \sqcup B_2 \sqcup C_4, a_2 < b_2, b_1 < c_2$;
$NP_{6.14} = A_3 \sqcup B_3 \sqcup C_3, b_1 < c_1, b_2 < c_2, b_3 < c_3$;
$NP_{6.15} = A_3 \sqcup B_3 \sqcup C_3, a_3 < b_3, b_1 < c_1, b_2 < c_2$;
$NP_{6.16} = A_5 \sqcup B_2 \sqcup C_2, a_1 < b_2$; $NP_{6.17} = A_2 \sqcup B_2 \sqcup C_5, b_1 < c_2, b_2 < c_3$;
$NP_{6.18} = A_1 \sqcup B_3 \sqcup C_5, a_1 < b_3, b_1 < c_5$;
$NP_{6.19} = A_1 \sqcup B_5 \sqcup C_3, a_1 < b_3, b_1 < c_3$;
$NP_{6.20} = A_2 \sqcup B_2 \sqcup C_5, a_1 < b_2, b_1 < c_2$;
$NP_{6.21} = A_2 \sqcup B_3 \sqcup C_4, b_1 < c_1, b_2 < c_2, b_3 < c_3$;
$NP_{6.22} = A_2 \sqcup B_3 \sqcup C_4, a_2 < b_3, b_1 < c_1, b_2 < c_2$;
$NP_{6.23} = A_4 \sqcup B_3 \sqcup C_2, a_4 < b_3, b_1 < c_1, b_2 < c_2$;
$NP_{6.24} = A_1 \sqcup B_2 \sqcup C_6, b_1 < c_2$; $NP_{6.25} = A_1 \sqcup B_2 \sqcup C_6, b_1 < c_6$;
$NP_{6.26} = A_1 \sqcup B_2 \sqcup C_6, b_1 < c_2, b_2 < c_3$;
$NP_{6.27} = A_1 \sqcup B_3 \sqcup C_5, a_1 < b_3, b_1 < c_1$;
$NP_{6.28} = A_1 \sqcup B_6 \sqcup C_2, a_1 < b_3, b_1 < c_2$;
$NP_{6.29} = A_1 \sqcup B_3 \sqcup C_5, b_1 < c_1, b_2 < c_2, b_3 < c_3$;
$NP_{6.30} = A_1 \sqcup B_4 \sqcup C_4, b_2 < c_1, b_3 < c_2, b_4 < c_3$;
$NP_{6.31} = A_1 \sqcup B_3 \sqcup C_5, a_1 < b_3, b_1 < c_1, b_2 < c_2$;
$NP_{6.32} = A_1 \sqcup B_7 \sqcup C_1, a_1 < b_3, b_1 < c_1$;
$NP_{6.33} = A_1 \sqcup B_7 \sqcup C_1, a_1 < b_7, b_1 < c_1$.

## 4. Calculation of the transitivity coefficients. Proof of Theorem 1

We first calculate the coefficients of transitivity $k_t$ of the $NP$-critical posets, which are indicated in Theorem 2. The coefficients $k_t$ are calculated up to the fifth decimal place. If the number of decimal places is less than five, then the decimal fraction is finite, and if it is five, then infinite. When two decimal fractions are equal up to five digits, then they are generally equal.

The following holds for the posets from Theorem 2:

| $N$ | $h$ | $n_e$ | $n_w$ | $k_t$ | $N$ | $h$ | $n_e$ | $n_w$ | $k_t$ |
|---|---|---|---|---|---|---|---|---|---|
| $NP_{1.3}$ | 1 | 0 | 0 | 0 | $NP_{1.2}$ | 2 | 4 | 4 | 0 |
| $NP_{2.2}$ | 2 | 5 | 5 | 0 | $NP_{2.4}$ | 3 | 4 | 5 | 0,2 |
| $NP_{1.1}$ | 2 | 6 | 6 | 0 | $NP_{2.1}$ | 3 | 5 | 7 | 0,28571 |
| $NP_{2.6}$ | 2 | 1 | 1 | 0 | $NP_{2.5}$ | 3 | 5 | 7 | 0,28571 |
| $NP_{2.7}$ | 2 | 3 | 3 | 0 | $NP_{2.3}$ | 3 | 4 | 7 | 0,42857 |

| $N$ | $h$ | $n_e$ | $n_w$ | $k_t$ | $N$ | $h$ | $n_e$ | $n_w$ | $k_t$ |
|---|---|---|---|---|---|---|---|---|---|
| $NP_{3.7}$ | 3 | 4 | 5 | 0,2 | $NP_{3.1}$ | 4 | 6 | 13 | 0,53846 |
| $NP_{3.10}$ | 3 | 7 | 9 | 0,22222 | $NP_{3.11}$ | 4 | 5 | 11 | 0,54545 |
| $NP_{3.8}$ | 3 | 5 | 7 | 0,28571 | $NP_{3.6}$ | 5 | 7 | 15 | 0,53333 |
| $NP_{3.9}$ | 3 | 6 | 9 | 0,33333 | $NP_{3.5}$ | 5 | 7 | 17 | 0,58824 |
| $NP_{3.12}$ | 4 | 5 | 9 | 0,44444 | $NP_{3.3}$ | 5 | 6 | 15 | 0,6 |
| $NP_{3.13}$ | 4 | 6 | 11 | 0,45455 | $NP_{3.4}$ | 5 | 6 | 15 | 0,6 |
| $NP_{3.14}$ | 4 | 6 | 11 | 0,45455 | $NP_{3.2}$ | 5 | 6 | 17 | 0,64706 |
| $NP_{3.15}$ | 4 | 6 | 11 | 0,45455 | | | | | |

| $N$ | $h$ | $n_e$ | $n_w$ | $k_t$ | $N$ | $h$ | $n_e$ | $n_w$ | $k_t$ |
|---|---|---|---|---|---|---|---|---|---|
| $NP_{4.17}$ | 4 | 7 | 11 | 0,36364 | $NP_{4.18}$ | 5 | 6 | 13 | 0,53846 |
| $NP_{4.13}$ | 4 | 6 | 10 | 0,4 | $NP_{4.3}$ | 5 | 7 | 16 | 0,5625 |
| $NP_{4.16}$ | 4 | 7 | 12 | 0,41667 | $NP_{4.4}$ | 5 | 7 | 16 | 0,5625 |
| $NP_{4.12}$ | 4 | 5 | 9 | 0,44444 | $NP_{4.2}$ | 5 | 7 | 17 | 0,58824 |
| $NP_{4.15}$ | 4 | 7 | 13 | 0,46154 | $NP_{4.26}$ | 6 | 7 | 17 | 0,58824 |
| $NP_{4.14}$ | 4 | 7 | 14 | 0,5 | $NP_{4.25}$ | 6 | 7 | 18 | 0,61111 |
| $NP_{4.1}$ | 4 | 7 | 15 | 0,53333 | $NP_{4.6}$ | 6 | 7 | 19 | 0,63158 |
| $NP_{4.20}$ | 5 | 7 | 13 | 0,46154 | $NP_{4.24}$ | 6 | 6 | 17 | 0,64706 |
| $NP_{4.19}$ | 5 | 6 | 12 | 0,5 | $NP_{4.5}$ | 6 | 7 | 20 | 0,65 |
| $NP_{4.22}$ | 5 | 7 | 14 | 0,5 | $NP_{4.23}$ | 6 | 6 | 18 | 0,66667 |
| $NP_{4.21}$ | 5 | 7 | 15 | 0,53333 | $NP_{4.10}$ | 7 | 8 | 24 | 0,66667 |
| $NP_{4.9}$ | 7 | 8 | 25 | 0,68 | $NP_{4.8}$ | 7 | 7 | 24 | 0,70833 |
| $NP_{4.11}$ | 7 | 8 | 25 | 0,68 | $NP_{4.7}$ | 7 | 7 | 25 | 0,72 |

| $N$ | $h$ | $n_e$ | $n_w$ | $k_t$ | $N$ | $h$ | $n_e$ | $n_w$ | $k_t$ |
|---|---|---|---|---|---|---|---|---|---|
| $NP_{5.15}$ | 4 | 8 | 14 | 0,42857 | $NP_{5.26}$ | 6 | 8 | 22 | 0,63636 |
| $NP_{5.14}$ | 4 | 7 | 14 | 0,5 | $NP_{5.23}$ | 6 | 7 | 20 | 0,65 |
| $NP_{5.16}$ | 4 | 8 | 16 | 0,5 | $NP_{5.6}$ | 7 | 8 | 24 | 0,66667 |
| $NP_{5.18}$ | 5 | 7 | 14 | 0,5 | $NP_{5.7}$ | 7 | 8 | 24 | 0,66667 |
| $NP_{5.19}$ | 5 | 8 | 16 | 0,5 | $NP_{5.29}$ | 7 | 7 | 22 | 0,68182 |
| $NP_{5.20}$ | 5 | 8 | 16 | 0,5 | $NP_{5.30}$ | 7 | 8 | 26 | 0,69231 |
| $NP_{5.1}$ | 5 | 8 | 18 | 0,55556 | $NP_{5.31}$ | 7 | 8 | 26 | 0,69231 |
| $NP_{5.2}$ | 5 | 8 | 18 | 0,55556 | $NP_{5.5}$ | 7 | 8 | 28 | 0,71429 |
| $NP_{5.21}$ | 5 | 8 | 18 | 0,55556 | $NP_{5.28}$ | 7 | 7 | 26 | 0,73077 |
| $NP_{5.17}$ | 5 | 7 | 16 | 0,5625 | $NP_{5.11}$ | 8 | 9 | 30 | 0,7 |
| $NP_{5.27}$ | 6 | 8 | 18 | 0,55556 | $NP_{5.9}$ | 8 | 8 | 30 | 0,73333 |
| $NP_{5.3}$ | 6 | 8 | 20 | 0,6 | $NP_{5.10}$ | 8 | 9 | 34 | 0,73529 |
| $NP_{5.4}$ | 6 | 8 | 20 | 0,6 | $NP_{5.12}$ | 8 | 9 | 34 | 0,73529 |
| $NP_{5.24}$ | 6 | 8 | 20 | 0,6 | $NP_{5.13}$ | 8 | 9 | 34 | 0,73529 |
| $NP_{5.25}$ | 6 | 8 | 20 | 0,6 | $NP_{5.8}$ | 8 | 8 | 34 | 0,76471 |
| $NP_{5.22}$ | 6 | 6 | 16 | 0,625 | | | | | |

| $N$ | $h$ | $n_e$ | $n_w$ | $k_t$ | $N$ | $h$ | $n_e$ | $n_w$ | $k_t$ |
|---|---|---|---|---|---|---|---|---|---|
| $NP_{6.11}$ | 4 | 8 | 13 | 0,38462 | $NP_{6.17}$ | 5 | 8 | 19 | 0,57895 |
| $NP_{6.14}$ | 4 | 9 | 15 | 0,4 | $NP_{6.2}$ | 6 | 9 | 21 | 0,57143 |
| $NP_{6.10}$ | 4 | 8 | 15 | 0,46667 | $NP_{6.27}$ | 6 | 8 | 19 | 0,57895 |
| $NP_{6.12}$ | 4 | 8 | 15 | 0,46667 | $NP_{6.25}$ | 6 | 7 | 17 | 0,58824 |
| $NP_{6.13}$ | 4 | 8 | 15 | 0,46667 | $NP_{6.31}$ | 6 | 9 | 23 | 0,60870 |
| $NP_{6.15}$ | 4 | 9 | 17 | 0,47059 | $NP_{6.28}$ | 6 | 8 | 21 | 0,61905 |
| $NP_{6.20}$ | 5 | 8 | 14 | 0,42857 | $NP_{6.29}$ | 6 | 9 | 25 | 0,64 |
| $NP_{6.16}$ | 5 | 7 | 13 | 0,46154 | $NP_{6.30}$ | 6 | 9 | 25 | 0,64 |
| $NP_{6.18}$ | 5 | 8 | 15 | 0,46667 | $NP_{6.24}$ | 6 | 7 | 21 | 0,66667 |
| $NP_{6.23}$ | 5 | 9 | 17 | 0,47059 | $NP_{6.26}$ | 6 | 8 | 25 | 0,68 |
| $NP_{6.1}$ | 5 | 9 | 19 | 0,52632 | $NP_{6.5}$ | 7 | 9 | 25 | 0,64 |
| $NP_{6.21}$ | 5 | 9 | 19 | 0,52632 | $NP_{6.6}$ | 7 | 9 | 25 | 0,64 |
| $NP_{6.22}$ | 5 | 9 | 19 | 0,52632 | $NP_{6.33}$ | 7 | 8 | 23 | 0,65217 |
| $NP_{6.19}$ | 5 | 8 | 17 | 0,52941 | $NP_{6.4}$ | 7 | 9 | 29 | 0,68966 |
| $NP_{6.7}$ | 7 | 9 | 29 | 0,68966 | $NP_{6.32}$ | 7 | 8 | 27 | 0,70370 |
| $NP_{6.8}$ | 7 | 10 | 33 | 0,69697 | $NP_{6.3}$ | 7 | 9 | 33 | 0,72727 |
| $NP_{6.9}$ | 7 | 10 | 33 | 0,69697 | | | | | |

For calculation the coefficients from the tables we need the following lemmas (see [17] or [18]).

**Lemma 1.** *Let* $S = S_1 \sqcup S_2$. *Then*

$$n_e(S) = n_e(S_1) + n_e(S_2), \quad n_w(S) = n_w(S_1) + n_w(S_2).$$

**Lemma 2.** *Let $S = A_m$. Then*

$$n_e(S) = m-1, \quad n_w(S) = \tfrac{(m-1)m}{2}.$$

**Lemma 3.** *Let $S = \{A_m \sqcup B_n, a_i < b_j\}$. Then*

$(a)$ $n_e(S) = m+n-1$;

$(b)$ $n_w(S) = \frac{(m-1)m+(n-1)n}{2} + i(n-j+1)$.

**Lemma 4.** *Let $S = \{A_m \sqcup B_n, a_i < b_j, a_{i'} < b_{j'}\}$, where $i < i', j < j'$. Then*

$(a)$ $n_e(S) = m+n$;

$(b)$ $n_w(S) = \frac{(m-1)m+(n-1)n}{2} + i'(n-j'+1) + i(j'-j)$.

**Lemma 5.** *Let $S = \{A_m \sqcup B_n \sqcup C_s, a_i < b_j, b_{j'} < c_k$, where $j > j'$. Then*

$(a)$ $n_e(S) = m+n+s-1$;

$(b)$ $n_w(S) = \frac{(m-1)m+(n-1)n+s(s-1)}{2} + i(n-j+1) + j'(s-k+1)$.

**Lemma 6.** *Let $S = \{A_m \sqcup B_n, a_i < b_j, a_{i+1} < b_{j+1}, a_{i+2} < b_{j+2}$. Then*

$(a)$ $n_e(S) = m+n+1$;

$(b)$ $n_w(S) = \frac{(m-1)m+(n-1)n}{2} + (i+2)n - i(j-1) - (2j+1)$.

**Lemma 7.** *Let $S = \{A_m \sqcup B_n \sqcup C_s, a_i < b_j, b_{j'} < c_k$, $b_{j'+1} < c_{k+1}$, where $j > j'+1$. Then*

$(a)$ $n_e(S) = m+n+s$;

$(b)$ $n_w(S) = \frac{(m-1)m+(n-1)n+s(s-1)}{2} + i(n-j+1) + (j'+1)(s-k) + j'$.

Namely, the transitivity coefficients indicated in the tables follows from the following Lemmas:

Lemmas 1, 2 for $NP_{3.7}$, Lemma 3 for $NP_{3.1} - NP_{3.4}$, Lemmas 1–3 for $NP_{3.8}, NP_{3.11}, NP_{3.12}$, Lemma 4 for $NP_{3.5} - NP_{3.6}$, Lemmas 1, 2, 4 for $NP_{3.13}$, Lemma 5 for $NP_{3.9}, NP_{3.14}, NP_{3.15}$;

Lemmas 1, 2 for $NP_{4.12}$, Lemma 3 for $NP_{4.1} - NP_{4.8}$, Lemmas 1–3 for $NP_{4.13}, NP_{4.18} - NP_{4.19}, NP_{4.23} - NP_{4.24}$, Lemma 4 for $NP_{4.9} - NP_{4.11}$, Lemmas 1, 2, 4 for $NP_{4.20}, NP_{4.25}, NP_{4.26}$, Lemma 5 for $NP_{4.14} - NP_{4.17}, NP_{4.21} - NP_{4.22}$;

Lemmas 1, 2 for $NP_{5.22}$, Lemma 3 for $NP_{5.1} - NP_{5.9}$, Lemmas 1–3 for $NP_{5.14}, NP_{5.17} - NP_{5.18}, NP_{5.23}, NP_{5.28} - NP_{5.29}$, Lemma 4 for $NP_{5.10} - NP_{5.13}$, Lemmas 1, 2, 4 for $NP_{5.15}, NP_{5.19}, NP_{5.24} - NP_{5.25}, NP_{5.30} - NP_{5.31}$, Lemma 5 for $NP_{5.16}, NP_{5.20} - NP_{5.21}, NP_{5.26} - NP_{5.27}$;

Lemmas 1–3 for $NP_{6.16}, NP_{6.24} - NP_{6.25}$; Lemma 4 for $NP_{6.1} - NP_{6.7}$; Lemmas 1, 2, 4 for $NP_{6.10} - NP_{6.11}, NP_{6.17}, NP_{6.26}$; Lemma 5 for $NP_{6.12} - NP_{6.13}, NP_{6.18} - NP_{6.20}, NP_{6.27} - NP_{6.28}, NP_{6.32} - NP_{6.33}$;

Lemma 6 for $NP_{6.8} - NP_{6.9}$; Lemmas 1, 6 for $NP_{6.14}, NP_{6.21}, NP_{6.29} - NP_{6.30}$; Lemma 7 for $NP_{6.15}, NP_{6.22} - NP_{6.23}, NP_{6.31}$.

The coefficients in the cases $NP_{1i}, NP_{2j}$ and $NP_{3.10}$ are proved by direct calculations.

Given the lexicographic notation in the tables, it is easy to check that Theorem 1 follows from them.

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