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RESEARCH ARTICLE

Network semigroups

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Abstract. We introduce the class of network semigroups. These are based on networks that extend the notion of a directed graph. This class properly contains the class of graph inverse semigroups. We investigate the structure of network semigroups. We show that two network semigroups are isomorphic if and only if the underlying networks are isomorphic.

Introduction

The concept of a *network* provides a fundamental framework for modelling complex systems in which entities and their interactions are represented abstractly. Traditionally, such systems are described by graphs, where vertices represent entities and edges encode pairwise (dyadic) interactions between them. This classical framework has proved highly successful in a wide range of areas, including algebraic structures associated with graphs [5, 21]. However, it has become increasingly clear that many real-world systems cannot be adequately captured by pairwise interactions alone. In particular, complex systems arising in biology, social

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sciences and data science often exhibit *higher-order interactions*, involving simultaneous relationships among more than two entities [7–9, 16].

Motivated by these developments, the theory of *higher-order networks* has emerged as a natural generalisation of classical graph theory. One prominent approach models higher-order interactions via simplicial complexes, where interactions are encoded by simplices of arbitrary dimension [6]. Another direction, which is particularly relevant in applications involving flows or transformations, considers networks in which both the source and the range of an interaction are subsets of vertices, not necessarily singletons [9]. Such structures extend directed graphs by allowing multi-source and multi-target relations, thereby providing a flexible framework for describing non-dyadic and nonlinear interactions.

In this paper we focus on a class of higher-order networks in which relations connect subsets of vertices to subsets of vertices. Our aim is to investigate the algebraic structures naturally associated with such networks, and in particular to study the semigroups arising from them and their structural properties.

Throughout, a *network* $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ consists of a set of vertices V, a set of relations T, together with mappings $\mathbf{s}, \mathbf{r} : T \to \mathcal{P}(V)$, where $\mathcal{P}(V)$ is the power set of V and any relation $t \in T$ is composed of an ordered pair $(\mathbf{s}(t), \mathbf{r}(t))$, where $\mathbf{s}(t)$ and $\mathbf{r}(t)$ are disjoint non-empty subsets of V, called the *source* and the *range* of t, respectively. If, in addition, each $t \in T$ satisfies the condition that $\mathbf{s}(t)$ and $\mathbf{r}(t)$ are singleton subsets of V, then Γ reduces to a (simple directed) graph, and we identify it with its underlying graph structure and refer to it simply as a *graph*. In this paper we restrict attention to networks with finitely many or countably infinitely many vertices and relations.

Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a graph. A *path* in Γ is a finite sequence $p = t_1 t_2 \cdots t_n$ of relations such that $\mathbf{r}(t_i) = \mathbf{s}(t_{i+1})$ for all $i = 1, \ldots, n-1$. Ash and Hall [5] introduced the class of *graph inverse semigroups* associated with such paths. These semigroups generalize the polycyclic monoids [20] and play an important role in the theory of graph algebras [21]. The algebraic investigation of structures generated by paths in graphs has developed into an active area of research, revealing deep connections with semigroup theory, ring theory, and operator algebras (see, for example, [1–5, 14, 17]).

For general networks, there is a natural extension of the notion of a path. Namely, a sequence $p = t_1 t_2 \cdots t_n$ of relations is admissible whenever $\mathbf{r}(t_i) \cap \mathbf{s}(t_{i+1}) \neq \emptyset$ for each $i = 1, \ldots, n-1$. This broader notion reflects the higher-order interactions inherent in networks. It is therefore natural to ask what algebraic structures arise from such generalized paths.

Motivated by the construction of graph inverse semigroups, we associate to each network a semigroup generated by paths and their formal inverses, subject to suitable relations. The resulting semigroup is obtained as a quotient of a path-generated semigroup and is shown to be a right $*$-abundant semigroup. In the sense of Fountain, this places it within the framework of unary (or $*$-) semigroups [13]. Such structures form a rich class encompassing, for example, right adequate, right h-adequate and right ample semigroups [11, 13]. Furthermore, in the special case where the network is a graph, the constructed semigroup coincides with the corresponding graph inverse semigroup (see Proposition 2), showing that our construction properly extends the classical theory.

The paper is organized as follows. In Section 1 we recall the necessary preliminaries on semigroup theory, with particular emphasis on right abundant semigroups. We also introduce paths in a network and discuss basic properties, together with the notion of network homomorphisms. In Section 2 we define, for a given network Γ, the semigroup Q_Γ generated by paths in Γ, and prove that Q_Γ is a right $*$-abundant semigroup (see Theorem 2). Furthermore, we construct two subsemigroups S_Γ and R_Γ of Q_Γ determined by paths satisfying additional conditions, and establish that S_Γ is right ample (see Proposition 4) and that R_Γ is a fundamental inverse semigroup (see Corollary 2), with $R_\Gamma \subseteq S_\Gamma \subseteq Q_\Gamma$. Section 3 is devoted to congruence-theoretic aspects. We define a proper ideal I of Q_Γ (and similarly for S_Γ), and show that the relation $\rho_I = (I \times I) \cup 1_{Q_\Gamma}$ is an idempotent-separating congruence on Q_Γ. We also provide sufficient conditions under which Q_Γ fails to be $*$-congruence-free as a unary semigroup (see Theorem 4). In addition, we obtain analogous sufficient conditions ensuring that S_Γ is not $*$-congruence-free and that R_Γ is not congruence-free (see Theorem 5). Finally, in Section 4 we investigate the structure of idempotents in Q_Γ via the natural partial order. As an application, we prove that two networks are isomorphic if and only if their associated network semigroups are isomorphic (see Theorem 6).

1. Preliminaries

For the convenience of the reader, we recall some basic definitions and results concerning (right) abundant semigroups and networks. Further details may be found in [9, 11, 12, 15].

1.1. (Right) abundant semigroups

We recall the definitions of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$. Let S be a semigroup. We denote the set of idempotents of S by $E(S)$. For $a, b \in S$, define

$$a \, \mathcal{L}^* \, b \iff \forall x, y \in S^1 (ax = ay \iff bx = by),$$

and

$$a \, \mathcal{R}^* \, b \iff \forall x, y \in S^1 (xa = ya \iff xb = yb).$$

Then $\mathcal{L}^*$ is a right congruence and $\mathcal{R}^*$ is a left congruence.

Lemma 1. *Let $e \in E(S)$ and $a \in S$. The following are equivalent:*

(i) *$e \, \mathcal{L}^* \, a$;*

(ii) *$ae = a$ and $ax = ay$ implies $ex = ey$ for all $x, y \in S^1$.*

The dual statement holds for $\mathcal{R}^*$.

Recall that $\mathcal{L}$ and $\mathcal{R}$ denote Green's relations on S. Then $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. Moreover, for regular elements $a, b \in S$,

$$a \, \mathcal{L}^* \, b \iff a \, \mathcal{L} \, b, \qquad a \, \mathcal{R}^* \, b \iff a \, \mathcal{R} \, b.$$

In particular, if S is regular, then $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$.

A semigroup S is called *right abundant* if every $\mathcal{L}^*$-class contains an idempotent.

Definition 1. A right abundant semigroup S is called *right $*$-abundant* if each $\mathcal{L}^*$-class of S contains a unique idempotent.

Let S be a right $*$-abundant semigroup. For each $a \in S$, denote by a^* the unique idempotent in the $\mathcal{L}^*$-class of a. Then * defines a unary operation on S, and hence S may be regarded as an algebra of type $(2, 1)$. In this setting, a homomorphism between right $*$-abundant semigroups is understood to be a mapping that preserves both the multiplication and the unary operation * (equivalently, the relation $\mathcal{L}^*$). When necessary, such homomorphisms will be referred to as $(2, 1)$-*morphisms.* In particular, every semigroup isomorphism between right $*$-abundant semigroups preserves the unary operation *.

Definition 2. A semigroup S is *right ample* (formerly *right type A*) if it is right abundant, its idempotents commute, and $ea = a(ea)^*$ for all $a \in S$ and $e \in E(S)$.

Since the idempotents of a right ample semigroup commute, they are closed under multiplication and hence form a semilattice. Furthermore, each $\mathcal{L}^*$-class contains at most one idempotent, and thus every right ample semigroup is right $*$-abundant. Moreover, when regarded as unary semigroups, the class of right ample semigroups constitutes a quasi-variety of right $*$-abundant semigroups [13].

Dually, one defines *left ample* semigroups. A semigroup is *ample* (formerly *type A*) if it is both left and right ample. In particular, every inverse semigroup is ample, where for $a \in S$, $a^\dagger = aa^{-1}$ and $a^* = a^{-1}a$, with $a^\dagger$ the unique idempotent in the $\mathcal{R}^*$-class of a and a^{-1} the inverse of a. Thus ample semigroups may be viewed as the abundant analogue of inverse semigroups.

Lemma 2. *Let S be a right ample semigroup with semilattice of idempotents $E(S)$, and let $\mathrm{Reg}(S)$ denote the set of regular elements of S. Then $\mathrm{Reg}(S)$ is an inverse semigroup and a subsemigroup of S.*

Proof. Since every idempotent is regular, $E(S) \subseteq \mathrm{Reg}(S)$. To show that $\mathrm{Reg}(S)$ is a subsemigroup, it suffices to prove closure under multiplication.

Let $a, b \in \mathrm{Reg}(S)$. We have $a \,\mathcal{L}^*\, a^*$ and $b \,\mathcal{L}^*\, b^*$. Since a, b, a^*, b^* are regular, it follows that $a \,\mathcal{L}\, a^*$ and $b \,\mathcal{L}\, b^*$. As $\mathcal{L}$ is a right congruence, we obtain $ab \,\mathcal{L}\, a^*b$, and hence $(ab)^* = (a^*b)^*$. Since S is right ample, we have $a^*b = b(a^*b)^*$. Applying the unary operation * to both sides yields $(a^*b)^* = (b(a^*b)^*)^*$. Using $b \,\mathcal{L}\, b^*$, it follows that $(b(a^*b)^*)^* = b^*(a^*b)^*$, and therefore $(ab)^* = (a^*b)^* = (b(a^*b)^*)^* = b^*(a^*b)^*$. Consequently, $ab \,\mathcal{L}\, a^*b = b(a^*b)^* \,\mathcal{L}\, b^*(a^*b)^* = (ab)^*$, so ab is $\mathcal{L}$-related to an idempotent. Hence $ab \in \mathrm{Reg}(S)$. Therefore $\mathrm{Reg}(S)$ is a subsemigroup of S. Since its idempotents form a semilattice, it follows that $\mathrm{Reg}(S)$ is an inverse semigroup. □

Lemma 3 ([15])**.** *Let S be a semigroup with set of idempotents $E(S)$, and let a be a regular element in S, where b is an inverse of a. Then $ab, ba \in E(S)$ and $ab \,\mathcal{R}\, a \,\mathcal{L}\, ba$. Moveover, if $E(S)$ is a semilattice and a is a regular element in S then the inverse of a is unique.*

1.2. Generation and presentation

We now recall the notion of a semigroup generated by a non-empty set X. The free monoid X^* on X consists of all words over X with operation of juxtaposition. We use ε to denote the empty word. The free semigroup

X^+ on X is $X^*\backslash\{\varepsilon\}$. A non-empty word is denoted by $x_1x_2\cdots x_n$, where $x_i \in X$ for $1 \leq i \leq n$. For any two words $\alpha = x_1x_2\cdots x_n$, $\beta = y_1y_2\cdots y_m$ of X^+, we use $\alpha\beta$ to denote the juxtaposition of α and β, that is $\alpha\beta = x_1x_2\cdots x_ny_1y_2\cdots y_m$. If $\alpha = \beta\mu$, where $\alpha, \beta, \mu \in X^*$, β is called a *prefix* of α, and a *proper prefix* if μ is not the empty word ε. For any two non-empty words α, β in X^+, we say that α, β are *prefix comparable* if one of α, β is a prefix of the other.

Let R be a binary relation on X^+. The quotient semigroup $X^+/R^\sharp$, where $R^\sharp$ is the smallest congruence on X^+ containing R, is said to be *the semigroup generated by X subject to relations R*. We use the formal equality $u = v$ to mean that $(u, v) \in R$. We denote the $R^\sharp$-class of $x \in X^+$ by $[x]$.

We conclude this subsection with basic notions on rewriting systems; see [10, 23] for details.

Let A be a non-empty set and $\to$ a binary relation on A. The pair $(A, \to)$ is called a *rewriting system*. Denote by $\xrightarrow{*}$ the reflexive and transitive closure of $\to$, and by $\overset{*}{\leftrightarrow}$ the equivalence relation generated by $\to$. For $x \in A$, write $[x]$ for its $\overset{*}{\leftrightarrow}$ -class. An element $x \in A$ is *irreducible* or *reduced* if there is no $y \in A$ with $x \to y$. If $x \xrightarrow{*} y$ and y is irreducible, then y is called a *normal form* of x. A rewriting system $(A, \to)$ is *noetherian* if there is no infinite sequence $x_0 \to x_1 \to x_2 \to \cdots$ in A. A rewriting system $(A, \to)$ is *confluent* if for all $w, x, y \in A$,

$$w \xrightarrow{*} x,\ w \xrightarrow{*} y \ \Rightarrow\ \exists z \in A \text{ such that } x \xrightarrow{*} z,\ y \xrightarrow{*} z,$$

and *locally confluent* if for all $w, x, y \in A$,

$$w \to x,\ w \to y \ \Rightarrow\ \exists z \in A \text{ such that } x \xrightarrow{*} z,\ y \xrightarrow{*} z.$$

Lemma 4 ([10])**.** *Let $(A, \to)$ be a rewriting system.*

(i) *If $(A, \to)$ is noetherian and confluent, then every $\overset{*}{\leftrightarrow}$-class contains a unique normal form.*

(ii) *If $(A, \to)$ is noetherian, then it is confluent if and only if it is locally confluent.*

Let S be a semigroup with presentation $\langle X : u_i = v_i, i \in I\rangle$, where $u_i, v_i \in X^+$. This presentation induces a rewriting system $(X^+, \to)$ with rules

$$xu_iy \to xv_iy \quad (x, y \in X^*,\ i \in I).$$

Then $\overset{*}{\leftrightarrow}$ is the congruence on X^+ generated by $R = \{(u_i, v_i) : i \in I\}$, and hence $S \cong X^+/\overset{*}{\leftrightarrow}$. In particular, if $(X^+, \rightarrow)$ is noetherian and confluent, then every element of S admits a unique normal form over X^+.

1.3. Networks

In this subsection we give some basic definitions and results of networks. For further details, of both background and technicalities, we refer the reader to [9] and [22].

Definition 3. A *network* is a quadruple $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ consisting of a non-empty set V of vertices, a non-empty set T of relations, and mappings $\mathbf{s}, \mathbf{r} : T \to \mathcal{P}(V)$, where $\mathcal{P}(V)$ denotes the power set of V, such that for each $t \in T$, the subsets $\mathbf{s}(t)$ and $\mathbf{r}(t)$ are non-empty and disjoint. Each element $t \in T$ is identified with the ordered pair $(\mathbf{s}(t), \mathbf{r}(t))$, where $\mathbf{s}(t)$ and $\mathbf{r}(t)$ are called the *source* and the *range* of t, respectively.

Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a network. If for all $t \in T$, the sets $\mathbf{s}(t)$ and $\mathbf{r}(t)$ are singletons, we identify Γ with the underlying simple directed graph and refer to it simply as a *graph*. In a graph, a relation corresponds to an edge connecting two vertices, whereas in a general network, a relation connects two disjoint non-empty subsets of V.

For a network $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$, define

$$T^0 = \{\, A \subseteq V \mid \exists\, t \in T,\ A = \mathbf{s}(t) \text{ or } A = \mathbf{r}(t) \,\} \cup V,$$

and for each $A \in T^0$, set

$$\mathbf{s}(A) = A = \mathbf{r}(A).$$

Remark 1. Here we include $V \subseteq T^0$ to account for isolated vertices $v \in V$ in the network Γ, i.e., those for which there exists no $t \in T$ with $v \in \mathbf{s}(t)$ or $v \in \mathbf{r}(t)$. If Γ is a graph, then $T^0 = V$, since the source and range of each relation are singleton sets.

Definition 4. Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a network. A *path* in Γ is a finite sequence $\alpha = t_1 t_2 \cdots t_n$ with $t_i \in T \cup T^0$ such that $\mathbf{r}(t_i) \cap \mathbf{s}(t_{i+1}) \neq \emptyset$ for $1 \leq i < n$. The *source* and *range* of α are defined by $\mathbf{s}(\alpha) = \mathbf{s}(t_1)$ and $\mathbf{r}(\alpha) = \mathbf{r}(t_n)$, respectively.

Elements of T^0 are regarded as trivial (empty) paths. Let $P(\Gamma)$ denote the set of all paths in Γ, together with a zero element 0.

Definition 5. A path $\alpha = t_1 t_2 \cdots t_n \in P(\Gamma) \setminus \{0\}$ is called *linear* if $\mathbf{r}(t_i) = \mathbf{s}(t_{i+1})$ for all $1 \leq i < n$.

Let $LP(\Gamma)$ denote the set of all linear paths in Γ. Then $T^0 \subseteq LP(\Gamma)$ and $0 \notin LP(\Gamma)$. If Γ is a graph, then every non-zero path is linear, and hence $P(\Gamma) = LP(\Gamma) \cup \{0\}$.

For non-zero paths $\alpha = t_1 \cdots t_n$ and $\beta = y_1 \cdots y_m$, their concatenation $\alpha\beta$ is defined as the sequence $t_1 \cdots t_n y_1 \cdots y_m$. Then the following lemma is immediate, and its proof is omitted.

Lemma 5. *Let* $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ *be a network. Then:*

(i) *if* $\alpha = t_1 \cdots t_n \in LP(\Gamma)$*, then* $t_i \cdots t_j \in LP(\Gamma)$ *for all* $1 \leq i < j \leq n$*;*

(ii) $\alpha\beta \in P(\Gamma) \setminus \{0\}$ *if and only if* $\mathbf{r}(\alpha) \cap \mathbf{s}(\beta) \neq \emptyset$*;*

(iii) *if* $\alpha, \beta \in LP(\Gamma)$ *with* $\mathbf{r}(\alpha) = \mathbf{s}(\beta)$*, then* $\alpha\beta \in LP(\Gamma)$*.*

Definition 6. Let $\Gamma = (V_\Gamma, T_\Gamma, \mathbf{s}, \mathbf{r})$ and $\Delta = (V_\Delta, T_\Delta, \mathbf{s}, \mathbf{r})$ be networks. A *homomorphism* $\phi : \Gamma \to \Delta$ is a pair of maps $\phi = (\phi_V, \phi_T)$, where $\phi_V : V_\Gamma \to V_\Delta$ and $\phi_T : T_\Gamma \to T_\Delta$, such that for all $t \in T_\Gamma$, $\mathbf{s}(t)\phi = \{v\phi_V : v \in \mathbf{s}(t)\} = \mathbf{s}(t\phi_T)$ and $\mathbf{r}(t)\phi = \{v\phi_V : v \in \mathbf{r}(t)\} = \mathbf{r}(t\phi_T)$.

Let $\Gamma = (V_\Gamma, T_\Gamma, \mathbf{s}, \mathbf{r})$ and $\Delta = (V_\Delta, T_\Delta, \mathbf{s}, \mathbf{r})$ be networks. A homomorphism $\phi = (\phi_V, \phi_T) : \Gamma \to \Delta$ is called an *isomorphism* if both ϕ_V and ϕ_T are bijections. In this case, Γ and Δ are said to be *isomorphic*, and we write $\Gamma \cong \Delta$.

Proposition 1. *Let* $\Gamma = (V_\Gamma, T_\Gamma, \mathbf{s}, \mathbf{r})$ *and* $\Delta = (V_\Delta, T_\Delta, \mathbf{s}, \mathbf{r})$ *be networks. If* $\phi = (\phi_V, \phi_T) : \Gamma \to \Delta$ *is an isomorphism, then its inverse* $\phi^{-1} = (\phi_V^{-1}, \phi_T^{-1}) : \Delta \to \Gamma$ *is also an isomorphism.*

Proof. Since $\phi = (\phi_V, \phi_T)$ is an isomorphism, both ϕ_V and ϕ_T are bijections, and hence ϕ_V^{-1} and ϕ_T^{-1} are well-defined bijections. It suffices to show that ϕ^{-1} is a homomorphism.

Let $t' \in T_\Delta$. Since ϕ_T is surjective, there exists $t \in T_\Gamma$ such that $t' = t\phi_T$. Then

$$\mathbf{s}(t') = \mathbf{s}(t\phi_T) = \{v\phi_V : v \in \mathbf{s}(t)\}.$$

Applying ϕ_V^{-1}, we obtain

$$\{w\phi_V^{-1} : w \in \mathbf{s}(t')\} = \mathbf{s}(t),$$

and hence $\mathbf{s}(t')\phi^{-1} = \mathbf{s}(t'\phi_T^{-1})$. Similarly, $\mathbf{r}(t')\phi^{-1} = \{w\phi_V^{-1} : w \in \mathbf{r}(t')\} = \mathbf{r}(t'\phi_T^{-1})$. Thus ϕ^{-1} is a homomorphism, and therefore an isomorphism. $\square$

2. The semigroup Q_Γ

The aim of this section is to construct a semigroup using paths in a network.

Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a network. The set of *inverse edges* is defined as

$$T^{-1} = \{\, t^{-1} \mid t \in T \,\},$$

which is in bijection with T and disjoint from T. For each $t \in T$, the corresponding $t^{-1} \in T^{-1}$ is a *formal symbol* with

$$\mathbf{s}(t^{-1}) = \mathbf{r}(t), \quad \mathbf{r}(t^{-1}) = \mathbf{s}(t),$$

and it is distinct from any element of T, even if $(\mathbf{r}(t), \mathbf{s}(t)) \in T$.

Moreover, for any vertex $A \in T^0$, we set $A^{-1} = A$.

Let $\{0\}$ be disjoint from $T^0 \cup T \cup T^{-1}$. Extend the source and range maps by defining

$$\mathbf{s}(0) = \mathbf{r}(0) = 0,$$

and define the involution on 0 by

$$0^{-1} = 0.$$

Definition 7. Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a network. The semigroup Q_Γ is given by the presentation $Q_\Gamma := \langle X : R \rangle$, where

$$X = T^0 \cup T \cup T^{-1} \cup \{0\},$$

and R consists of the following relations:

(NR1) $\mathbf{s}(t)t = t = t\mathbf{r}(t)$ for $t \in T^0 \cup T \cup T^{-1}$;

(NR2) $t_1 t_2 = 0$ if $\mathbf{r}(t_1) \cap \mathbf{s}(t_2) = \emptyset$ for $t_1, t_2 \in T^0 \cup T \cup T^{-1}$;

(NR3) $t_1^{-1} t_2 = 0$ if $t_1 \neq t_2$ for $t_1, t_2 \in T$;

(NR4) $t^{-1} t = \mathbf{r}(t)$ for $t \in T$;

(NR5) $t^{-1} A = 0$ if $\mathbf{s}(t) \neq A$ for $t \in T$ and $A \in T^0$;

(NR6) $0x = 0 = x0$ for all $x \in X$.

By (NR6), the semigroup Q_Γ defined in Definition 7 has a zero element 0. For $A, B \in T^0$, we regard the product AB as a path from A to B whenever $A \cap B \neq \emptyset$.

For a graph Γ, the associated graph inverse semigroup was introduced by Ash and Hall [5] as the set of pairs (α, β) with $\alpha, \beta \in P(\Gamma)$ and $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$. Paterson [21] subsequently gave a presentation of this semigroup and proved that the two constructions are isomorphic. In [21], the graph inverse semigroup $I(\Gamma)$ of a graph $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ is defined as the semigroup generated by $T^0 \cup T \cup T^{-1}$ together with a zero 0, subject to the relations:

(V) $uv = \delta_{uv} u$ for all $u, v \in T^0$;

(E1) $\mathbf{s}(t)t = t = t\mathbf{r}(t)$ for each $t \in T$;

(E2) $\mathbf{r}(t)t^{-1} = t^{-1} = t^{-1}\mathbf{s}(t)$ for each $t \in T$;

(CK1) $t_1^{-1}t_2 = \delta_{t_1 t_2}\mathbf{r}(t_1)$ for all $t_1, t_2 \in T$,

where δ is Kronecker Delta.

Remark 2. Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a *graph*, that is, a network in which $\mathbf{s}(t), \mathbf{r}(t) \in V$ for all $t \in T$. Then the semigroup Q_Γ defined by (NR1)–(NR6) coincides with the graph inverse semigroup $I(\Gamma)$. Indeed, in this case $T^0 = V$, and all sources and ranges are single vertices. Consequently:

(i) By (NR2), for $u, v \in V$ we have $uv = 0$ whenever $u \neq v$, while $uu = u$ by (NR1). Hence $uv = \delta_{uv} u$, which is precisely the vertex relation (V).

(ii) Relation (NR1) gives $\mathbf{s}(t)t = t = t\mathbf{r}(t)$ for all $t \in T$, which is exactly (E1), and dually yields (E2) for t^{-1}.

(iii) Relations (NR3) and (NR4) combine to give

$$t_1^{-1}t_2 = \begin{cases} \mathbf{r}(t_1) & t_1 = t_2, \\ 0 & t_1 \neq t_2, \end{cases}$$

which is precisely the Cuntz–Krieger relation (CK1).

(iv) Relation (NR5) reduces to

$$t^{-1}A = 0 \quad \text{for } A \in V,\ A \neq \mathbf{s}(t),$$

which simply enforces that products involving t^{-1} and a vertex vanish unless the vertex matches the source of t. This is consistent with the vertex relation (V).

(v) Relation (NR6) provides a zero element.

Proposition 2. *If* Γ *is a graph then* Q_Γ *is the graph inverse semigroup* $I(\Gamma)$*.*

Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a network. If $\alpha = t_1t_2\cdots t_n \in P(\Gamma) \setminus \{0\}$, where $t_i \in T \cup T^0$, then α is a word over $T^0 \cup T \cup T^{-1}$. We define

$$\alpha^{-1} = t_n^{-1} \cdots t_2^{-1}t_1^{-1}.$$

If Γ is a graph, then every element of the graph inverse semigroup $I(\Gamma)$ admits a unique representation of the form $\alpha\beta^{-1}$, where $\alpha, \beta \in P(\Gamma)$. Motivated by this, we show that each element of Q_Γ has a unique normal form $\alpha\beta^{-1}$, analogous to the graph inverse semigroup case.

Let Γ be a network and consider $X = T^0 \cup T \cup T^{-1} \cup \{0\}$. Define the rewriting system $(X^+, \to)$ using the following reduction rules (R1)-(R6) corresponding to (NR1)-(NR6):

$$\begin{array}{lll}
(\mathrm{R1}): & \mathbf{s}(t)t \to t, \quad t\mathbf{r}(t) \to t, & t \in T^0 \cup T \cup T^{-1}, \\
(\mathrm{R2}): & xy \to 0, & x, y \in T^0 \cup T \cup T^{-1},\ \mathbf{r}(x) \cap \mathbf{s}(y) = \emptyset, \\
(\mathrm{R3}): & x^{-1}y \to 0, & x, y \in T,\ x \neq y, \\
(\mathrm{R4}): & x^{-1}y \to \mathbf{r}(y), & x, y \in T, x = y, \\
(\mathrm{R5}): & x^{-1}y \to 0, & x \in T, y \in T^0,\ \mathbf{s}(x) \neq y, \\
(\mathrm{R6}): & 0x \to 0, \quad x0 \to 0 & \forall x \in X.
\end{array}$$

Proposition 3. *The rewriting system* $(X^+, \to)$ *is confluent.*

Proof. Let $X = T^0 \cup T \cup T^{-1} \cup \{0\}$ and consider the rewriting system $(X^+, \to)$ defined by rules (R1)-(R6). To prove confluence, it suffices to consider the one-step overlapping case

$$(t_1t_2)t_3 = t_1(t_2t_3), \quad t_1, t_2, t_3 \in X,$$

that is, considering the situation

$$\begin{tikzcd}
 & t_1t_2t_3 \arrow[dl] \arrow[dr] & \\
ut_3 & & t_1v
\end{tikzcd}$$

we must show in each case that there exists w with

$$\begin{tikzcd}
ut_3 \arrow[dr, "*"'] & & t_1v \arrow[dl, "*"] \\
 & w &
\end{tikzcd}\ ,$$

where $\xrightarrow{*}$ is the reflexive and transitive closure of $\to$,

We consider all possible combinations of rules applied to t_1t_2 and t_2t_3. Since $0x = x0 = 0$ for all $x \in X$, if at least one of t_1, t_2, t_3 is 0, then$(t_1t_2)t_3 = 0 = t_1(t_2t_3)$. Hence, it suffices to consider the case where t_1t_2 and t_2t_3 satisfy one of the relations (R1)–(R5). Observe that it is impossible for both t_1t_2 and t_2t_3 to satisfy (R3), (R4), or (R5) simultaneously. Therefore, there are exactly seven nontrivial cases to consider.

In each of the following cases, the dual situation (obtained by interchanging the roles of t_1t_2 and t_2t_3) can be verified analogously, and will therefore be omitted.

We verify in each case that there exists an element w such that $ut_3{}^*{\to}w$ and $t_1v \xrightarrow{*} w$, where $t_1t_2 \to u$ and $t_2t_3 \to v$.

Case 1. Both t_1t_2 and t_2t_3 satisfy (R1).

Suppose that $t_1t_2 \to u$ and $t_2t_3 \to v$ by (R1). Then four subcases arise.

(i) Suppose that $t_1 = \mathbf{s}(t_2)$ and $t_2 = \mathbf{s}(t_3)$. Then $t_1 = t_2 = \mathbf{s}(t_3) \in T^0$, $u = t_2$ and $v = t_3$. Hence

$$ut_3 = t_2t_3 \to t_3, \qquad t_1v = t_1t_3 \to t_3.$$

Thus both expressions reduce to $w = t_3$.

(ii) Suppose that $t_1 = \mathbf{s}(t_2)$ and $t_3 = \mathbf{r}(t_2)$. Then $u = v = t_2$. Hence

$$ut_3 = t_2t_3 \to t_2, \qquad t_1v = t_1t_2 \to t_2,$$

and we take $w = t_2$.

(iii) Suppose that $t_2 = \mathbf{r}(t_1)$ and $t_2 = \mathbf{s}(t_3)$. Then $u = t_1$ and $v = t_3$. Hence

$$ut_3 = t_1t_3, \qquad t_1v = t_1t_3,$$

so both expressions coincide. Let $w = t_1t_3$.

(iv) Suppose that $t_2 = \mathbf{r}(t_1)$ and $t_3 = \mathbf{r}(t_2)$. Then $t_3 = t_2 = \mathbf{r}(t_1) \in T^0$, $u = t_1$ and $v = t_2$. Hence

$$ut_3 = t_1t_3 \to t_1, \qquad t_1v = t_1t_2 \to t_1.$$

Thus both reduce to $w = t_1$.

Case 2. t_1t_2 satisfies (R1) and t_2t_3 satisfies (R2).

Suppose that $t_1t_2 \to u$ and $t_2t_3 \to v$ by (R1) and (R2), respectively.

(i) If $t_1 = \mathbf{s}(t_2)$ and $\mathbf{r}(t_2) \cap \mathbf{s}(t_3) = \emptyset$, then $u = t_2$ and $v = 0$. Hence

$$ut_3 = t_2t_3 \to 0, \qquad t_1v = t_10 \to 0,$$

so we take $w = 0$.

(ii) If $t_2 = \mathbf{r}(t_1)$ and $\mathbf{r}(t_2) \cap \mathbf{s}(t_3) = \emptyset$, then $\mathbf{r}(t_1) = t_2 = \mathbf{r}(t_2) \in T^0$, $u = t_1$ and $v = 0$. Hence

$$ut_3 = t_1t_3 \to 0, \qquad t_1v = t_10 \to 0,$$

and again $w = 0$.

Case 3. t_1t_2 satisfies (R1) and t_2t_3 satisfies (R3).

In this case $t_1 = \mathbf{s}(t_2)$, $t_2 \in T^{-1}$, $t_3 \in T$ and $t_2 \neq t_3^{-1}$. Then $u = t_2$ and $v = 0$. Hence

$$ut_3 = t_2t_3 \to 0, \qquad t_1v = t_10 \to 0,$$

so $w = 0$.

Case 4. t_1t_2 satisfies (R1) and t_2t_3 satisfies (R4).

Here $t_1 = \mathbf{s}(t_2)$ and $t_2 = t_3^{-1}$. Then $t_1 = \mathbf{r}(t_3)$, $u = t_2$ and $v = \mathbf{r}(t_3)$. Hence

$$ut_3 = t_2t_3 \to \mathbf{r}(t_3), \qquad t_1v = t_1\mathbf{r}(t_3) \to \mathbf{r}(t_3).$$

Thus $w = \mathbf{r}(t_3)$.

Case 5. t_1t_2 satisfies (R1) and t_2t_3 satisfies (R5).

Here $t_1 = \mathbf{s}(t_2)$, $t_2 \in T^{-1}$, $t_3 \in T^0$ and $\mathbf{r}(t_2) \neq t_3$. Then $u = t_2$ and $v = 0$. Hence

$$ut_3 = t_2t_3 \to 0, \qquad t_1v = t_10 \to 0,$$

so $w = 0$.

Case 6. t_1t_2 satisfies (R2) and t_2t_3 satisfies (R2), (R3), or (R5).

In all such cases we have $u = 0$ and $v = 0$, and hence

$$ut_3 \to 0, \qquad t_1 v \to 0,$$

so $w = 0$.

Case 7. t_1t_2 satisfies (R2) and t_2t_3 satisfies (R4).

Then $\mathbf{r}(t_1) \cap \mathbf{s}(t_2) = \emptyset$, $\mathbf{s}(t_2) = \mathbf{r}(t_3)$, and so $u = 0$ and $v = \mathbf{r}(t_3)$. Hence

$$ut_3 = 0t_3 \to 0, \qquad t_1 v = t_1\mathbf{r}(t_3) = t_1\mathbf{s}(t_2) \to 0,$$

so $w = 0$.

This completes the verification. □

The reader may ask why we do not define the product on T^0 by $AB = A \cap B$. We show that such a definition leads to a failure of confluence. Let $t \in T$ and let $A \in T^0$ satisfy $\mathbf{r}(t^{-1}) \subsetneq A$. Suppose that the product on T^0 is given by set-theoretic intersection, that is, $AB = A \cap B$ for all $A, B \in T^0$. Then, using (NR1) and (NR5), we obtain

$$t^{-1}\mathbf{r}(t^{-1})A = (t^{-1}\mathbf{r}(t^{-1}))A \to t^{-1}A \to 0.$$

On the other hand, interpreting the product via intersection yields

$$t^{-1}\mathbf{r}(t^{-1})A = t^{-1}\big(\mathbf{r}(t^{-1})A\big) = t^{-1}\big(\mathbf{r}(t^{-1}) \cap A\big) = t^{-1}\mathbf{r}(t^{-1}) \to t^{-1},$$

where we use $\mathbf{r}(t^{-1}) \cap A = \mathbf{r}(t^{-1})$. Hence two distinct irreducible outcomes are obtained, and so $(X^+, \to)$ fails to be confluent.

By contrast, the rewriting system $(X^+, \to)$, where $X = T^0 \cup T \cup T^{-1} \cup \{0\}$, is clearly noetherian. It follows from Lemma 4 that every element of Q_Γ admits a unique normal form as a word over X.

We next describe these normal forms. Observe that if $\alpha \in P(\Gamma) \setminus \{0\}$, then any reduction of α involves only relations of type (R1), and hence preserves both $\mathbf{s}(\alpha)$ and $\mathbf{r}(\alpha)$. Consequently, every non-zero path reduces to a unique irreducible path.

A path $\alpha = t_1t_2 \cdots t_n$ with $t_i \in T \cup T^0$ $(1 \le i \le n)$ is said to be *reduced* (or *irreducible*) if no adjacent pair admits a reduction, that is, $t_i \ne \mathbf{r}(t_{i-1})$ and $t_{i-1} \ne \mathbf{s}(t_i)$ for all $2 \le i \le n$.

Lemma 6. *Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a network. For any $\alpha \in P(\Gamma) \setminus \{0\}$, let α' be the unique reduced path with $[\alpha] = [\alpha']$. Then $\mathbf{s}(\alpha) = \mathbf{s}(\alpha')$ and $\mathbf{r}(\alpha) = \mathbf{r}(\alpha')$.*

We remark that the reduction of a linear path is again linear, and that 0 is irreducible. Define

$$RP(\Gamma) = \{\alpha \in P(\Gamma) : \alpha \text{ is reduced}\},$$

and

$$RLP(\Gamma) = \{\alpha \in LP(\Gamma) : \alpha \text{ is reduced}\}.$$

Then $T \cup T^0 \subseteq RLP(\Gamma) \subseteq RP(\Gamma)$, and

$$RLP(\Gamma) = \{\, t_1 \cdots t_n : t_i \in T,\ \mathbf{r}(t_i) = \mathbf{s}(t_{i+1})\ (1 \le i < n)\,\} \cup T^0.$$

Lemma 7. *If $\alpha = t_1 \cdots t_n \in RLP(\Gamma)$, then every subword $t_i \cdots t_j$ $(1 \le i < j \le n)$ also lies in $RLP(\Gamma)$.*

Theorem 1. *Each element of Q_Γ has a unique normal form of one of the following types:*

$$\text{(i) } [\alpha], \qquad \text{(ii) } [\beta^{-1}], \qquad \text{(iii) } [\alpha\beta^{-1}], \qquad \text{(iv) } [0],$$

where $\alpha \in RP(\Gamma)$, $\beta \in RLP(\Gamma) \setminus T^0$, and in (iii) *one has $\mathbf{r}(\alpha) \cap \mathbf{r}(\beta) \neq \emptyset$.*

In particular, for $A \in T^0$, the normal form of $[A]$ is $[A]$, while $[A^{-1}]$ reduces to $[A]$.

Proof. Let $[w] \in Q_\Gamma$. By Proposition 3, we may assume that w is reduced. If $w \neq 0$, then w contains no subword of the form $x^{-1}y$ with $x \in T$, $y \in T \cup T^0$, nor of the form $x^{-1}y^{-1}$ with $x, y \in T$ such that yx is not linear. It follows that w is of the form α, β^{-1}, or $\alpha\beta^{-1}$, where $\alpha \in RP(\Gamma)$, $\beta \in RLP(\Gamma) \setminus T^0$, and in the latter case $\mathbf{r}(\alpha) \cap \mathbf{r}(\beta) \neq \emptyset$.

Finally, if $A \in T^0$, we have $A^{-1} = A$, and hence any occurrence of A^{-1} reduces to A. Therefore, the normal form of $[A]$ is $[A]$, and $[A^{-1}]$ has the same normal form. □

A word $\alpha\beta^{-1}$ with $\alpha \in RP(\Gamma)$, $\beta \in RLP(\Gamma)$ and $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$ is called a *right normal form.* In particular, 0 is in right normal form.

Each element of types (i)–(iii) in Theorem 1 can be written uniquely in the form $[\alpha\beta^{-1}]$ with $\alpha \in RP(\Gamma)$, $\beta \in RLP(\Gamma)$ and $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$. More precisely, one has

$$[\alpha] = [\alpha\beta^{-1}] \quad \text{with } \beta = \mathbf{r}(\alpha),$$

and

$$[\beta^{-1}] = [\alpha\beta^{-1}] \quad \text{with } \alpha = \mathbf{r}(\beta).$$

Furthermore, if an element is given in the form $[\alpha\beta^{-1}]$, then

$$[\alpha\beta^{-1}] = \begin{cases} [\alpha\beta^{-1}] & \text{if } \mathbf{r}(\alpha) = \mathbf{r}(\beta), \\ [\mu\beta^{-1}] & \text{otherwise,} \end{cases}$$

where $\mu = \alpha\,\mathbf{r}(\beta)$.

Corollary 1. *Each element of Q_Γ admits a unique representative of the form $\alpha\beta^{-1}$ in right normal form, where $\alpha \in RP(\Gamma)$, $\beta \in RLP(\Gamma)$ and $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$.*

Lemma 8. *If $\alpha \in LP(\Gamma) \setminus \{0\}$, then $[\alpha^{-1}\alpha] = [\mathbf{r}(\alpha)]$.*

Proof. The claim is immediate if $\alpha \in T \cup T^0$ by (NR1) and (NR4). Let $\alpha = t_1 \cdots t_n$ with $\mathbf{r}(t_i) = \mathbf{s}(t_{i+1})$ for $1 \leq i < n$. Then repeated applications of (NR4) and (NR1) yield

$$[\alpha^{-1}\alpha] = [t_n^{-1} \cdots t_1^{-1} t_1 \cdots t_n] = [t_n^{-1} t_n] = [\mathbf{r}(t_n)] = [\mathbf{r}(\alpha)].$$

□

Lemma 9. *Let $[\alpha\beta^{-1}], [\mu\nu^{-1}] \in Q_\Gamma \setminus \{[0]\}$ be such that $\alpha\beta^{-1}$ and $\mu\nu^{-1}$ are in right normal form, where $\alpha, \mu \in RP(\Gamma)$ and $\beta, \nu \in RLP(\Gamma)$. Then*

$$[\alpha\beta^{-1}][\mu\nu^{-1}] = \begin{cases} [\alpha\mu\nu^{-1}] & \text{if } \beta = \mathbf{r}(\alpha) \in T^0 \text{ and } \mathbf{r}(\alpha) \cap \mathbf{s}(\mu) \neq \emptyset, \\ [\alpha(\nu\beta)^{-1}] & \text{if } \beta \notin T^0,\ \mu = \mathbf{r}(\nu) = \mathbf{s}(\beta), \\ [\alpha\nu^{-1}] & \text{if } \beta \notin T^0 \text{ and } \mu = \beta, \\ [\alpha\xi\nu^{-1}] & \text{if } \beta \notin T^0 \text{ and } \mu = \beta\xi \text{ for some } \xi \in RP(\Gamma), \\ [\alpha(\nu\eta)^{-1}] & \text{if } \beta \notin T^0 \text{ and } \beta = \mu\eta \text{ for some } \eta \in RLP(\Gamma), \\ [0] & \text{otherwise.} \end{cases}$$

Proof. Let $[\alpha\beta^{-1}], [\mu\nu^{-1}] \in Q_\Gamma \setminus \{[0]\}$ be such that $\alpha\beta^{-1}$ and $\mu\nu^{-1}$ are in right normal form. We proceed by a case analysis, according to whether $\beta \in T^0$ or $\beta \notin T^0$, and whether $\mu \in T^0$ or $\mu \notin T^0$.

Case 1. $\beta \in T^0$. Then $\beta = \mathbf{r}(\alpha)$.

If $\mathbf{r}(\alpha) \cap \mathbf{s}(\mu) = \emptyset$, then by (NR2), $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha\mathbf{r}(\alpha)\mu\nu^{-1}] = [0]$. If $\mathbf{r}(\alpha) \cap \mathbf{s}(\mu) \neq \emptyset$, then by (NR1), $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha\mathbf{r}(\alpha)\mu\nu^{-1}] = [\alpha\mu\nu^{-1}]$.

Case 2. $\beta \notin T^0$ and $\mu \in T^0$. Then $\mu = \mathbf{r}(\nu)$.

If $\mathbf{s}(\beta) = \mu$, then by (NR1), $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha\beta^{-1}\nu^{-1}] = [\alpha(\nu\beta)^{-1}]$. Otherwise, by (NR5), the product is $[0]$.

Case 3. $\beta \notin T^0$ and $\mu \notin T^0$.

(i) Suppose that β and μ are prefix comparable.

(a1) If $\mu = \beta$, then using Lemma 8, $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$ and (NR1), we get $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha\nu^{-1}]$.

(a2) If $\mu = \beta\xi$ for some $\xi \in RP(\Gamma)$, then using Lemma 8, $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$ and (NR1),

$$[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha(\beta^{-1}\beta)\xi\nu^{-1}] = [\alpha\mathbf{r}(\beta)\xi\nu^{-1}] = [\alpha\xi\nu^{-1}].$$

(a3) If $\beta = \mu\eta$ for some $\eta \in RLP(\Gamma)$, then $\mu \in RLP(\Gamma)$ and $\mathbf{r}(\mu) = \mathbf{s}(\eta)$. By Lemma 8 and (NR1),

$$\begin{aligned}[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha\eta^{-1}\mu^{-1}\mu\nu^{-1}] = [\alpha\eta^{-1}\mathbf{r}(\mu)\nu^{-1}] &= [\alpha\eta^{-1}\nu^{-1}] \\ &= [\alpha(\nu\eta)^{-1}].\end{aligned}$$

(ii) Suppose that β and μ are not prefix comparable. Write $\beta = x_1x_2\cdots x_m$ and $\mu = y_1y_2\cdots y_n$, where $x_i \in T$ and $y_j \in T \cup T^0$.

(a1) If $x_1 \neq y_1$, then $\beta^{-1}\mu \to 0$.

Indeed, if $y_1 \in T$, then $x_1^{-1}y_1 \to 0$ by (NR3). If $y_1 \in T^0$, then either $y_1 \neq \mathbf{s}(x_1)$, so $x_1^{-1}y_1 \to 0$ by (NR5), or $y_1 = \mathbf{s}(x_1)$, in which case, since μ is reduced, $y_1 \neq \mathbf{s}(y_2)$ and hence

$$x_1^{-1}y_1y_2 \to x_1^{-1}\mathbf{s}(y_2)y_2 \to 0$$

by (NR5). Thus $\beta^{-1}\mu \to 0$, and so $[\alpha\beta^{-1}][\mu\nu^{-1}] = [0]$.

(a2) Otherwise, let $k \geq 2$ be minimal such that $x_k \neq y_k$ and $x_j = y_j$ for $1 \leq j \leq k-1$. Then

$$\beta^{-1}\mu = x_m^{-1}\cdots x_{k+1}^{-1}x_k^{-1}(x_{k-1}^{-1}\cdots x_1^{-1}x_1\cdots x_{k-1})y_k\cdots y_n.$$

Since $\beta \in RLP(\Gamma)$, we have $\mathbf{r}(x_{i-1}) = \mathbf{s}(x_i)$, and hence

$$x_{k-1}^{-1}\cdots x_1^{-1}x_1\cdots x_{k-1} \to \mathbf{r}(x_{k-1})$$

by Lemma 8. Thus

$$\beta^{-1}\mu \to x_m^{-1}\cdots x_{k+1}^{-1}x_k^{-1}y_k\cdots y_n.$$

Since $x_k \neq y_k$, if $y_k \in T$ then $x_k^{-1}y_k \to 0$ by (NR3); if $y_k \in T^0$, then reducedness of μ implies $\mathbf{r}(y_{k-1}) \neq y_k$, but $x_{k-1} = y_{k-1}$ and $\mathbf{r}(x_{k-1}) = \mathbf{s}(x_k)$, so $\mathbf{s}(x_k) = \mathbf{r}(y_{k-1}) \neq y_k$. Hence $x_k^{-1}y_k \to 0$ by (NR5). Therefore $\beta^{-1}\mu \to 0$.

Consequently, $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha(\beta^{-1}\mu)\nu^{-1}] = [0]$.

Combining all cases yields the result. □

Lemma 10. *The set of idempotents of Q_Γ is*

$$E(Q_\Gamma) = \{[\alpha\alpha^{-1}] : \alpha \in RLP(\Gamma)\} \cup \{[0]\}.$$

Moreover,

$$E = E(Q_\Gamma) \setminus \{[A] : A \in T^0\} = \{[\alpha\alpha^{-1}] : \alpha \in RLP(\Gamma) \setminus T^0\} \cup \{[0]\}$$

is a subsemilattice of Q_Γ.

Proof. It is immediate that $[0] \in E(Q_\Gamma)$. By Lemma 8 together with (NR1), for every $\alpha \in RLP(\Gamma)$ the element $[\alpha\alpha^{-1}]$ is idempotent.

Conversely, let $[\alpha\beta^{-1}] \in Q_\Gamma \setminus \{[0]\}$ be idempotent, where $\alpha\beta^{-1}$ is in right normal form. Then $[\alpha\beta^{-1}]^2 = [\alpha\beta^{-1}]$, and hence, by Lemma 9, one of the following cases occurs.

(i) If $\beta = \mathbf{r}(\alpha) \in T^0$, then $[\alpha] = [\alpha\beta^{-1}] = [\alpha\beta^{-1}]^2 = [\alpha\alpha]$, which implies that $\alpha \in T^0$. Thus $\alpha = \beta \in RLP(\Gamma)$.

(ii) Assume now that $\beta \notin T^0$. If $\alpha = \mathbf{r}(\beta) = \mathbf{s}(\beta) \in T^0$, then $[\alpha\beta^{-1}] = [\beta^{-1}]$, which is not idempotent, a contradiction. Hence $\alpha \notin T^0$. If $\alpha = \beta$, then clearly $[\alpha\beta^{-1}] = [\alpha\alpha^{-1}]$. Otherwise, either $\alpha = \beta\xi$ for some $\xi \in RP(\Gamma)$ or $\beta = \alpha\eta$ for some $\eta \in RLP(\Gamma)$.

If $\alpha = \beta\xi$, then $[\alpha\beta^{-1}] = [\alpha\beta^{-1}]^2 = [\alpha\xi\beta^{-1}]$, and since $\alpha\beta^{-1}$ is in right normal form, it follows that $[\alpha] = [\alpha\xi]$ or $[\xi\beta^{-1}] = [\beta^{-1}]$. As $\alpha \in RP(\Gamma)$ and $\beta \in RLP(\Gamma) \setminus \{[0]\}$, this forces $\xi = \mathbf{r}(\alpha) = \mathbf{r}(\beta)$, contradicting the reducedness of $\alpha = \beta\xi$.

If $\beta = \alpha\eta$, then $[\alpha\beta^{-1}] = [\alpha(\beta\eta)^{-1}]$, so $[\beta] = [\beta\eta]$. Since $\beta \notin T^0$, we deduce that $\eta = \mathbf{r}(\beta) \in T^0$, again contradicting the reducedness of $\beta = \alpha\eta$.

Hence neither of the above cases can occur, and we conclude that $\alpha = \beta \in RLP(\Gamma)$. Therefore $[\alpha\beta^{-1}] = [\alpha\alpha^{-1}]$, and so $E(Q_\Gamma) = \{[\alpha\alpha^{-1}] : \alpha \in RLP(\Gamma)\} \cup \{[0]\}$.

Finally, let $[\alpha\alpha^{-1}], [\beta\beta^{-1}] \in E \setminus \{[0]\}$. By Lemma 9,

$$[\alpha\alpha^{-1}][\beta\beta^{-1}] = \begin{cases} [\beta\beta^{-1}] & \text{if } \beta = \alpha\xi \text{ for some } \xi \in RLP(\Gamma), \\ [\alpha\alpha^{-1}] & \text{if } \alpha = \beta \text{ or } \alpha = \beta\eta \text{ for some } \eta \in RLP(\Gamma), \\ [0] & \text{otherwise.} \end{cases}$$

In particular, the product is commutative and again lies in E. Hence E is a semilattice. $\square$

Remark 3. Let $A, B \in T^0$ with $A \neq B$. Then $[A], [B] \in E(Q_\Gamma)$. However, if $A \cap B \neq \emptyset$, then $[AB] \notin E(Q_\Gamma)$, and moreover $[A][B] \neq [B][A]$. It follows that, in general, $E(Q_\Gamma)$ is neither closed under multiplication nor commutative; in particular, $E(Q_\Gamma)$ need not form a subsemigroup of Q_Γ.

Lemma 11. *Let* $[\alpha\beta^{-1}] \in Q_\Gamma$ *be such that* $\alpha\beta^{-1}$ *is in right normal form. Then* $[\alpha\beta^{-1}]$ *is regular if and only if* $\alpha \in RLP(\Gamma)$ *or* $[\alpha\beta^{-1}] = [0]$.

Proof. It is clear that $[0]$ is regular. Suppose that $\alpha \in RLP(\Gamma)$. Then, using (NR1) and Lemma 8, we obtain

$$\begin{aligned}[\alpha\beta^{-1}][\beta\alpha^{-1}][\alpha\beta^{-1}] &= [\alpha\mathbf{r}(\beta)(\alpha^{-1}\alpha)\beta^{-1}] = [\alpha(\alpha^{-1}\alpha)\beta^{-1}] \\ &= [\alpha\mathbf{r}(\alpha)\beta^{-1}] = [\alpha\beta^{-1}],\end{aligned}$$

where $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$. Thus $[\beta\alpha^{-1}]$ is an inverse of $[\alpha\beta^{-1}]$, and hence $[\alpha\beta^{-1}]$ is regular.

Conversely, let $[\alpha\beta^{-1}] \in Q_\Gamma \backslash \{[0]\}$ be regular, and suppose that $[\mu\nu^{-1}]$ is an inverse of $[\alpha\beta^{-1}]$, where both $\alpha\beta^{-1}$ and $\mu\nu^{-1}$ are in right normal form. Then $[\alpha\beta^{-1}][\mu\nu^{-1}]$ is a non-zero idempotent, and so $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\zeta\zeta^{-1}]$ for some $\zeta \in RLP(\Gamma)$. By Lemma 9, it follows that α is a prefix of ζ, and hence $\alpha \in RLP(\Gamma)$ by Lemma 7. $\square$

Lemma 12. *Let* $\alpha, \beta \in RLP(\Gamma)$ *with* $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$. *Then* $[\beta\alpha^{-1}]$ *is the unique inverse of* $[\alpha\beta^{-1}]$ *in* Q_Γ.

Proof. Since $\alpha, \beta \in RLP(\Gamma)$ and $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$, both $\alpha\beta^{-1}$ and $\beta\alpha^{-1}$ are in right normal form. By Lemma 11, $[\beta\alpha^{-1}]$ is an inverse of $[\alpha\beta^{-1}]$.

To prove uniqueness, let $[\mu\nu^{-1}]$ be any inverse of $[\alpha\beta^{-1}]$. Then

$$[\alpha\beta^{-1}][\mu\nu^{-1}][\alpha\beta^{-1}] = [\alpha\beta^{-1}], \tag{1}$$

and $[\mu\nu^{-1}]$ is regular. Hence, by Lemma 11, $\mu \in RLP(\Gamma)$.

We analyse the product $[\alpha\beta^{-1}][\mu\nu^{-1}]$.

Case 1. $\beta \in T^0$. Then $\beta = \mathbf{r}(\alpha)$, and by Lemma 9, $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha\mu\nu^{-1}]$, whenever $\mathbf{r}(\alpha) \cap \mathbf{s}(\mu) \neq \emptyset$. Hence

$$[\alpha\beta^{-1}][\mu\nu^{-1}][\alpha\beta^{-1}] = [\alpha\mu\nu^{-1}][\alpha\beta^{-1}].$$

Applying Lemma 9 again, the left component of the product $[\alpha\mu\nu^{-1}][\alpha\beta^{-1}]$ is $\alpha\mu$ (or an extension thereof). For (1) to hold, this must coincide with α, which forces $\mu = \mathbf{r}(\alpha) = \beta$. Substituting $\mu = \beta$, we obtain $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha\beta^{-1}][\beta\nu^{-1}] = [\alpha\nu^{-1}]$, and hence

$$[\alpha\nu^{-1}][\alpha\beta^{-1}] = [\alpha\beta^{-1}].$$

Since $\alpha \in RLP(\Gamma)$, a further application of Lemma 9 yields $\nu = \alpha$. Thus $[\mu\nu^{-1}] = [\beta\alpha^{-1}]$.

Case 2. $\beta \notin T^0$. Since $\beta, \mu \in RLP(\Gamma)$, Lemma 9 shows that a non-zero product can occur only in one of the following cases:

(i) $\mu = \mathbf{r}(\nu) = \mathbf{s}(\beta)$;

(ii) $\mu = \beta$;

(iii) $\mu = \beta\xi$ for some $\xi \in RP(\Gamma)$;

(iv) $\beta = \mu\eta$ for some $\eta \in RLP(\Gamma)$.

If (i) holds, then $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha(\nu\beta)^{-1}]$, and hence

$$[\alpha\beta^{-1}][\mu\nu^{-1}][\alpha\beta^{-1}] = [\alpha(\nu\beta)^{-1}][\alpha\beta^{-1}].$$

For (1) to hold, Lemma 9 forces $\nu\beta = \alpha$, and thus $[\nu^{-1}] = [\beta\alpha^{-1}]$. Consequently,

$$[\mu\nu^{-1}] = [\mathbf{s}(\beta)\,\beta\alpha^{-1}] = [\beta\alpha^{-1}].$$

If (iii) holds with non-trivial ξ, then $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha\xi\nu^{-1}]$, and hence

$$[\alpha\beta^{-1}][\mu\nu^{-1}][\alpha\beta^{-1}] = [\alpha\xi\nu^{-1}][\alpha\beta^{-1}].$$

By Lemma 9, the left component is $\alpha\xi$ (or an extension thereof), which cannot equal α. Hence (1) cannot hold.

If (iv) holds with non-trivial η, then $[\alpha\beta^{-1}][\mu\nu^{-1}] = [\alpha(\nu\eta)^{-1}]$, and similarly the product in (1) cannot reduce to $[\alpha\beta^{-1}]$, a contradiction.

Therefore the only possible case is (ii), namely $\mu = \beta$. Substituting into (1), we obtain $[\alpha\beta^{-1}][\beta\nu^{-1}][\alpha\beta^{-1}] = [\alpha\beta^{-1}]$. As in Case 1, repeated application of Lemma 9 yields $\nu = \alpha$.

Thus in all cases $\mu = \beta$ and $\nu = \alpha$, and hence $[\mu\nu^{-1}] = [\beta\alpha^{-1}]$. Therefore $[\beta\alpha^{-1}]$ is the unique inverse of $[\alpha\beta^{-1}]$. □

Lemma 13. *Let* $[\alpha\beta^{-1}] \in Q_\Gamma \setminus \{[0]\}$ *be regular, where* $\alpha\beta^{-1}$ *is in right normal form. Then* $\alpha, \beta \in RLP(\Gamma)$ *and* $[\alpha\alpha^{-1}]\,\mathcal{R}\,[\alpha\beta^{-1}]\,\mathcal{L}\,[\beta\beta^{-1}]$*, where* $[\alpha\alpha^{-1}], [\beta\beta^{-1}] \in E(Q_\Gamma)$*.*

Lemma 14. *Let* $[\alpha\beta^{-1}] \in Q_\Gamma \setminus \{[0]\}$ *be such that* $\alpha\beta^{-1}$ *is in right normal form. Then* $[\alpha\beta^{-1}]\,\mathcal{L}^*\,[\beta\beta^{-1}]$.

Proof. Let $[\alpha\beta^{-1}] \in Q_\Gamma \setminus \{[0]\}$ with $\alpha\beta^{-1}$ in right normal form. Then $\beta \in RLP(\Gamma)$, and hence $[\beta\beta^{-1}] \in E(Q_\Gamma)$ by Lemma 10. Moreover, $[\alpha\beta^{-1}][\beta\beta^{-1}] = [\alpha\beta^{-1}]$.

To verify that $[\alpha\beta^{-1}]\,\mathcal{L}^*\,[\beta\beta^{-1}]$, let $[x_1y_1^{-1}], [x_2y_2^{-1}] \in Q_\Gamma$ be such that $x_1y_1^{-1}$ and $x_2y_2^{-1}$ are in right normal form and $[\alpha\beta^{-1}][x_1y_1^{-1}] = [\alpha\beta^{-1}][x_2y_2^{-1}]$. We show that $[\beta\beta^{-1}][x_1y_1^{-1}] = [\beta\beta^{-1}][x_2y_2^{-1}]$.

First assume that $\beta = \mathbf{r}(\alpha) \in T^0$. Then $[\beta\beta^{-1}] = [\mathbf{r}(\alpha)]$. By Lemma 9, the equality $[\alpha\beta^{-1}][x_1y_1^{-1}] = [\alpha\beta^{-1}][x_2y_2^{-1}]$ implies either $\mathbf{r}(\alpha) \cap \mathbf{s}(x_1) = \emptyset$ and $\mathbf{r}(\alpha) \cap \mathbf{s}(x_2) = \emptyset$, or $\mathbf{r}(\alpha) \cap \mathbf{s}(x_1) \neq \emptyset$ and $\mathbf{r}(\alpha) \cap \mathbf{s}(x_2) \neq \emptyset$. In the former case, $[\mathbf{r}(\alpha)][x_1y_1^{-1}] = [\mathbf{r}(\alpha)][x_2y_2^{-1}] = [0]$. In the latter case, $[\alpha x_1y_1^{-1}] = [\alpha x_2y_2^{-1}]$, and since $\alpha \in RP(\Gamma)$ it follows that $[\mathbf{r}(\alpha)x_1y_1^{-1}] = [\mathbf{r}(\alpha)x_2y_2^{-1}]$. Thus $[\beta\beta^{-1}][x_1y_1^{-1}] = [\beta\beta^{-1}][x_2y_2^{-1}]$.

Assume now that $\beta \notin T^0$. We proceed by a case analysis according to whether x_1 or x_2 belongs to T^0.

(i) If $x_1, x_2 \in T^0$, then using Lemma 9, the equality $[\alpha\beta^{-1}][x_1y_1^{-1}] = [\alpha\beta^{-1}][x_2y_2^{-1}]$ implies that either $x_1 = x_2 = \mathbf{s}(\beta)$ or $x_1 \neq \mathbf{s}(\beta)$ and $x_2 \neq \mathbf{s}(\beta)$. In the former case, the equality $[\alpha\beta^{-1}][x_1y_1^{-1}] = [\alpha\beta^{-1}][x_2y_2^{-1}]$ yields $[\alpha(y_1\beta)^{-1}] = [\alpha(y_2\beta)^{-1}]$, and since $\alpha \in RP(\Gamma)$ we obtain

$$[\mathbf{r}(\alpha)(y_1\beta)^{-1}] = [\mathbf{r}(\alpha)(y_2\beta)^{-1}].$$

Using $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$, this gives $[\beta(y_1\beta)^{-1}] = [\beta(y_2\beta)^{-1}]$, that is,

$$[\beta\beta^{-1}][x_1y_1^{-1}] = [\beta\beta^{-1}][x_2y_2^{-1}].$$

In the latter case, both products $[\beta\beta^{-1}][x_iy_i^{-1}]$ are equal to $[0]$ for $i = 1, 2$.

(ii) Suppose that either $x_1 \in T^0$ and $x_2 \notin T^0$, or $x_1 \notin T^0$ and $x_2 \in T^0$. Since the latter situation is dual to the former, it suffices to consider the case where $x_1 \in T^0$ and $x_2 \notin T^0$. By Lemma 9, the equality $[\alpha\beta^{-1}][x_1y_1^{-1}] = [\alpha\beta^{-1}][x_2y_2^{-1}]$ implies that either $x_1 = \mathbf{s}(\beta)$ and x_2 is prefix-comparable with β, or $x_1 \neq \mathbf{s}(\beta)$ and x_2 is not prefix-comparable with β. In the latter case, both products are equal to $[0]$, and the conclusion is immediate.

Assume therefore that $x_1 = \mathbf{s}(\beta)$ and that x_2 is prefix-comparable with β. By Lemma 9, $[\alpha\beta^{-1}][x_1y_1^{-1}] = [\alpha(y_1\beta)^{-1}]$ and there exist three subcases for x_2 and β.

(a1) If $\beta = x_2$, then, by Lemma 9, $[\alpha\beta^{-1}][x_2y_2^{-1}] = [\alpha y_2^{-1}]$. Hence $[\alpha(y_1\beta)^{-1}] = [\alpha y_2^{-1}]$. Since $\alpha \in RP(\Gamma)$, we get

$$[\mathbf{r}(\alpha)(y_1\beta)^{-1}] = [\mathbf{r}(\alpha)y_2^{-1}],$$

and so $[\beta\mathbf{r}(\alpha)(y_1\beta)^{-1}] = [\beta\mathbf{r}(\alpha)y_2^{-1}]$. Noting that $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$, we obtain $[\beta(y_1\beta)^{-1}] = [\beta y_2^{-1}]$, that is, $[\beta\beta^{-1}][x_1y_1^{-1}] = [\beta\beta^{-1}][x_2y_2^{-1}]$ as required.

(a2) If $x_2 = \beta\mu_2$ for some $\mu_2 \in RP(\Gamma)$, then

$$[\alpha\beta^{-1}][x_2y_2^{-1}] = [\alpha\mu_2y_2^{-1}],$$

and the same argument as in (a1) gives

$$[\alpha(y_1\beta)^{-1}] = [\alpha\mu_2y_2^{-1}] \;\Rightarrow\; [\beta(y_1\beta)^{-1}] = [\beta\mu_2y_2^{-1}].$$

So $[\beta\beta^{-1}][x_1y_1^{-1}] = [\beta\beta^{-1}][x_2y_2^{-1}]$.

(a3) If $\beta = x_2\xi_2$ for some $\xi_2 \in RLP(\Gamma)$, then $[\alpha\beta^{-1}][x_2y_2^{-1}] = [\alpha(y_2\xi_2)^{-1}]$, and arguing as in (a1) we obtain

$$[\alpha(y_1\beta)^{-1}] = [\alpha(y_2\xi_2)^{-1}] \;\Rightarrow\; [\beta(y_1\beta)^{-1}] = [\beta(y_2\xi_2)^{-1}].$$

So $[\beta\beta^{-1}][x_1y_1^{-1}] = [\beta\beta^{-1}][x_2y_2^{-1}]$.

(iii) We now suppose that both $x_1, x_2 \notin T^0$. If β is prefix comparable with exactly one of x_1 and x_2, then Lemma 9 yields $[\alpha\beta^{-1}][x_1y_1^{-1}] \neq [\alpha\beta^{-1}][x_2y_2^{-1}]$, a contradiction. Hence β is either prefix comparable with both x_1 and x_2, or with neither.

If β is not prefix comparable with either x_1 or x_2, then both $[\alpha\beta^{-1}][x_iy_i^{-1}]$ and $[\beta\beta^{-1}][x_iy_i^{-1}]$ are equal to $[0]$ for $i = 1, 2$.

Otherwise, β is prefix comparable with both x_1 and x_2. It suffices to consider the case where $x_1 = \beta\mu_1$ and $x_2 = \beta\mu_2$ for some $\mu_1, \mu_2 \in RP(\Gamma)$, the remaining cases being analogous. Then $[\alpha\beta^{-1}][x_iy_i^{-1}] = [\alpha\mu_iy_i^{-1}] \neq [0]$ for $i = 1, 2$, and the hypothesis yields $[\alpha\mu_1y_1^{-1}] = [\alpha\mu_2y_2^{-1}]$. Since $\alpha \in RP(\Gamma)$, we obtain $[\mathbf{r}(\alpha)\mu_1y_1^{-1}] = [\mathbf{r}(\alpha)\mu_2y_2^{-1}]$, and hence $[\beta\mu_1y_1^{-1}] = [\beta\mu_2y_2^{-1}]$ as $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$. Noting that $[\beta\beta^{-1}][x_iy_i^{-1}] = [\beta\mu_iy_i^{-1}]$ for $i = 1, 2$, we conclude that $[\beta\beta^{-1}][x_1y_1^{-1}] = [\beta\beta^{-1}][x_2y_2^{-1}]$.

Therefore $[\alpha\beta^{-1}]\,\mathcal{L}^*\,[\beta\beta^{-1}]$ by Lemma 1. □

Lemma 15. *Suppose that* $[\alpha\beta^{-1}], [\mu\nu^{-1}] \in Q_\Gamma \setminus \{[0]\}$, *where* $\alpha\beta^{-1}$ *and* $\mu\nu^{-1}$, *are in right normal form. Then:*

(i) $[\alpha\beta^{-1}]\,\mathcal{L}^*\,[\mu\nu^{-1}]$ *if and only if* $\beta = \nu$;

(ii) *if* $\alpha, \mu \in RLP(\Gamma)$, *then* $[\alpha\beta^{-1}]\,\mathcal{R}\,[\mu\nu^{-1}]$ *if and only if* $\alpha = \mu$.

Proof. (i) By Lemma 14, we have $[\alpha\beta^{-1}]\,\mathcal{L}^*\,[\beta\beta^{-1}]$ and $[\mu\nu^{-1}]\,\mathcal{L}^*\,[\nu\nu^{-1}]$. Hence $[\alpha\beta^{-1}]\,\mathcal{L}^*\,[\mu\nu^{-1}]$ if and only if $[\beta\beta^{-1}]\,\mathcal{L}^*\,[\nu\nu^{-1}]$. Since $[\beta\beta^{-1}]$, $[\nu\nu^{-1}] \in E(Q_\Gamma)$, Lemma 11 implies that $\mathcal{L}^*$ coincides with $\mathcal{L}$ on idempotents, and thus $[\beta\beta^{-1}]\,\mathcal{L}\,[\nu\nu^{-1}]$. It is immediate that $\beta = \nu$ implies

$[\alpha\beta^{-1}]\,\mathcal{L}^*\,[\mu\nu^{-1}]$. Conversely, assume that $[\beta\beta^{-1}]\,\mathcal{L}\,[\nu\nu^{-1}]$. We distinguish three cases.

Case 1. If $\beta = \mathbf{r}(\alpha) \in T^0$ and $\nu = \mathbf{r}(\mu) \in T^0$, then $[\beta\beta^{-1}] = [\beta]$ and $[\nu\nu^{-1}] = [\nu]$. The relation $[\beta]\,\mathcal{L}\,[\nu]$ means that $[\beta][\nu] = [\beta]$ and $[\nu][\beta] = [\nu]$, which forces $\mathbf{r}(\beta) = \mathbf{r}(\nu)$, and hence $\beta = \nu$.

Case 2. If $\beta, \nu \in RLP(\Gamma) \setminus T^0$, then by Lemma 10 the idempotents commute, that is, $[\beta\beta^{-1}][\nu\nu^{-1}] = [\nu\nu^{-1}][\beta\beta^{-1}]$. The condition $[\beta\beta^{-1}]\,\mathcal{L}\,[\nu\nu^{-1}]$ implies $\beta = \nu$.

Case 3. If one of β, ν lies in T^0 and the other lies in $RLP(\Gamma) \setminus T^0$, then, say $\beta \in T^0$ and $\nu \notin T^0$, we have $[\beta\beta^{-1}] = [\beta] \neq [\beta\beta^{-1}][\nu\nu^{-1}]$, so $([\beta\beta^{-1}], [\nu\nu^{-1}]) \notin \mathcal{L}$, a contradiction. The symmetric case is analogous. Hence this situation cannot occur.

Thus $\beta = \nu$, and the proof of (i) is complete.

(ii) Let $\alpha, \mu \in RLP(\Gamma)$. Then $[\alpha\beta^{-1}]$ and $[\mu\nu^{-1}]$ are regular by Lemma 11. By Lemma 13, we have $[\alpha\beta^{-1}]\,\mathcal{R}\,[\alpha\alpha^{-1}]$ and $[\mu\nu^{-1}]\,\mathcal{R}\,[\mu\mu^{-1}]$. Hence $[\alpha\beta^{-1}]\,\mathcal{R}\,[\mu\nu^{-1}]$ if and only if $[\alpha\alpha^{-1}]\,\mathcal{R}\,[\mu\mu^{-1}]$. Arguing as in part (i), this is equivalent to $\alpha = \mu$. □

Lemma 16. *Suppose that* $\mu, \eta \in RP(\Gamma)$, $y \in RLP(\Gamma)$, $\mu = y\eta$, *and* $[\mu\nu^{-1}] \in Q_\Gamma \setminus \{[0]\}$ *with* $\mu\nu^{-1}$ *in right normal form. Then* $([yy^{-1}], [\mu\nu^{-1}]) \notin \mathcal{R}^*$.

Proof. Assume that $\mu, \eta \in RP(\Gamma)$, $y \in RLP(\Gamma)$ and $\mu = y\eta$, and let $[\mu\nu^{-1}] \in Q_\Gamma \setminus \{[0]\}$ be such that $\mu\nu^{-1}$ is in right normal form.

Let $[\alpha\beta^{-1}] \in Q_\Gamma \setminus \{[0]\}$, and choose $[xy^{-1}], [x_1y_1^{-1}] \in Q_\Gamma$ in right normal form such that $\alpha = y\xi = y_1\xi_1$ and $x\xi = x_1\xi_1$ for some $\xi, \xi_1 \in RLP(\Gamma)$, where y_1 and μ are not prefix comparable. Then

$$[xy^{-1}][\alpha\beta^{-1}] = [x\xi\beta^{-1}] = [x_1\xi_1\beta^{-1}] = [x_1y_1^{-1}][\alpha\beta^{-1}],$$

whereas

$$[xy^{-1}][\mu\nu^{-1}] = [x\eta\nu^{-1}] \neq [0] = [x_1y_1^{-1}][\mu\nu^{-1}].$$

It follows that $([\alpha\beta^{-1}], [\mu\nu^{-1}]) \notin \mathcal{R}^*$.

Now argue by contradiction. If $([yy^{-1}], [\mu\nu^{-1}]) \in \mathcal{R}^*$, then also $([yy^{-1}], [\alpha\beta^{-1}]) \in \mathcal{R}^*$, and hence $([\alpha\beta^{-1}], [\mu\nu^{-1}]) \in \mathcal{R}^*$, contradicting the above. Therefore $([yy^{-1}], [\mu\nu^{-1}]) \notin \mathcal{R}^*$. □

Let $[yy^{-1}] \in E(Q_\Gamma) \setminus \{[0]\}$. By Lemma 10, we have $y \in RLP(\Gamma)$. Let $\mu \in RP(\Gamma) \setminus RLP(\Gamma)$ be a non-zero element. Then either y is a prefix of μ, or y is not prefix-comparable with μ. If y is a prefix of μ, then by Lemma 16, the idempotent $[yy^{-1}]$ need not be $\mathcal{R}^*$-related to $[\mu\nu^{-1}] \in Q_\Gamma$. If y is not prefix-comparable with μ, then by Lemma 9 we have $[yy^{-1}][\mu\nu^{-1}] = [0]$. It follows that there exists an $\mathcal{R}^*$-class of Q_Γ which contains no idempotent, and hence Q_Γ is not left abundant. On the other hand, by Lemma 15 (i) and Lemma 14, each $\mathcal{L}^*$-class of Q_Γ contains a unique idempotent. Therefore, we obtain the following.

Theorem 2. *The semigroup Q_Γ is a right $*$-abundant semigroup with zero.*

Put

$$S_\Gamma = \{[\alpha\beta^{-1}] \in Q_\Gamma :\ \alpha \in RP(\Gamma),\ \beta \in RLP(\Gamma)\backslash T^0,\ \mathbf{r}(\alpha) = \mathbf{r}(\beta)\} \cup \{[0]\}.$$

Then the set

$$E = E(Q_\Gamma) \setminus \{[A] : A \in T^0\} = \{[\alpha\alpha^{-1}] : \alpha \in RLP(\Gamma) \setminus T^0\} \cup \{[0]\}$$

coincides with the set of idempotents of S_Γ. It follows from Lemma 9 that S_Γ is a subsemigroup of Q_Γ. Moreover, since $\beta \in RLP(\Gamma) \setminus T^0$ for every $[\alpha\beta^{-1}] \in S_\Gamma$, the multiplication in S_Γ can be described as follows: for all $[\alpha\beta^{-1}], [\mu\nu^{-1}] \in S_\Gamma$,

$$[\alpha\beta^{-1}][\mu\nu^{-1}] = \begin{cases} [\alpha(\nu\beta)^{-1}] & \text{if } \mu = \mathbf{r}(\nu) = \mathbf{s}(\beta), \\ [\alpha\nu^{-1}] & \text{if } \beta = \mu, \\ [\alpha\xi\nu^{-1}] & \text{if } \mu = \beta\xi \text{ for some } \xi \in RP(\Gamma), \\ [\alpha(\nu\eta)^{-1}] & \text{if } \beta = \mu\eta \text{ for some } \eta \in RLP(\Gamma), \\ [0] & \text{otherwise.} \end{cases}$$

Proposition 4. *The semigroup S_Γ is a right ample semigroup with zero.*

Proof. By Lemma 10 and Lemma 14, the semigroup S_Γ is right abundant and its set of idempotents forms the semilattice $E = \{[\alpha\alpha^{-1}] : \alpha \in RLP(\Gamma) \setminus T^0\} \cup \{[0]\}$. Thus it suffices to verify the right ample identity $ea = a(ea)^*$ for all $e \in E$ and $a \in S_\Gamma \setminus \{[0]\}$.

Let $e = [\xi\xi^{-1}] \in E$ and $a = [\alpha\beta^{-1}] \in S_\Gamma \setminus \{[0]\}$. By Lemma 9, the product $[\xi\xi^{-1}][\alpha\beta^{-1}]$ is equal to $[\alpha\beta^{-1}]$ if $\alpha = \xi$ or $\alpha = \xi\mu$ for some $\mu \in RP(\Gamma)$, is equal to $[\xi(\beta\eta)^{-1}]$ if $\xi = \alpha\eta$ for some $\eta \in RLP(\Gamma)$, and is

[0] otherwise. Consequently, $([\xi\xi^{-1}][\alpha\beta^{-1}])^* = [\beta\beta^{-1}]$ in the first case, $[(\beta\eta)(\beta\eta)^{-1}]$ in the second case, and [0] otherwise.

A direct computation, again using Lemma 9, shows that $[\alpha\beta^{-1}][\beta\beta^{-1}] = [\alpha\beta^{-1}]$ and $[\alpha\beta^{-1}][\beta\eta(\beta\eta)^{-1}] = [\xi(\beta\eta)^{-1}]$ whenever $\xi = \alpha\eta$, while in the remaining case both products are equal to [0]. It follows that $[\alpha\beta^{-1}]([\xi\xi^{-1}][\alpha\beta^{-1}])^*$ coincides with $[\xi\xi^{-1}][\alpha\beta^{-1}]$ in all cases. Hence $ea = a(ea)^*$, and therefore S_Γ is right ample with zero. □

By Lemma 11, if $\alpha, \beta \in RLP(\Gamma)$ with $\mathbf{r}(\alpha) = \mathbf{r}(\beta)$, then $[\alpha\beta^{-1}]$ is regular. Define

$$R_\Gamma = \{[\alpha\beta^{-1}] \in S_\Gamma : \alpha, \beta \in RLP(\Gamma) \setminus T^0 \text{ and } \mathbf{r}(\alpha) = \mathbf{r}(\beta)\} \cup \{[0]\}.$$

Then $E \subseteq R_\Gamma$, and by Lemma 2 the set R_Γ is an inverse semigroup of S_Γ. Moreover, by Lemma 15, the Green's relation $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ is trivial on R_Γ. Recall that a semigroup is *fundamental* if the only congruence contained in $\mathcal{H}$ is the identity congruence [19]. Hence we obtain the following.

Corollary 2. *The semigroup R_Γ is a fundamental inverse semigroup and a subsemigroup of S_Γ.*

3. The congruence-free case

The aim of this section is to investigate congruence-free conditions on Q_Γ. We will consider congruences to be semigroup congruences, unless stated otherwise. Congruences compatible with both the multiplication and the unary operation $*$ will be referred to as $*$-*congruences*.

We begin by recalling that a semigroup is said to be *congruence-free* if it admits precisely two congruences, namely the identity congruence and the universal congruence. A congruence ρ on a semigroup S with set of idempotents $E(S)$ is called *idempotent-separating* if $e\rho \neq f\rho$ for all distinct $e, f \in E(S)$. A non-empty subset I of S is an *ideal* if $SI \subseteq I$ and $IS \subseteq I$, and such an ideal is said to be *proper* if $I \neq S$.

Lemma 17. *Let $I = \{[\alpha\beta^{-1}] \in Q_\Gamma : \ \alpha \in RP(\Gamma) \setminus RLP(\Gamma)\} \cup \{[0]\}$. Then I is a proper ideal of Q_Γ. Moreover, the relation $\rho_I = (I \times I) \cup 1_{Q_\Gamma}$ is an idempotent-separating congruence on Q_Γ.*

Proof. By Lemma 9, the set I is closed under multiplication in Q_Γ, and clearly $Q_\Gamma \setminus I \neq \emptyset$, so $I \neq Q_\Gamma$. Let $[\alpha\beta^{-1}] \in I$ and $[\mu\nu^{-1}] \in Q_\Gamma$. Since $\alpha \in RP(\Gamma) \setminus RLP(\Gamma)$, an application of Lemma 9 shows that $[\alpha\beta^{-1}][\mu\nu^{-1}] \in I$.

To verify that $Q_\Gamma I \subseteq I$, consider $[\mu\nu^{-1}][\alpha\beta^{-1}]$. Since $\alpha \notin RLP(\Gamma)$ and $\nu \in RLP(\Gamma)$, Lemma 7 implies that $\nu \neq \alpha$ and $\nu \neq \alpha\eta$ for any $\eta \in RLP(\Gamma)$. Hence, by Lemma 9, the product $[\mu\nu^{-1}][\alpha\beta^{-1}]$ is equal to $[\mu\alpha\beta^{-1}]$ if $\nu = \mathbf{r}(\mu)$ and $\mathbf{r}(\mu) \cap \mathbf{s}(\alpha) \neq \emptyset$, or to $[\mu\xi\nu^{-1}]$ if $\alpha = \nu\xi$ for some $\xi \in RP(\Gamma)$, and is $[0]$ otherwise.

If $\nu = \mathbf{r}(\mu)$ and $\mathbf{r}(\mu) \cap \mathbf{s}(\alpha) \neq \emptyset$, then $\mu\alpha \in RP(\Gamma) \setminus RLP(\Gamma)$ since $\alpha \in RP(\Gamma) \setminus RLP(\Gamma)$, and hence $[\mu\alpha\beta^{-1}] \in I$. If $\alpha = \nu\xi$ for some $\xi \in RP(\Gamma)$, then either $\mathbf{r}(\nu) \neq \mathbf{s}(\xi)$ with $\mathbf{r}(\nu) \cap \mathbf{s}(\xi) \neq \emptyset$, or $\mathbf{r}(\nu) = \mathbf{s}(\xi)$ and $\xi \in RP(\Gamma) \setminus RLP(\Gamma)$. In the former case, since $\mathbf{r}(\mu) = \mathbf{r}(\nu)$, we have $\mathbf{r}(\mu) \neq \mathbf{s}(\xi)$ and $\mathbf{r}(\mu) \cap \mathbf{s}(\xi) \neq \emptyset$, so $\mu\xi \in RP(\Gamma) \setminus RLP(\Gamma)$ and thus $[\mu\xi\nu^{-1}] \in I$. In the latter case, $\mu\xi \in RP(\Gamma) \setminus RLP(\Gamma)$ again follows directly, and hence $[\mu\xi\nu^{-1}] \in I$. In the remaining case the product is $[0] \in I$. Therefore $Q_\Gamma I \subseteq I$, and so I is an ideal of Q_Γ; moreover, it is proper.

Finally, by Lemma 10, the only idempotent contained in I is $[0]$. It follows immediately that ρ_I is an idempotent-separating congruence on Q_Γ. □

A similar construction yields a proper ideal of S_Γ.

Lemma 18. *Let* $I = \{[\alpha\beta^{-1}] \in S_\Gamma : \ \alpha \in RP(\Gamma) \setminus RLP(\Gamma)\} \cup \{[0]\}$*. Then* I *is a proper ideal of* S_Γ*, and* $\rho_I = (I \times I) \cup 1_{S_\Gamma}$ *is an idempotent-separating congruence on* S_Γ*.*

Proof. The proof is analogous to that of Lemma 17, and is therefore omitted. □

According to Lemma 11, Lemma 17 and Lemma 18, the semigroups Q_Γ and S_Γ fail to be congruence-free whenever they contain non-regular elements. Indeed, if $t \in T$ satisfies $|\mathbf{s}(t)| > 1$ and $v \in \mathbf{s}(t)$, then $vt \in RP(\Gamma) \setminus RLP(\Gamma)$, whence $[vtt^{-1}] \in S_\Gamma \subseteq Q_\Gamma$ is a non-regular element. This yields the following result.

Theorem 3. *Let* $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ *be a network. If there exists* $t \in T$ *with* $|\mathbf{s}(t)| > 1$*, then neither* Q_Γ *nor* S_Γ *is congruence-free.*

Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a network. For each subset $A \subseteq V$, the cardinality of the set $\{t \in T : \mathbf{s}(t) = A\}$ is called the *out-index* of A in Γ, and is denoted by $o(A)$.

Let S be a right abundant semigroup. An ideal I of S is called a $*$*-ideal* if it is closed under the relation $\mathcal{L}^*$. Viewing S as a unary semigroup, it is straightforward to verify that if I is a proper $*$-ideal of

S, then $\rho_I = (I \times I) \cup 1_S$ is a unary semigroup congruence on S. Indeed, if $a \, \rho_I \, b$, then either $a = b$, in which case $a^* = b^*$ trivially, or $a, b \in I$, and hence $a^*, b^* \in I$ since I is closed under the unary operation *; thus $a^* \, \rho_I \, b^*$.

A right $*$-abundant semigroup is said to be *$*$-congruence-free* if it admits only two unary semigroup congruences, namely the identity congruence and the universal congruence.

Lemma 19. *Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a network and let $t \in T$ be such that $o(\mathbf{r}(t)) = 0$ and there exists no $A \in T^0 \setminus V$ with $\mathbf{r}(t) \subseteq A$. Then the principal ideal $I = Q_\Gamma[tt^{-1}]Q_\Gamma$ generated by $[tt^{-1}]$ is a proper $*$-ideal of Q_Γ.*

Proof. Let $t \in T$ satisfy the stated hypotheses. We first show that $|\mathbf{r}(t)| = 1$. Indeed, if $|\mathbf{r}(t)| > 1$, then $\mathbf{r}(t) \in T^0 \setminus V$ and trivially $\mathbf{r}(t) \subseteq \mathbf{r}(t)$, contradicting the assumption. Hence $\mathbf{r}(t)$ is a singleton.

We derive the explicit description of the ideal $I = Q_\Gamma[tt^{-1}]Q_\Gamma$ using Lemma 9.

Let $[\alpha\beta^{-1}], [\mu\nu^{-1}] \in Q_\Gamma \setminus \{[0]\}$ be such that $\alpha\beta^{-1}$ and $\mu\nu^{-1}$ are in right normal form. We first compute the product $[\alpha\beta^{-1}][tt^{-1}]$. Since $t, \beta \in RLP(\Gamma)$, Lemma 9 shows that a non-zero product can occur only if either $\beta = \mathbf{r}(\alpha) \in T^0$ and $\mathbf{r}(\alpha) \cap \mathbf{s}(\alpha) \neq \emptyset$ or $\beta \notin T^0$ and t are prefix comparable, namely in the cases $\beta = t$, $\beta = t\xi$ for some $\xi \in RP(\Gamma)$, or $t = \beta\eta$ for some $\eta \in RLP(\Gamma)$.

In the former case, we get $[\alpha\beta^{-1}][tt^{-1}] = [att^{-1}]$. In the later case, the assumption $o(\mathbf{r}(t)) = 0$ and $|\mathbf{r}(t)| = 1$ exclude any non-trivial extension at $\mathbf{r}(t)$. Consequently, neither $\beta = t\xi$ with non-trivial ξ nor $t = \beta\eta$ with non-trivial η can occur. Hence only the case $\beta = t$ yields a non-zero product, and then $[\alpha\beta^{-1}][tt^{-1}] = [\alpha t^{-1}]$. Thus every non-zero element of $Q_\Gamma[tt^{-1}]$ is of the form $[\gamma t^{-1}]$ with $\gamma \in RP(\Gamma)$ and $\mathbf{r}(\gamma) = \mathbf{r}(t)$.

Next, consider the product $[\gamma t^{-1}][\mu\nu^{-1}]$. By Lemma 9, a non-zero product can occur either when $\mu = \mathbf{r}(\nu) = \mathbf{s}(t)$ or when t and μ are prefix comparable. In the former case we obtain $[\gamma t^{-1}][\mu\nu^{-1}] = [\gamma(\nu t)^{-1}]$, therefore the resulting element again has $\mathbf{r}(\gamma) = \mathbf{r}(\nu t) = \mathbf{r}(t)$.

In the latter case, prefix comparability yields $\mu = t$, $\mu = t\xi$ for some $\xi \in RP(\Gamma)$, or $t = \mu\eta$ for some $\eta \in RLP(\Gamma)$. As before, the condition $o(\mathbf{r}(t)) = 0$ excludes the possibilities $\mu = t\xi$ and $t = \mu\eta$, so that necessarily $\mu = t$. In this case we obtain $[\gamma t^{-1}][\mu\nu^{-1}] = [\gamma\nu^{-1}]$, and clearly $\mathbf{r}(\gamma) = \mathbf{r}(\nu) = \mathbf{r}(t)$.

Combining the above, every non-zero element of $Q_\Gamma[tt^{-1}]Q_\Gamma$ is of the form $[\alpha\beta^{-1}]$ with $\alpha \in RP(\Gamma)$, $\beta \in RLP(\Gamma)$ and $\mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t)$.

Therefore,

$$Q_\Gamma[tt^{-1}]Q_\Gamma = \{[\alpha\beta^{-1}] : \alpha \in RP(\Gamma),\ \beta \in RLP(\Gamma),\\ \mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t)\} \cup \{[0]\}.$$

By Lemma 15(i), each $\mathcal{L}^*$-class in Q_Γ is determined by the second component in right normal form; hence I is closed under $\mathcal{L}^*$ and therefore is a $*$-ideal. Finally, since $\mathbf{s}(t) \cap \mathbf{r}(t) = \emptyset$, we have $[\mathbf{s}(t)] \notin I$, so that I is a proper ideal of Q_Γ. $\square$

The following consequence is immediate.

Theorem 4. *Let $\Gamma = (V, T, \mathbf{s}, \mathbf{r})$ be a network. If there exists $t \in T$ such that $o(\mathbf{r}(t)) = 0$ and there is no $A \in T^0 \setminus V$ with $\mathbf{r}(t) \subseteq A$, then Q_Γ is not $*$-congruence-free as a unary semigroup.*

Lemma 20. *Let $\Gamma = (V, T, \mathbf{r}, \mathbf{s})$ be a network with $|T| > 1$, and let $t, q \in T$ be such that $o(\mathbf{r}(t)) = 0$, $\mathbf{r}(t) \neq \mathbf{r}(q)$, and there exists no $A \in T^0 \setminus V$ with $\mathbf{r}(t) \subseteq A$. Then the principal ideal generated by $[tt^{-1}]$ is a proper $*$-ideal of S_Γ.*

Proof. Let $t, q \in T$ satisfy the stated hypotheses. As in the proof of Lemma 19, the assumptions imply that $|\mathbf{r}(t)| = 1$, and hence $\mathbf{r}(t) \in V$.

Using the multiplication rule in S_Γ together with the condition $o(\mathbf{r}(t)) = 0$, the same argument as in Lemma 19 shows that non-zero products involving $[tt^{-1}]$ arise only in the trivial prefix-comparable case. Consequently, every non-zero element of $S^1_\Gamma[tt^{-1}]S^1_\Gamma$ is of the form $[\alpha\beta^{-1}]$ with $\alpha \in RP(\Gamma)$, $\beta \in RLP(\Gamma) \setminus T^0$, and $\mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t)$. Hence

$$S^1_\Gamma[tt^{-1}]S^1_\Gamma = \{[\alpha\beta^{-1}] : \alpha \in RP(\Gamma),\ \beta \in RLP(\Gamma) \setminus T^0,\\ \mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t)\} \cup \{[0]\}.$$

By Lemma 15(i), the set I is closed under $\mathcal{L}^*$, and thus forms a $*$-ideal of S_Γ. To see that I is proper, observe that $\mathbf{r}(q) \neq \mathbf{r}(t)$, so $[qq^{-1}] \notin I$, and hence $I \neq S_\Gamma$. $\square$

Lemma 21. *Let $\Gamma = (V, T, \mathbf{r}, \mathbf{s})$ be a network with $|T| > 1$, and let $t, q \in T$ be such that $o(\mathbf{r}(t)) = 0$, $\mathbf{r}(t) \neq \mathbf{r}(q)$, and there exists no $A \in T^0 \setminus V$ with $\mathbf{r}(t) \subseteq A$. Then the principal ideal $R_\Gamma[tt^{-1}]R_\Gamma$ generated by $[tt^{-1}]$ is a proper ideal of R_Γ, where*

$$R_\Gamma[tt^{-1}]R_\Gamma = \{[\alpha\beta^{-1}] : \alpha, \beta \in RLP(\Gamma) \setminus T^0,\ \mathbf{r}(\alpha) = \mathbf{r}(\beta) = \mathbf{r}(t)\} \cup \{[0]\}.$$

Proof. The proof is analogous to that of Lemma 20, restricting to elements whose components lie in $RLP(\Gamma)\backslash T^0$, and is therefore omitted. □

Theorem 5. *Let* $\Gamma = (V, T, \mathbf{r}, \mathbf{s})$ *be a network with* $|T| > 1$*, and suppose that there exist* $t, q \in T$ *such that* $o(\mathbf{r}(t)) = 0$*,* $\mathbf{r}(t) \neq \mathbf{r}(q)$*, and there exists no* $A \in T^0 \setminus V$ *with* $\mathbf{r}(t) \subseteq A$*. Then* S_Γ *is not* $*$*-congruence-free as a unary semigroup, and* R_Γ *is not congruence-free.*

Proof. By Lemma 20, the semigroup S_Γ admits a proper $*$-ideal I. Hence the relation $\rho_I = (I \times I) \cup 1_{S_\Gamma}$ defines a non-trivial unary semigroup congruence on S_Γ, so S_Γ is not $*$-congruence-free.

Similarly, by Lemma 21, the semigroup R_Γ admits a proper ideal, which induces a non-trivial congruence. Therefore R_Γ is not congruence-free. □

4. Homomorphisms

In this section we investigate the relationship between network homomorphisms and semigroup homomorphisms.

Let S be a semigroup with set of idempotents $E(S)$. The *natural partial order* on S is defined by

$$a \leq b \iff a = xb = by \text{ and } xa = a$$

for some $x, y \in S^1$ (see [18]).

When restricted to $E(S)$, the natural partial order takes the familiar form: for all $e, f \in E(S)$, we have $e \leq f$ if and only if $e = ef = fe$. In particular, $E(S)$ becomes a partially ordered set under $\leq$. Moreover, if $E(S)$ is a semilattice, then the order simplifies further, and for all $e, f \in E(S)$ we have

$$e \leq f \quad \text{if and only if } e = ef.$$

Lemma 22. *Let* $E(Q_\Gamma)$ *be the set of all idempotents of* Q_Γ *and let* $\leq$ *be the natural partial order on* Q_Γ*. Then the following statements hold.*

(i) *An idempotent* $[\alpha\alpha^{-1}]$ *is maximal in* $E(Q_\Gamma)$ *with respect to* $\leq$ *if and only if* $\alpha \in T^0$*.*

(ii) *An idempotent* $[\alpha\alpha^{-1}]$ *is maximal in* $E = E(Q_\Gamma) \setminus \{[A] : A \in T^0\}$ *with respect to* $\leq$ *if and only if* $\alpha \in T$*.*

Proof. (i) Suppose that $\alpha \in T^0$ and $[\alpha\alpha^{-1}] \leq [\mu\mu^{-1}]$ for some $\mu \in RLP(\Gamma)$. Then $[\mu\mu^{-1}] \neq [0]$ and $[\alpha\alpha^{-1}] = [\alpha\alpha^{-1}][\mu\mu^{-1}] = [\mu\mu^{-1}][\alpha\alpha^{-1}]$. Since $\alpha \in T^0$, we have $\alpha^{-1} = \alpha$ and hence $[\alpha\alpha^{-1}] = [\alpha]$. It follows that $[\alpha] = [\alpha\mu\mu^{-1}] = [\mu\mu^{-1}\alpha]$. If $\mu \in RLP(\Gamma) \setminus T^0$, this is impossible. Hence $\mu \in T^0$, and the above equalities reduce to $[\alpha] = [\alpha\mu] = [\mu\alpha]$, which implies $\alpha = \mu$. Thus $[\alpha\alpha^{-1}]$ is maximal in $E(Q_\Gamma)$.

Conversely, let $\alpha \in RLP(\Gamma)$ and suppose that $[\alpha\alpha^{-1}]$ is maximal in $E(Q_\Gamma)$. Then $[\alpha\alpha^{-1}] \neq [0]$, and $[\alpha\alpha^{-1}] \leq [\mathbf{s}(\alpha)]$, since $[\alpha\alpha^{-1}] = [\mathbf{s}(\alpha)\alpha\alpha^{-1}] = [\alpha\alpha^{-1}\mathbf{s}(\alpha)]$. By maximality, we obtain $[\alpha\alpha^{-1}] = [\mathbf{s}(\alpha)]$, and hence $\alpha = \mathbf{s}(\alpha) \in T^0$.

(ii) By Lemma 10, the set $E = E(Q_\Gamma)\setminus\{[A] : A \in T^0\}$ is a semilattice. Let $\alpha \in T$ and suppose that $[\alpha\alpha^{-1}] \leq [\mu\mu^{-1}]$ for some $\mu \in RLP(\Gamma) \setminus T^0$. Then $[\mu\mu^{-1}] \neq [0]$ and $[\alpha\alpha^{-1}] = [\alpha\alpha^{-1}][\mu\mu^{-1}]$. By the proof of Lemma 10, this implies that either $\alpha = \mu$ or $\alpha = \mu x$ for some $x \in RLP(\Gamma)$. Since $\alpha \in T$ and $\mu \in RLP(\Gamma) \setminus T^0$, the latter possibility cannot occur, and hence $\alpha = \mu$. Therefore $[\alpha\alpha^{-1}]$ is maximal in E.

Conversely, suppose that $\alpha \in RLP(\Gamma) \setminus T^0$ and $[\alpha\alpha^{-1}]$ is maximal in E. Then $[\alpha\alpha^{-1}] \neq [0]$. Thus we may write $\alpha = t\beta$ for some $t \in T$ and $\beta \in RLP(\Gamma)$. By the proof of Lemma 10, we have $[\alpha\alpha^{-1}] \leq [tt^{-1}]$, and since $[tt^{-1}] \in E$, maximality yields $[\alpha\alpha^{-1}] = [tt^{-1}]$. As both $\alpha\alpha^{-1}$ and tt^{-1} are in right normal form, it follows that $\alpha = t \in T$. □

Theorem 6. *Let* $\Gamma = (V_\Gamma, T_\Gamma, \mathbf{s}, \mathbf{r})$ *and* $\Delta = (V_\Delta, T_\Delta, \mathbf{s}, \mathbf{r})$ *be networks. Then* $\Gamma \cong \Delta$ *if and only if* $Q_\Gamma \cong Q_\Delta$.

Proof. It suffices to prove the non-trivial direction. Suppose that $\theta : Q_\Gamma \to Q_\Delta$ is a semigroup isomorphism. Let $E(Q_\Gamma)$ and $E(Q_\Delta)$ denote the sets of idempotents of Q_Γ and Q_Δ, respectively, and let $\leq$ denote the natural partial order on each semigroup. Then θ restricts to an order-isomorphism from $(E(Q_\Gamma), \leq)$ onto $(E(Q_\Delta), \leq)$.

We first identify the vertices. By Lemma 22(i), the maximal elements of $E(Q_\Gamma)$ are precisely the idempotents of the form $[A]$ with $A \subseteq V_\Gamma$, and analogously for Q_Δ. Hence θ induces a bijection

$$\theta : \{[A] : A \subset V_\Gamma\} \longrightarrow \{[A'] : A' \subseteq V_\Delta\}.$$

We claim that for each $v \in V_\Gamma$, the element $[v]\theta$ corresponds to a singleton vertex. Indeed, write $[v]\theta = [B]$ with $B \subseteq V_\Delta$. If $|B| > 1$, choose distinct $u_1, u_2 \in B$. Then $[u_i][B] \neq [0]$ for $i = 1, 2$. By Lemma 22(i), there exist $A_1, A_2 \in T^0_\Gamma$ such that $[A_i]\theta = [u_i]$. It follows that

$$[A_i v]\theta = [A_i]\theta[v]\theta = [u_i][B] \neq [0] \quad (i = 1, 2),$$

whence $[A_iv] \neq [0]$ and so $v \in A_i$ for $i = 1, 2$. Thus $A_1 \cap A_2 \neq \emptyset$, and hence $[A_1A_2] \neq [0]$. However,

$$[A_1A_2]\theta = [A_1]\theta[A_2]\theta = [u_1][u_2] = [0],$$

a contradiction. Therefore B is a singleton. Consequently, θ induces a bijection

$$\varphi : V_\Gamma \to V_\Delta, \qquad v \mapsto v' \text{ where } [v]\theta = [v'].$$

Next, we identify the relations. By Lemma 22(ii), the maximal elements of $E(Q_\Gamma) \setminus \{[A] : A \in T^0_\Gamma\}$ are precisely the idempotents $[tt^{-1}]$ with $t \in T_\Gamma$, and similarly for Δ. Hence θ induces a bijection

$$\theta : \{[tt^{-1}] : t \in T_\Gamma\} \longrightarrow \{[qq^{-1}] : q \in T_\Delta\}.$$

Let $t \in T_\Gamma$ and suppose that $[tt^{-1}]\theta = [qq^{-1}]$ for some $q \in T_\Delta$. Write

$$[t]\theta = [xy^{-1}],$$

where xy^{-1} is in right normal form, that is, $x \in RP(\Delta)$, $y \in RLP(\Delta)$ and $\mathbf{r}(x) = \mathbf{r}(y)$.

We claim that $x \in RLP(\Delta)$. Since θ is a semigroup isomorphism, $[t]\theta$ is a regular element of Q_Δ with inverse $[t^{-1}]\theta$. By Lemma 11, an element of the form $[xy^{-1}]$ admits a (non-zero) inverse only if $x \in RLP(\Delta)$. If $x \in RP(\Delta) \setminus RLP(\Delta)$, then $[xy^{-1}]$ has no non-zero inverse in Q_Δ, which contradicts the fact that $[t^{-1}]\theta$ is an inverse of $[t]\theta$. Therefore $x \in RLP(\Delta)$, as required. Then by Lemma 12, $[t^{-1}]\theta = [yx^{-1}] \neq [0]$, and hence

$$[tt^{-1}]\theta = [t]\theta[t^{-1}]\theta = [xy^{-1}][yx^{-1}] = [xx^{-1}].$$

Thus $[qq^{-1}] = [xx^{-1}]$, and so $q = x$. Moreover,

$$[\mathbf{r}(t)]\theta = [t^{-1}t]\theta = [yx^{-1}xy^{-1}] = [yy^{-1}],$$

which implies that $y \in T^0_\Delta$ and hence $y = \mathbf{r}(y)$. Therefore

$$[t]\theta = [xy^{-1}] = [x] = [q].$$

It follows that θ induces a bijection

$$\psi : T_\Gamma \to T_\Delta, \qquad t \mapsto q \text{ where } [t]\theta = [q].$$

Finally, we verify that φ and ψ preserve the source and range maps. Let $t \in T_\Gamma$. From the preceding computations we have

$$[\mathbf{r}(t)]\theta = [\mathbf{r}([t]\theta)] \quad \text{and} \quad [\mathbf{s}(t)]\theta = [\mathbf{s}([t]\theta)].$$

We now justify the description of $[\mathbf{r}(t)]\theta$. Clearly,

$$[\mathbf{r}(t)]\theta \subseteq \{[v]\theta : v \in \mathbf{r}(t)\}.$$

Conversely, let $v \in \mathbf{r}(t)$. Then $[tv] \neq [0]$, and since θ is a homomorphism, it follows that $[tv]\theta \neq [0]$. Hence $[v]\theta \in \mathbf{r}([t]\theta)$, and therefore $[v]\theta \in [\mathbf{r}(t)]\theta$. Thus

$$[\mathbf{r}(t)]\theta = \{[v]\theta : v \in \mathbf{r}(t)\}.$$

Combining this with the identity $[\mathbf{r}(t)]\theta = [\mathbf{r}([t]\theta)]$, we obtain

$$[\mathbf{r}(t)]\theta = \{[v]\theta : v \in \mathbf{r}(t)\} = [\mathbf{r}([t]\theta)].$$

Since θ induces a bijection between $\{[v] : v \in V_\Gamma\}$ and $\{[v'] : v' \in V_\Delta\}$, it follows that

$$\varphi(\mathbf{r}(t)) = \mathbf{r}(\psi(t)) \quad \text{and} \quad \varphi(\mathbf{s}(t)) = \mathbf{s}(\psi(t)).$$

Therefore (φ, ψ) is an isomorphism of networks from Γ to Δ.

The converse implication is immediate, since any network isomorphism induces a semigroup isomorphism between the associated semigroups. This completes the proof. □

Since the unary operation $*$ on Q_Γ is determined by the semigroup structure, namely a^* is the unique idempotent in the $\mathcal{L}^*$-class of a, every semigroup isomorphism between network semigroups automatically preserves the $*$-operation. Therefore the isomorphisms considered in Theorem 6 may equivalently be viewed as semigroup isomorphisms or as $*$-semigroup isomorphisms.

References

[1] Abrams, G., Aranda Pino, G.: The Leavitt path algebra of a graph. J. Algebra **293**(2), 319–334 (2005). https://doi.org/10.1016/j.jalgebra.2005.07.028

[2] Abrams, G., Dokuchaev, M., Nam, T.G.: Realizing corners of Leavitt path algebras as Steinberg algebras, with corresponding connections to graph C^*-algebras. J. Algebra **593**, 72–104 (2022). https://doi.org/10.1016/j.jalgebra.2021.11.004

[3] Abrams, G., Nam, T.G.: Corners of Leavitt path algebras of finite graphs are Leavitt path algebras. J. Algebra **547**, 494–518 (2020). https://doi.org/10.1016/j.jalgebra.2019.11.020

[4] Ara, P., Moreno, M.A., Pardo, E.: Nonstable K-theory for graph algebras. Algebr. Represent. Theory **10**(2), 157-178 (2007). https://doi.org/10.1007/s10468-006-9044-z

[5] Ash, C.J., Hall, T.E.: Inverse semigroups on graphs. Semigroup Forum **11**, 140–145 (1975). https://doi.org/10.1007/BF02195262

[6] Bianconi, G.: Higher-Order Networks. Cambridge University Press (2022). https://doi.org/10.1017/9781108770996

[7] Battiston, F., Cencetti, G., Iacopini, I., Latora, V., Lucas, M., Patania, A., Young, J.G., Petri, G.: Networks beyong pairwise interactions: Structure and dynamics. Phys. Rep. **874**, 1–92 (2020). https://doi.org/10.1016/j.physrep.2020.05.004

[8] Bick, C., Gross, E., Harrington, A.H., Schaub, T.M.: What are higher-order networks? SIAM Rev. **65**(3), 686–731 (2023). https://doi.org/10.1137/21M1414024

[9] Bowers, P.M., Cokus, S.J., Eisenberg, D., et al. Use of logic relationships to decipher protein network organization. Sci. **306**(5705), 2246–2249 (2004). https://doi.org/10.1126/science.1103330

[10] Book, R.V., Otto, F.: String-Rewriting Systems. In: String-Rewriting Systems. Text and Monographs in Computer Science. Springer, New York, NY (1993). https://doi.org/10.1007/978-1-4613-9771-7_3

[11] Fountain, J.B.: Adequate semigroups. Proc. Edinburgh Math. Soc. **22**(2), 113–125 (1979). https://doi.org/10.1017/S0013091500016230

[12] Fountain, J.B.: Abundant semigroups. Proc. Lond. Math. Soc. **44**(s3), 103–129 (1982). https://doi.org/10.1112/plms/s3-44.1.103

[13] Fountain, J.B.: Free right h-adequate semigroups. In: Jürgensen, H., Lallement, G., Weinert, H.J. (eds) Semigroups Theory and Applications. Lecture Notes in Mathematics, vol. **1320**, 97–120. Springer, Berlin, Heidelberg (1988). https://doi.org/10.1007/BFb0083424

[14] Hazrat, R., Nam, T.G.: Unital algebras being Morita equivalent to weighted Leavitt path algebras. J. Algebr. Comb. **62**, 28 (2025). https://doi.org/10.1007/s10801-025-01454-y

[15] Howie, J.M.: An Introduction to Semigroup Theory. Academic Press, London (1976).

[16] Klamt, S., Haus, U.-U., Theis, F.: Hypergraphs and cellular networks, PLoS Comput. Biol. **5**(5), e1000385 (2009). https://doi.org/10.1371/journal.pcbi.1000385

[17] Meakin, J., Milan, D., Wang, Z.P.: On a class of inverse semigroups related to Leavitt path algebras. Adv. Math. **384**, 107729 (2021). https://doi.org/10.1016/J.AIM.2021.107729

[18] Mitsch, H.: A natural partial order for semigroups. Proc. Amer. Math. Soc. **97**(3), 384–388 (1986). https://doi.org/10.1090/S0002-9939-1986-0840614-0

[19] Munn, W.D.: Fundamental inverse semigroups. Quart. J. Math. Oxford **21**(2), 157–170 (1970). https://doi.org/10.1093/qmath/21.2.157

[20] Nivat, M., Perrot, J.F.: Une generalisation du monoide bicclique. C. R. Acad. Sci. Paris **271**, 824–827 (1970).

[21] Paterson, A.L.T.: Graph inverse semigroups, groupoids and their C^*-algebras. J. Operator Theory **48**(3), 645–662 (2002). http://www.jstor.org/stable/24715591

[22] Wang, Y.H., Meng, D.Z.: Quasi-semilattices on networks. Axioms **12**(10), 943 (2023). https://doi.org/10.3390/axioms12100943

[23] Yang, D., Gould, V.A.R.: Free idempotent generated semigroups over bands and biordered sets with trivial products. Int. J. Algebra Comput. **26**(3), 473–507 (2016). https://doi.org/10.1142/S021819671650020X

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