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RESEARCH ARTICLE

On $C\sqrt{J}$ rings

Dinesh Udar and Shiksha Saini

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Abstract. A ring R in which every $x \in R$ is expressible as addition of $c \in C(R)$ and an element $z \in \sqrt{J(R)}$, where $\sqrt{J(R)} = \{z : z^n \in J(R) \text{ for some n} \geq 1\}$ is called a $C\sqrt{J}$ ring. The class of $C\sqrt{J}$ rings is a proper subclass of the class of CU rings and includes CN and CJ rings properly. After developing the basic properties of $C\sqrt{J}$ rings, we connect them with several other rings. We examine how this concept behaves across different extensions like matrices, Dorroh extension, $T[R, S]$, and $T(R, M)$. We investigate the condition to characterize the group rings as a $C\sqrt{J}$ ring.

Introduction

In this paper, unless explicitly mentioned, R is assumed to be an associative ring having unity ($1 \neq 0$). Recall that the notion of *clean rings* was introduced by Nicholson in [12] and was defined as the rings in which all the elements $x \in R$ are expressible as the addition of an idempotent e and unit u satisfying $x = e + u$. On replacing the idempotent e by a nilpotent n, we say R is a *nil clean ring*, the notion unveiled by Diesl in [4]. In [7], CU rings were introduced recently. We say R is a CU ring if every element possesses a CU decomposition, i.e., for every $x \in R$, there is some unit u and a central element c so that $x = c + u$.

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When a nilpotent n replaces u, we say that R is a CN ring. Similarly, on replacing u by j, an element from the Jacobson radical $J(R)$, R is said to be a CJ ring. The class of CN and CJ rings was introduced in [8] and [10], respectively. Additionally, in [15], Wang and Chen defined $\sqrt{J(R)} = \{z \in R : z^n \in J(R) \text{ for some n} \geq 1\}$. Note that $\sqrt{J(R)}$ is a subset of R, which may even fail to be a subring of R in some cases. For more properties of $\sqrt{J(R)}$, [14] can be referred.

All the above constructions motivated us to introduce and explore: $C\sqrt{J}$ rings. The class of $C\sqrt{J}$ rings is a proper subclass of the class of CU rings and includes CN and CJ rings properly. The structural knowledge and elementary properties will be helpful in further study of ring theory and in defining relationships between various structures more elaborately. Now we define $C\sqrt{J}$ rings.

Definition 1. Any element r of a ring R is said to have a $C\sqrt{J}$ decomposition if there exists $c \in C(R)$ and an element z from $\sqrt{J(R)}$, satisfying $r = c + z$. A ring in which every element is $C\sqrt{J}$ is called a $C\sqrt{J}$ ring. Hence, for a $C\sqrt{J}$ ring R, $R = C(R) + \sqrt{J(R)}$.

The main outline of our paper is: In Section 1, we highlight the properties exhibited by $C\sqrt{J}$ rings. We prove that the class of $C\sqrt{J}$ rings is a proper subclass of the class of CU rings, and properly contains the class of CN and CJ rings. Moving ahead, we show that all *uniquely nil clean rings* are $C\sqrt{J}$. We also prove that every strongly π-regular ring having trivial idempotents is a CU ring. We characterize the ring of all upper triangular matrices over R, i.e., $T_n(R)$, as $C\sqrt{J}$ rings. Section 2 focuses on the extensions of $C\sqrt{J}$ rings. We discuss about *Dorroh extension* $D(\mathbb{Z}, R)$ and then exhibit the conditions under which certain subrings of $M_3(R)$ like $H_{(p,q)}(R)$ are $C\sqrt{J}$. Here, $M_n(R)$ represents the matrix ring of order n. In Section 3, we first obtain the condition when $\sqrt{J(RG)} \cap R = \sqrt{J(R)}$. Then we investigate the conditions to characterize the group ring RG as a $C\sqrt{J}$ ring.

We represent the set of all idempotents, center and nilpotents of R by $N(R)$, $C(R)$, and $I(R)$, respectively. Also, $T_n(R)$ and $D_n(R)$ represent the ring of all $n \times n$ upper triangular matrices and the upper triangular matrices of order n whose diagonal entries are equal, respectively. Refer [9, 11] for any other unexplained definition or notation.

1. Properties of $C\sqrt{J}$ rings

We present some examples of $C\sqrt{J}$ rings and $C\sqrt{J}$ elements.

1. Every commutative ring is a $C\sqrt{J}$ ring.
2. The radical ring is always a $C\sqrt{J}$ ring.
3. The nilpotent and unipotent elements of a ring R are $C\sqrt{J}$ elements.
4. The center of a ring R is always a $C\sqrt{J}$ subring of R.

Theorem 1. *Under a homomorphism ϕ, the $C\sqrt{J}$ decomposition of ring R is preserved.*

Proof. For a $C\sqrt{J}$ ring R, if $x \in \sqrt{J(R)}$, then following [13, Lemma 3.5], under the homomorphism $\phi : R \to P$, $\phi(x) \in \sqrt{J(P)}$. Now, let $a = c + z \in R$ be a $C\sqrt{J}$ decomposition, where c is a central element and $z \in \sqrt{J(R)}$. Then $\phi(a) = \phi(c) + \phi(z)$, where $\phi(c) \in C(P)$ and $\phi(z) \in \sqrt{J(P)}$. □

Building upon prior studies, we define, *uniquely* $C\sqrt{J}$ ring as a ring in which for every $a \in R$, $a = c + z$, where $c \in C(R)$ and $z \in \sqrt{J(R)}$ uniquely.

Theorem 2. *When R is a non-commutative $C\sqrt{J}$ ring, then the following properties are equivalent:*

1. *R is a uniquely $C\sqrt{J}$ ring.*
2. *$C(R) \cap \sqrt{J(R)} = \{0\}$.*

Proof. (1) ⇒ (2) Let $x \in C(R) \cap \sqrt{J(R)}$ and $z \in \sqrt{J(R)}$. If $y = 1 + z$, as R is a uniquely $C\sqrt{J}$ ring, this decomposition of y should be unique. However, if $y = (1 - x) + (x - z)$, then $1 - x \in C(R)$. From [13, Proposition 2.2(vi)], we have $x - z \in \sqrt{J(R)}$. Hence, we have obtained two $C\sqrt{J}$ decompositions of y, which is not possible. Hence, $x = 0$, as required.

(2) ⇒ (1) If possible, let $c + y = d + z$ where $c, d \in C(R)$ and $y, z \in \sqrt{J(R)}$, then $c - d = y - z$ and this gives $c = d$ and $y = z$. □

We now establish the relationship among CN, CU, CJ and $C\sqrt{J}$ rings. As the set of all nilpotents and $J(R)$ is a subset of $\sqrt{J(R)}$, evidently, the class of CN and CJ rings is a subclass of the class of $C\sqrt{J}$ rings, and the inclusion is proper. We list the following examples to validate the same:

Example 1. 1. Consider a field K, and

$$D_2(K) = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} : x, y \in K \right\}$$

for any $A = \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \in D_2(K), A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} + \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ is central in $D_2(K)$ and $\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$ is a nilpotent and hence, $\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in \sqrt{J(D_2(K))}$. Hence, $D_2(K)$ is a $C\sqrt{J}$ ring. However, from [10, Example 2.4], $D_2(K)$ is not a CJ ring.

2. Identified by $\sigma(x) = x^2$, where σ denotes the k-endomorphism of $k[\![x]\!]$, suppose $R = k[\![x]\!][\![t; \sigma]\!]$. Then R is a $C\sqrt{J}$ ring, although it is not a CN ring.

It is imperative to note that if $\sqrt{J(R)} \subseteq C(R)$, then $\sqrt{J(R)}$ and $J(R)$ are equal. So in this case, if R is a $C\sqrt{J}$ ring, it is equivalent to R being a CJ ring.

Theorem 3. *Every $C\sqrt{J}$ ring is a CU ring.*

Proof. In a $C\sqrt{J}$ ring R, for any $a \in R$, $a = c + z$, for some central element c and $z \in \sqrt{J(R)}$. As $z \in \sqrt{J(R)}$, we have, $z^n \in J(R)$ for some $n \geq 1$ and hence $1 - z^n \in U(R)$. Hence,

$$1 - z^n = (1 - z)(1 + z + z^2 + \cdots + z^{n-1}) \in U(R).$$

This gives $1 - z$ is invertible in R and hence, $z - 1$ is also invertible in R. Hence, $a = c + z = (c + 1) + (z - 1)$, where $c + 1$ is central in R and $z - 1 \in U(R)$. □

The converse of Theorem 3 does not hold.

Example 2. Consider a non-commutative division ring, $H(\mathbb{R}) = \{a_0 + a_1 i + a_2 j + a_3 k : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$, satisfying the condition $i^2 = j^2 = k^2 = ijk = -1$. Here, $U(H(\mathbb{R}))$ are non-zero elements of $H(\mathbb{R})$ and $\mathbb{R}$ is the center. If $a = 0$, we have, $0 = 5 + (-5)$ and for any non-zero $a \in H(\mathbb{R})$, $a = 0 + a$ gives a CU decomposition of any $a \in H(\mathbb{R})$ and therefore, $H(\mathbb{R})$ is a CU ring. From [2, Theorem 3.6], $\sqrt{J(H(\mathbb{R}))} = \{0\}$ and this gives that $H(\mathbb{R})$ is not a $C\sqrt{J}$ ring.

Hence, the class of $C\sqrt{J}$ rings forms a proper generalization of CJ and CN rings and a proper subclass of CU rings.

$$\begin{array}{c} CN \\ \Downarrow \\ CJ \Rightarrow \quad C\sqrt{J} \Rightarrow \quad CU \end{array}$$

Now, we recall *strongly π-regular* rings. If for every element x in R, there exists $n \in \mathbb{Z}$ and $a \in R$ such that $x^n = x^{n+1}a$, then R is known as a *strongly π-regular* ring.

Proposition 1. *If the only idempotents possessed by a strongly π-regular ring are trivial, then R is a CU ring.*

Proof. By following the proof of [5, Proposition 2.2], we obtain that under the given conditions, the ring is local and hence is a CU ring. □

The converse of Proposition 1 is false, as let $R = \{\frac{p}{q} \in \mathbb{Q} : q = 2x + 1,\ x \in \mathbb{Z},\ x \geq 1\}$, then R fails to be a strongly π-regular ring because of nil $J(R)$. However, being a local ring, we obtain that R is a CU ring.

Proposition 2. *For any $a \in R$, the following are equivalent:*

(i) element a has $C\sqrt{J}$ decomposition;

(ii) for every invertible p in R, pap^{-1} has $C\sqrt{J}$ decomposition.

Proof. (i) $\Rightarrow$ (ii) If $a \in R$ has $C\sqrt{J}$ decomposition, then $a = c + z$, wherein c is from the center and z lies in $\sqrt{J(R)}$. Let $p \in R$ be invertible, then $pap^{-1} = pcp^{-1} + pzp^{-1}$. Clearly pcp^{-1} is a central element. By [13, Proposition 2.2(iv)], $pzp^{-1} \in \sqrt{J(R)}$. Hence, pap^{-1} has $C\sqrt{J}$ decomposition.

(ii) $\Rightarrow$ (i) If p is any invertible element, then let pap^{-1} have a $C\sqrt{J}$ decomposition, then $pap^{-1} = c + z$, where c is central element and $z \in \sqrt{J(R)}$. Then $a = p^{-1}cp + p^{-1}zp$. By [13, Proposition 2.2(iv)], for any unit p, $p^{-1}zp \in \sqrt{J(R)}$. Next, if x is any arbitrary element from R, as c is central in R, we have $(p^{-1}cp)x = cx = xc = x(p^{-1}cp)$. This implies $p^{-1}cp$ is a central element. Hence, we have obtained a decomposition of a which is $C\sqrt{J}$, as required. □

Corollary 1. *If $M_n(D)$ is a $C\sqrt{J}$ ring, then $C(D)$ has more than n elements, where D is a division ring.*

Proof. Moving by the method of contradiction, suppose $|C(D)| < n$. Let $A = diag(c_1, c_2, \ldots, c_n)$, where $c_i \in C(D)\ \forall\ i = 1, 2, 3, \ldots, n$. Hence for some $c \in C(D)$, $\det(A - cI) = 0$. This implies that D is not a $C\sqrt{J}$ ring. So, the total number of central elements is more than n. □

Theorem 4. *When R is commutative, $A \in M_n(R)$ admits a $C\sqrt{J}$ decomposition if and only if for some $c \in R$, $A - cI_n \in \sqrt{J(M_n(R))}$.*

Proof. Let $A \in M_n(R)$ admits $C\sqrt{J}$ decomposition. Then there exists a $C' \in C(M_n(R))$ and $Z \in \sqrt{J(M_n(R))}$ so that $A = C' + Z$. Being a central element, $C' = cI_n$ for some $c \in R$. This gives us $A - cI_n = Z \in \sqrt{J(M_n(R))}$ for some $c \in R$.

Now, let $A - cI_n \in \sqrt{J(M_n(R))}$ for some $c \in R$. As cI_n is central in $M_n(R)$, we have $A = cI_n + (A - cI_n)$. Hence A admits a $C\sqrt{J}$ decomposition. □

Corollary 2. *For a commutative ring R, $a \in T_n(R)$ admits a $C\sqrt{J}$ decomposition if and only if for some $c \in R$, $A - cI_n \in \sqrt{J(T_n(R))}$.*

Proposition 3. *The following are equivalent in a commutative ring R:*

(i) $T_2(R)$ is a $C\sqrt{J}$ ring;

(ii) there exist some $c \in R$ so that $a - c, b - c \in \sqrt{J(R)}$ for any $a, b \in R$.

Proof. (i) ⇒ (ii) For any $a, b \in R$, we consider a matrix $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ from $T_2(R)$. Then by Corollary 2, $\begin{pmatrix} a-c & 0 \\ 0 & b-c \end{pmatrix} \in \sqrt{J(T_2(R))}$ for some $c \in R$ and hence, $a - c, b - c \in \sqrt{J(R)}$.

(ii) ⇒ (i) For arbitrary elements a, b of R, let there exists some $c \in R$ satisfying $a - c \in \sqrt{J(R)}$ and $b - c \in \sqrt{J(R)}$. Then for some $r \in R$, let $A = \begin{pmatrix} a & r \\ 0 & b \end{pmatrix} \in T_2(R)$. This gives $A = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} a-c & r \\ 0 & b-c \end{pmatrix}$. Let $C' = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ and $Z = \begin{pmatrix} a-c & r \\ 0 & b-c \end{pmatrix}$, then we get $A = C' + Z$, where, $C' \in C(T_2(R))$ and $Z \in \sqrt{J(T_2(R)}$. □

Example 3. It can be observed that $T_2(\mathbb{Z})$ is not a $C\sqrt{J}$ ring. If $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ such that $a \neq b$, then there is no $c \in R$ satisfying $a - c, b - c = 0$.

We first recollect *uniquely nil clean* rings. A nil clean ring with unique $e \in I(R)$ and $n \in N(R)$ for every $x \in R$ satisfying $x = e + n$ is called a uniquely nil clean ring.

Lemma 1. *For a uniquely nil clean ring R, if $e \in I(R)$, then R and eRe are $C\sqrt{J}$ rings.*

Proof. In a uniquely nil clean ring, let $a \in R$ and e be an idempotent. Then $a - e = n$, with $n \in N(R)$. Then by using [4, Lemma 5.5], e is central in R and hence, R is a $C\sqrt{J}$ ring. Then eRe posses a $C\sqrt{J}$ decomposition follows from [4, Lemma 5.7] in hand. □

However, note that every $C\sqrt{J}$ ring is not a uniquely nil clean ring, like $\mathbb{Z}$.

Theorem 5. *For a ring R and an integer $n \geq 1$, R qualifies as a $C\sqrt{J}$ ring if and only if $D_n(R)$ also possesses a $C\sqrt{J}$ decomposition.*

Proof. For the necessary part, let any $a \in R$. Hence, for some central element c and $z \in \sqrt{J(R)}$, we have $a = c + z$. Let

$$A = \begin{pmatrix} a & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & a & b_{23} & \dots & b_{2n} \\ 0 & 0 & a & \dots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a \end{pmatrix} \in D_n(R).$$

Substituting a by $c + z$ in A shows A as sum of central matrix

$$diag(c, c, c, \dots, c)$$

and

$$Z = \begin{pmatrix} z & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & z & b_{23} & \dots & b_{2n} \\ 0 & 0 & z & \dots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & z \end{pmatrix}.$$

This ensues $z \in \sqrt{J(R)}$. On further using the definition of $\sqrt{J(R)}$, $z^m \in J(R)$. This gives $Z^m \in J(D_n(R))$, i.e., $Z \in \sqrt{J(D_n(R))}$. Hence $D_n(R)$ is a $C\sqrt{J}$ ring.

Conversely, consider $D_n(R)$ to be a $C\sqrt{J}$ ring. Thus, $A = C + Z$, where $C = diag(c, c, c, \ldots, c)$ and

$$Z = \begin{pmatrix} z & b_{12} & b_{13} & \ldots & b_{1n} \\ 0 & z & b_{23} & \ldots & b_{2n} \\ 0 & 0 & z & \ldots & b_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & z \end{pmatrix} \in \sqrt{J(D_n(R))}.$$

Then, $Z^m \in J(D_n(R))$ implies $z^m \in J(R)$, which finally gives $z \in \sqrt{J(R)}$. This results in $a = c + z$, where $c \in C(R)$ and $z \in \sqrt{J(R)}$. □

2. Extensions of $C\sqrt{J}$ rings

We begin by recalling the *Dorroh extension* of ring R by $\mathbb{Z}$, where $\mathbb{Z}$ is the ring of integers. It is denoted $D(\mathbb{Z}, R)$ and equals $\mathbb{Z} \oplus R$, where addition is determined componentwise and multiplication is defined as follows for any $(n_1, r_1), (n_2, r_2) \in D(\mathbb{Z}, R)$:

$$(n_1, r_1)(n_2, r_2) = (n_1 n_2, n_1 r_2 + n_2 r_1 + r_1 r_2).$$

The unity is $(1, 0)$ and the center is $\{(n, c) : n \in \mathbb{Z}, c \in C(R)\}$. It is easy to verify that $\sqrt{J(D(\mathbb{Z}, R))} = \sqrt{J(R)}$.

Proposition 4. *For any ring R, $R = C(R) + \sqrt{J(R)}$ if and only if $D(\mathbb{Z}, R)$ is a $C\sqrt{J}$ ring.*

Proof. Let $R = C(R) + \sqrt{J(R)}$ and any $(n, r) \in D(\mathbb{Z}, R)$. Then, $r = c + z$, where $c \in C(R)$ and $z \in \sqrt{J(R)}$. This gives $(n, r) = (n, c) + (0, z)$ where (n, c) is central and $(0, z) \in \sqrt{J(D(\mathbb{Z}, R))}$.

Now let $D(\mathbb{Z}, R)$ be a $C\sqrt{J}$ ring and for any $r \in R$, $(0, r) \in D(\mathbb{Z}, R)$. Then $(0, r) = (0, c) + (0, z)$ where $(0, c)$ is central in $D(\mathbb{Z}, R)$ and $(0, z) \in \sqrt{J(D(\mathbb{Z}, R))}$. This proves that R is a $C\sqrt{J}$ ring. □

Consider a ring R and S, its subring. Then,

$$T[R, S] = \{(x_1, x_2, x_3, \ldots, x_n, s, s, s, \ldots) : x_j \in R, s \in S, 1 \leq j \leq n, n \geq 1\}.$$

It is a ring with componentwise addition and componentwise multiplication. Observe that $C(T[R, S]) = T[C(R), C(R) \cap C(S)]$, $J(T[R, S]) = T[J(R), J(R) \cap J(S)]$.

Proposition 5. *For a ring R and its subring S,*

$$\sqrt{J(T[R,S])} = T[\sqrt{J(R)}, \sqrt{J(R)} \cap \sqrt{J(S)}].$$

Proof. Let

$$x = (r_1, r_2, \dots, r_n, s, s, \dots) \in \sqrt{J(T[R,S])}.$$

This results in

$$x^k = (r_1^k, r_2^k, \dots, r_n^k, s^k, s^k, \dots) \in J(T[R,S]).$$

From the above equation, we obtain that for every i, $r_i^k \in J(R)$ and $s^k \in J(R) \cap J(S)$. Thus, we have $r_i \in \sqrt{J(R)}$ and $s \in \sqrt{J(R)} \cap \sqrt{J(S)}$. Therefore, we obtain

$$x = (r_1, r_2, \dots, r_n, s, s, \dots) \in T[\sqrt{J(R)}, \sqrt{J(R)} \cap \sqrt{J(S)}]$$

and this provides us the inequality

$$\sqrt{J(T[R,S])} \subseteq T[\sqrt{J(R)}, \sqrt{J(R)} \cap \sqrt{J(S)}].$$

Now, assume that

$$x = (r_1, r_2, \dots, r_n, s, s, \dots) \in T[\sqrt{J(R)}, \sqrt{J(R)} \cap \sqrt{J(S)}].$$

Hence, for every i, we have $r_i \in \sqrt{J(R)}$ and $s \in \sqrt{J(R)} \cap \sqrt{J(S)}$. Then, there exists some positive integer k so that $r_i^p \in J(R)\ \forall\ i$ and $s^p \in J(R) \cap J(S)$. Hence, we have

$$x^p = (r_1^p, r_2^p, \dots, r_n^p, s^p, s^p, \dots) \in T[J(R), J(R) \cap J(S)].$$

This results in

$$x^p = (r_1^p, r_2^p, \dots, r_n^p, s^p, s^p, \dots) \in J(T[R,S])$$

and hence, $x \in \sqrt{J(T[R,S])}$. Hence, we have the inequality,

$$\sqrt{J(T[R,S])} \subseteq T[\sqrt{J(R)}, \sqrt{J(R)} \cap \sqrt{J(S)}].$$

□

Proposition 6. *For a ring R and subring S, if $T[R,S]$ is a $C\sqrt{J}$ ring, then R and S are $C\sqrt{J}$. The converse is true when $J(S) \subseteq J(R)$.*

Proof. Let $T[R,S]$ be a $C\sqrt{J}$ ring, consider an $x \in R$ and hence, define $X = (x,0,0,0,\dots)$ in $T[R,S]$. As $T[R,S]$ is a $C\sqrt{J}$ ring, $X = C + Z$, where $C = (c_1, c_2, c_3, \dots, c_n, s, s, s, \dots)$ is central in $T[R,S]$ and

$$Z = (z_1, z_2, z_3, \dots, z_n, t, t, t, \dots)$$

is in $\sqrt{J[T(R,S)]}$. This gives us a $C\sqrt{J}$ decomposition $x = c_1 + z_1$ of x where $c_1 \in C(R)$ and $z \in \sqrt{J(R)}$ and hence R is a $C\sqrt{J}$ ring. Now, we assume $x \in S$. Following a similar process, we get S is a $C\sqrt{J}$ ring. Next, let R and S are $C\sqrt{J}$ rings and suppose $Y = (x_1, x_2, x_3, \dots, x_n, s, s, s, \dots)$ be some element in $T[R,S]$. Then $x_i = c_j + z_j$ for $j = 1, 2, \dots, n$, where $c_j \in C(R)$, $z_j \in \sqrt{J(R)}$ and $s = c + z$, where $z \in \sqrt{J(R)} \cap \sqrt{J(S)}$, $c \in C(R) \cap C(S)$. This shows that $C = (c_1, c_2, c_3, \dots, c_n, c, c, c \dots) \in C(T[R,S])$. As $J(S) \subseteq J(R)$, $Z = (z_1, z_2, z_3, \dots, z_n, z, z, \dots) \in \sqrt{J(T[R,S])}$. Hence, $T[R,S]$ has a $C\sqrt{J}$ decomposition and therefore is $C\sqrt{J}$, as required. □

Consider a ring R and let p, q be central in R. We define a subring of $M_3(R)$, $S = H_{(p,q)}(R)$ which is defined as:

$$H_{(p,q)}(R) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in M_3(R) : a - d = pc, d - f = qe; a, c, d, e, f \in R \right\}.$$

Hence, if $A = (a_{ij}) \in H_{(p,q)}(R)$, let $a_{3,3} = f$, then $a_{2,2} = qe + f$ and $a_{1,1} = pc + qe + f$. This gives that any element of S is of the form:

$$\begin{pmatrix} pc + qe + f & 0 & 0 \\ c & qe + f & e \\ 0 & 0 & f \end{pmatrix}.$$

Following [7], if p, q are central, then the centre and the set of all units of S are given by

$$C(S) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in S : c, e, f \in C(R) \right\},$$

$$U(S) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in S : a, d, f \in U(R); c, e \in R \right\}.$$

From [10], the Jacobson radical of S is given by

$$J(S) = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in S : a, d, f \in J(R); c, e \in R \right\}.$$

Lemma 2. *Let $p, q \in C(R)$ for a ring R. Then*

$$\sqrt{J(H_{(p,q)}(R))} = \left\{ \begin{pmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{pmatrix} \in H_{(p,q)}(R) : a, d, f \in \sqrt{J(R)}; c, e \in R \right\}.$$

Proof. Assuming $a, d, f \in \sqrt{J(R)}$ gives $a^{n_1}, d^{n_2}, f^{n_3} \in J(R)$ for some positive integers $n_1,\ n_2,\ n_3$, respectively. Let $l = max\{n_1, n_2, n_3\}$ and hence $a^n, d^n, f^n \in J(R)$. Also, let $B = \begin{pmatrix} m_1 & 0 & 0 \\ n & m_2 & s \\ 0 & 0 & m_3 \end{pmatrix}$, for arbitrary $m_1, m_2, m_3, n,\ s$ in R. This gives

$$I - A^l B = \begin{pmatrix} 1 - a^l m_1 & 0 & 0 \\ * & 1 - d^l m_2 & * \\ 0 & 0 & 1 - f^l m_3 \end{pmatrix}.$$

As $1 - a^l m_1, 1 - d^l m_2, 1 - f^l m_3 \in U(R)$, we have $I - A^l B$ is invertible and hence $A \in \sqrt{J(H_{(p,q)}(R))}$.

On the contrary if $A \in \sqrt{J(H_{(p,q)}(R))}$ and let $B = \begin{pmatrix} m_1 & 0 & 0 \\ n & m_2 & s \\ 0 & 0 & m_3 \end{pmatrix}$. Now for some $l \geq 1$, as $I - A^l B = \begin{pmatrix} 1 - a^l m_1 & 0 & 0 \\ * & 1 - d^l m_2 & * \\ 0 & 0 & 1 - f^l m_3 \end{pmatrix} \in U(H_{(p,q)}(R))$, $1 - a^l m_1, 1 - d^l m_2, 1 - f^l m_3$ are units and this results in $a^l, d^l, f^l \in J(R)$ and hence $a, d, f \in \sqrt{J(R)}$. □

Theorem 6. *Let p, q be central in R. If*

$$H_{(p,q)}(R) = C(H_{(p,q)}(R)) + \sqrt{J(H_{(p,q)}(R))},$$

then $R = C(R) + \sqrt{J(R)}$. The converse holds if for any $A = (a_{ij}) \in H_{(p,q)}(R)$, $a_{3,3} \in C(R)$.

Proof. Suppose $H_{(p,q)}(R)$ is a $C\sqrt{J}$ ring. For any $a \in R$, let $A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in H_{(p,q)}(R)$. Then, as $H_{(p,q)}(R)$ is a $C\sqrt{J}$ ring, there exists a

$$C = \begin{pmatrix} m_1 & 0 & 0 \\ n_1 & p_1 & q_1 \\ 0 & 0 & r_1 \end{pmatrix} \in C(H_{(p,q)}(R))$$

and $Z = \begin{pmatrix} m_2 & 0 & 0 \\ n_2 & p_2 & q_2 \\ 0 & 0 & r_2 \end{pmatrix} \in \sqrt{J(H_{(p,q)}(R))}$ such that A have a $C\sqrt{J}$ decomposition given by, $A = C + Z$. Hence, we obtain that m_1 is central and $m_2 \in \sqrt{J(R)}$. This gives a $C\sqrt{J}$ decomposition of a, $a = m_1 + m_2$, proving R is a $C\sqrt{J}$ ring. Now, assume that R is a $C\sqrt{J}$ ring and let $A = (a_{ij}) \in H_{(p,q)}(R)$ such that $a_{3,3}$ is central in R. For any $c, e \in R$ and $f \in C(R)$, let

$$A = \begin{pmatrix} pc + qe + f & 0 & 0 \\ c & qe + f & e \\ 0 & 0 & f \end{pmatrix}$$

be some element in $H_{(p,q)}(R)$. As R is a $C\sqrt{J}$ ring, we can have $c = c_1 + c_2$, $e = e_1 + e_2$, where $c_1, e_1 \in C(R)$ and $c_2, e_2 \in \sqrt{J(R)}$ respectively. Hence, we have

$$A = \begin{pmatrix} pc_1 + qe_1 + f & 0 & 0 \\ c_1 & qe_1 + f & e_1 \\ 0 & 0 & f \end{pmatrix} + \begin{pmatrix} pc_2 + qe_2 & 0 & 0 \\ c_2 & qe_2 & e_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

By taking

$$C = \begin{pmatrix} pc_1 + qe_1 + f & 0 & 0 \\ c_1 & pe_1 + f & e_1 \\ 0 & 0 & f \end{pmatrix} \text{ and } Z = \begin{pmatrix} pc_2 + qe_2 & 0 & 0 \\ c_2 & qe_2 & e_2 \\ 0 & 0 & 0 \end{pmatrix},$$

we have $A = C + Z$. As $c_1, e_1, f \in C(R)$ and $c_2, e_2 \in \sqrt{J(R)}$, we get $C \in C(H_{(p,q)}(R))$ and $Z \in \sqrt{J(H_{(p,q)}(R))}$, respectively. □

Remark 1. By Lemma 1, every uniquely nil clean ring is a $C\sqrt{J}$ ring. The converse need not be true. On considering $H_{(0,0)}(\mathbb{Z})$, following Theorem 6, $H_{(0,0)}(\mathbb{Z})$ is a $C\sqrt{J}$ ring. Suppose $A = \begin{pmatrix} a & 0 & 0 \\ c & a & d \\ 0 & 0 & a \end{pmatrix} \in H_{(0,0)}(\mathbb{Z})$ admits a uniquely nil clean decomposition. Thus $A = E + N$ uniquely, where $E = \begin{pmatrix} e & 0 & 0 \\ c_1 & e & d_1 \\ 0 & 0 & e \end{pmatrix}$ is an idempotent and $N = \begin{pmatrix} n & 0 & 0 \\ c_2 & n & d_2 \\ 0 & 0 & n \end{pmatrix}$ is a nilpotent in $H_{(0,0)}(\mathbb{Z})$. However, note that any element a of $\mathbb{Z}$ accommodates a uniquely nil clean decomposition only if $a = 0$ or $a = 1$. Hence, $H_{(0,0)}(\mathbb{Z})$ fails to be a uniquely nil clean ring.

For a ring R, where p, q are central, let $S = L_{(p,q)}(R)$ be a subring of $M_3(R)$ defined as:

$$L_{(p,q)}(R) = \left\{ \begin{pmatrix} a & 0 & 0 \\ pc & d & qe \\ 0 & 0 & f \end{pmatrix} \in M_3(R) : a, c, d, e, f \in R \right\}.$$

Following [8, Lemma 3.1(2)], the central elements of $L_{(p,q)}(R)$ is given by

$$C(S) = \left\{ \begin{pmatrix} a & 0 & 0 \\ pc & d & qe \\ 0 & 0 & f \end{pmatrix} : pa = pd,\ qd = qf;\ a, d, f \in C(R) \right\}$$

and from [10, Lemma 3.2], the $U(S)$ and $J(S)$ is given by

$$U(S) = \left\{ \begin{pmatrix} a & 0 & 0 \\ pc & d & qe \\ 0 & 0 & f \end{pmatrix} : a, d, f \in U(R);\ c, e \in R \right\},$$

$$J(S) = \left\{ \begin{pmatrix} a & 0 & 0 \\ pc & d & qe \\ 0 & 0 & f \end{pmatrix} : a, d, f \in J(R);\ c, e \in R \right\}$$

respectively.

Lemma 3. *Let R be a ring and p, q be central in R. Then*

$$\sqrt{J(L_{(p,q)}(R))} = \left\{ \begin{pmatrix} a & 0 & 0 \\ pc & d & qe \\ 0 & 0 & f \end{pmatrix} \in M_3(R) : a, d, f \in \sqrt{J(R)} \right\}.$$

Proof. Let $a, d, f \in \sqrt{J(R)}$. Then for some $n_1, n_2, n_3 \geq 1$, $a^{n_1}, d^{n_2}, f^{n_3} \in J(R)$. If $l = max\{n_1, n_2, n_3\}$, then $a^l, d^l, f^l \in J(R)$. For any

$$A = \begin{pmatrix} a & 0 & 0 \\ pc & d & qe \\ 0 & 0 & f \end{pmatrix} \in L_{(p,q)}(R),$$

let $B = \begin{pmatrix} m_1 & 0 & 0 \\ n & m_2 & q \\ 0 & 0 & m_3 \end{pmatrix} \in L_{(p,q)}(R)$ giving,

$$I - A^l B = \begin{pmatrix} 1 - a^l m_1 & 0 & 0 \\ * & 1 - d^l m_2 & 0 \\ 0 & 0 & 1 - f^l m_3 \end{pmatrix} \in L_{(p,q)}(R).$$

As $1-a^lm_1, 1-d^lm_2, 1-f^lm_3$ are units in R, $I-a^lB$ is a unit in $L_{(p,q)}(R)$ and hence, $A \in \sqrt{J(L_{(p,q)}(R))}$.

Conversely, assume $A = \begin{pmatrix} a & 0 & 0 \\ pc & d & qe \\ 0 & 0 & f \end{pmatrix} \in \sqrt{J(L_{(p,q)}(R))}$. Then, for some $k \geq 1$, we get, $A^k \in J(L_{(p,q)}(R))$. Let $B = \begin{pmatrix} m_1 & 0 & 0 \\ n & m_2 & q \\ 0 & 0 & m_3 \end{pmatrix} \in L_{(p,q)}(R)$. Then $I - A^kB \in U(L_{(p,q)}(R))$. So, $1-a^km_1, 1-d^km_2, 1-f^km_3 \in U(R)$ and this results in $a, d, f \in \sqrt{J(R)}$. □

Lemma 4. *Let p, q be central in R. If $L_{(p,q)}(R)$ have a $C\sqrt{J}$ decomposition, then so does R.*

Proof. Suppose $L_{(p,q)}(R)$ is a $C\sqrt{J}$ ring. For any $a \in R$, let $A = diag(a,a,a)$ be some element of $L_{(p,q)}(R)$. As A accounts a $C\sqrt{J}$ decomposition, there exists some central element $C = \begin{pmatrix} s & 0 & 0 \\ pc_1 & s & qe_1 \\ 0 & 0 & s \end{pmatrix}$ in $L_{(p,q)}(R)$ and

$$Z = \begin{pmatrix} z & 0 & 0 \\ pc_2 & z & qe_2 \\ 0 & 0 & z \end{pmatrix} \in \sqrt{J(L_{(p,q)}(R))}.$$

This shows that $a = p + z$, where p is central in R and $z \in \sqrt{J(R)}$ and hence, R is a $C\sqrt{J}$ ring. □

However, the converse of Lemma 4 is not true. We will show that $A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in L_{(1,1)}(\mathbb{Z})$ does not have any $C\sqrt{J}$ decomposition in $L_{(1,1)}(\mathbb{Z})$. If A has any $C\sqrt{J}$ decomposition, then let $A = C + Z$, where $C \in C(L_{(1,1)}(\mathbb{Z}))$ and $Z \in \sqrt{J(L_{(1,1)}(\mathbb{Z}))}$. If $C = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$, then $Z = \begin{pmatrix} 2-c & 0 & 0 \\ 2 & 3-c & 0 \\ 0 & 0 & 4-c \end{pmatrix}$, which is not possible.

Let $S = V_2(L_{(p,q)}(R))$ be a subring of $L_{(p,q)}(R)$ defined as

$$S = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & qe \\ 0 & 0 & a \end{pmatrix} \in L_{(p,q)}(R) : a, e \in R \right\}.$$

Lemma 5. *If R is any ring with central elements p and q, then*

$$\sqrt{J(V_2(L_{(p,q)}(R)))} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & qe \\ 0 & 0 & a \end{pmatrix} \in L_{(p,q)}(R) : a \in \sqrt{J(R)} \right\}.$$

Proof. The proof follows as in Lemma 3. □

Proposition 7. *Let R be any ring. Then R possesses a $C\sqrt{J}$ decomposition if and only if $V_2(L_{(p,q)}(R))$ also have a $C\sqrt{J}$ decomposition.*

Proof. Let R be a $C\sqrt{J}$ ring and $A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & qe \\ 0 & 0 & a \end{pmatrix} \in V_2(L_{(p,q)}(R))$. Then there exists some $c \in C(R)$ and $z \in \sqrt{J(R)}$ such that $a = c + z$. Therefore, we have $C = diag(c, c, c)$ is central in $V_2(L_{(p,q)}(R))$ and $Z = \begin{pmatrix} z & 0 & 0 \\ 0 & z & qe \\ 0 & 0 & z \end{pmatrix} \in \sqrt{J(V_2(L_{(p,q)}(R)))}$, so that $A = C + Z$. This results in a $C\sqrt{J}$ decomposition of A and hence, $V_2(L_{(p,q)}(R))$ is a $C\sqrt{J}$ ring. Conversely, assume $V_2(L_{(p,q)}(R))$ is a $C\sqrt{J}$ ring and $a \in R$. Let

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \in V_2(L_{(p,q)}(R)).$$

Then, there is $C\sqrt{J}$ decomposition of A such that

$$A = \begin{pmatrix} c & 0 & 0 \\ 0 & c & qe_1 \\ 0 & 0 & c \end{pmatrix} + \begin{pmatrix} z & 0 & 0 \\ 0 & z & qe_2 \\ 0 & 0 & z \end{pmatrix},$$

where $\begin{pmatrix} c & 0 & 0 \\ 0 & c & qe_1 \\ 0 & 0 & c \end{pmatrix}$ is a central element and

$$\begin{pmatrix} z & 0 & 0 \\ 0 & z & qe_2 \\ 0 & 0 & z \end{pmatrix} \in \sqrt{J(V_2(L_{(p,q)}(R)))}.$$

This implies $z \in \sqrt{J(R)}$. Also, c is central in R. This proves that R is $C\sqrt{J}$. □

Proposition 8. *For a $C\sqrt{J}$ ring R, let $\{a, d, f\} \subseteq R$ with $C\sqrt{J}$ decompositions $a = c_1 + z_1$, $d = c_2 + z_2$ and $f = c_3 + z_3$, respectively, where c_1, c_2, c_3 are central in R and $z_1, z_2, z_3 \in \sqrt{J(R)}$. If $pc_1 = pc_2$ and $qc_2 = qc_3$, then $L_{(p,q)}(R)$ is a $C\sqrt{J}$ ring.*

Proof. Let $A = \begin{pmatrix} a & 0 & 0 \\ pc & d & qe \\ 0 & 0 & f \end{pmatrix} \in L_{(p,q)}(R)$ and $a = c_1+z_1$, $d = c_2+z_2$ and $f = c_3 + z_3$, respectively, where c_1, c_2, c_3 are central in R and $z_1, z_2, z_3 \in \sqrt{J(R)}$. As $pc_1 = pc_2$ and $qc_2 = qc_3$, we obtain $A = C + Z$, with central element $C = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}$ and $Z = \begin{pmatrix} z_1 & 0 & 0 \\ pc & z_2 & qe \\ 0 & 0 & z_3 \end{pmatrix} \in \sqrt{J(L_{(p,q)}(R))}$. □

Assume s is central in a ring R. The generalized matrix ring $K_s(R)$ over R is defined as the ring of 2×2 matrices of the form $\begin{pmatrix} R & R \\ R & R \end{pmatrix}$. It is a ring with the operations performed componentwise addition and multiplication given by:

$$\begin{pmatrix} a_1 & x_1 \\ y_1 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & x_2 \\ y_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_1 & sy_1x_2 + b_1b_2 \end{pmatrix}. \tag{1}$$

According to [6], for a commutative ring R, the center of $K_0(R)$, denoted by $C(K_0(R))$ consists of all scalar matrices. Additionally, from [6], for a commutative ring R,

$$J(K_0(R)) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in J(R) \right\}.$$

We aim to extend this result to $\sqrt{J(K_0(R))}$ as mentioned below.

Lemma 6. *Let R be a commutative ring. Then*

$$\sqrt{J(K_0(R))} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(R) : a, d \in \sqrt{J(R)} \right\}.$$

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sqrt{J(K_0(R))}$. Hence, $A^n \in J(K_0(R))$. Following the multiplication described for $K_s(R)$ in equation (1), if $s = 0$, we

have the diagonal entries of A^n as a^n and d^n, respectively. Hence, we have

$$A^n = \begin{pmatrix} a^n & * \\ * & d^n \end{pmatrix} \in J(K_0(R)).$$

This gives us $a^n, d^n \in J(R)$ and therefore, resulting in a, d as elements from the Jacobson radical.

Now, we assume that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0(R) : a, d \in \sqrt{J(R)} \right\}.$$

This gives positive integers m, k such that $a^m, d^k \in J(R)$. If $n = max\{m, k\}$, then $a^n, d^n \in J(R)$. The diagonal entries of A^n will be a^n, d^n and by following the multiplication defined in equation (1), for $s = 0$, we have

$$A^n = \begin{pmatrix} a^n & * \\ * & d^n \end{pmatrix} \in J(K_0(R)).$$

Hence, $A \in \sqrt{J(K_0(R))}$. □

Remark 2. Given a $C\sqrt{J}$ ring R, $K_0(R)$ may fail to be a $C\sqrt{J}$ ring. For instance, if $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in K_0(\mathbb{Z})$ has any $C\sqrt{J}$ decomposition. Let $C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$ be the central element and $Z = \begin{pmatrix} 2-c & 0 \\ 0 & 1-c \end{pmatrix} \in \sqrt{J(K_0(\mathbb{Z}))}$ such that,

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} 2-z & 0 \\ 0 & 1-c \end{pmatrix}.$$

Hence, $\begin{pmatrix} 2-c & 0 \\ 0 & 1-c \end{pmatrix}^n$ for some $n \geq 1$. Hence, $(2-c)^n$ and $(1-c)^n$ are elements of $J(R)$. This gives $2-c,\ 1-c \in \sqrt{J(\mathbb{Z})}$, and hence, following [13, Corollary 2.6 and Lemma 2.3], $(2-c) + (-(1-c)) = 1 \in \sqrt{J(\mathbb{Z})}$, which is a contradiction.

Theorem 7. *If R is a commutative ring, then R is a $C\sqrt{J}$ if and only if $D_n(K_0(R))$ is a $C\sqrt{J}$ ring.*

Suppose on a ring R, M is a bimodule. Then $T(R, R, M)$ is a matrix ring of the form $\begin{pmatrix} R & M \\ 0 & R \end{pmatrix}$. $T(R, M)$ is a subring of $T(R, R, M)$ with equal diagonal entries, which is isomorphic to the trivial extension R

and M, i.e., $R \propto M$. Recall that a trivial extension of R and M is a ring consisting of elements of the form (x, y), where $x \in R$ and $y \in M$. The operations are: addition defined componentwise and multiplication defined as

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2),$$

where (r_1, m_1) and (r_2, m_2) are from $R \propto M$. The $J(R \propto M)$ is the set of elements (x, y) of $R \propto M$, where $x \in J(R)$. The group of units is the elements of $R \propto M$, where the first component of (x, y) is from $U(R)$. Utilizing the definition of $\sqrt{J(R)}$, it can be easily proved that

$$\sqrt{J(R \propto M)} = \{(r, m) : r \in \sqrt{J(R)},\ m \in M\}.$$

Theorem 8. *For a ring R and bimodule M, R is a $C\sqrt{J}$ ring iff $R \propto M$ is a $C\sqrt{J}$ ring.*

Proof. We begin by assuming R as a $C\sqrt{J}$ ring. Let $(x, y) \in R \propto M$. Then $x \in R$ and there exists some central element $c \in R$ and $z \in \sqrt{J(R)}$ satisfying $x = c+z$. Then $(x, y) = (c, 0)+(z, y)$, where $(c, 0) \in C(R \propto M)$ and $(z, y) \in \sqrt{J(R \propto M)}$. Hence, $R \propto M$ is a $C\sqrt{J}$ ring.

If $R \propto M$ is a $C\sqrt{J}$ ring and (x, y) is an arbitrary element from $R \propto M$. Then, there exists $(c, m_1) \in C(R \propto M)$ and $(z, m_2) \in \sqrt{J(R \propto M)}$ satisfying $(x, y) = (c, m_1) + (z, m_2)$. As $(z, m_2) \in \sqrt{J(R \propto M)}$, we have $z \in \sqrt{J(R)}$. Also, as $(c, m_1) \in C(R \propto M)$, for any element $(p, q) \in R \propto M$,

$$(c, m_1)(p, q) = (p, q)(c, m_1) \implies (cp, cq + m_1 p) = (pc, pm_1 + qc).$$

This results in $cp = pc$ implying $c \in C(R)$. Therefore, for any element $x \in R$, we have $x = c + z$, where $c \in C(R)$ and $z \in \sqrt{J(R)}$. □

Note that $T(R, M) \cong R \propto M$. If the ideal generated by x^2 in $R[x]$ is (x^2), then $T(R, R) \cong R[x]/(x^2)$.

Corollary 3. *The following conditions are equivalent for a ring R:*

(i) R is a $C\sqrt{J}$ ring;

(ii) $T(R, M)$ is a $C\sqrt{J}$ ring;

(iii) $T(R, R)$ and $R \propto R$ are $C\sqrt{J}$ rings;

(iv) $R[x]/(x^2)$ is a $C\sqrt{J}$ ring.

3. Group rings

We first recollect the definition of a group ring RG defined on a ring R over a group G. Elements of RG are the finite linear combinations of the form $\alpha = \sum_{g\in G} r_g g$. The *support* of α is $\text{supp}\alpha = \{g \in G : r_g \neq 0\}$. The augmentation map from RG to R is represented using ω given by $\omega(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g$. The kernel of ω is denoted by $\Delta_R(G)$, also known as the *augmentation ideal* and $RG/\Delta_R(G) \cong R$. If any finite subset in G generates a finite subgroup, we say G is a *locally finite group*.

Note that a subring S of a ring R is called a unit closed subring if $U(R) \cap S \subseteq U(S)$. It is well known that if H is a subgroup of a group G, then RH is a unit closed subring of RG. We will now extend the result proved by Connel in [3] that for any group G and ring a R, $J(RG) \cap R \subseteq J(R)$ and equality holds when G is a locally finite group. In [1], Chen et al. proved that $J(R)G \subseteq J(RG)$ in a simplified manner.

Proposition 9. *For a ring R and group G, $\sqrt{J(RG)} \cap R \subseteq \sqrt{J(R)}$. Equality is held if G is locally finite.*

Proof. Suppose H is a subgroup of G and r is an element in $\sqrt{J(RG)}\cap R$. Then $r^n \in J(RG)$ for some $n \geq 1$. Hence, $1 - r^n x \in U(RG)$, for every $x \in RG$ and some $n \geq 1$. This implies that for every $x \in RH$, $1 - r^n x \in U(RG)$ for some $n \geq 1$. Also, as $r \in R$ and $x \in H$, we have $1 - r^n x \in RH$. Now, due to the fact that RH is a unit closed subring of RG, $U(RG) \cap RH \subseteq U(RH)$ and therefore, we obtain $1 - r^n x \in U(RH)$ for every $x \in RH$ and some $n \geq 1$. Above developments lead us to conclude that $r \in \sqrt{J(RH)}$. By taking $H = \{1\}$, we see that $\sqrt{J(RG)} \cap R \subseteq \sqrt{J(R)}$.

For the converse, let G be a locally finite group and $z \in \sqrt{J(R)}$. This gives $z^n \in J(R)$ for some $n \geq 1$. Let $r \in RG$ and $\text{supp}(r) = \{1, g_2, g_3, \ldots g_n\}$. We see if the solution of

$$(1 + z^n r)y = 1$$

exists in RG, where $r = \sum_{i=1}^{n} r_i g_i$ and $y = \sum_{i=1}^{n} y_i g_i$, with $g_1 = 1$. Solving the above equation results in a system of linear equations of the form $BY = C$, where

$$B = \begin{pmatrix} 1+z^n r_1 & z^n r_{12} & z^n r_{13} & \dots & z^n r_{1n} \\ z^n r_{21} & 1+z^n r_1 & z^n r_{23} & \dots & z^n r_{2n} \\ z^n r_{31} & z^n r_{32} & 1+z^n r_1 & \dots & z^n r_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ z^n r_{n1} & z^n r_{n2} & z^n r_{3n} & \dots & 1+z^n r_1 \end{pmatrix},$$

$Y = \begin{pmatrix} y_1 & y_2 & \dots & y_n \end{pmatrix}^T$ and $C = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix}^T$. Any r_{ij} in B represents the permutation of $r_1,\ r_2,\ r_3, \dots,\ r_n$. If the ring is commutative, $\det(B) = 1 + z^n b$ for some $b \in R$. As $z \in \sqrt{J(R)}$, we get $1 + z^n b \in U(R)$ and hence the solution exists. Now, the above system of linear equations is of the form $Y + DY = C$, where

$$D = \begin{pmatrix} z^n r_1 & z^n r_{12} & z^n r_{13} & \dots & z^n r_{1n} \\ z^n r_{21} & z^n r_1 & z^n r_{23} & \dots & z^n r_{2n} \\ z^n r_{31} & z^n r_{32} & z^n r_1 & \dots & z^n r_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ z^n r_{n1} & z^n r_{n2} & z^n r_{3n} & \dots & z^n r_1 \end{pmatrix}.$$

As $D \in J(M_n(R))$, $Y = (1 + D)^{-1} C$ is the solution. Hence, $z^n \in J(RG)$ and thus $z \in \sqrt{J(RG)}$. This proves that when G is locally finite, $\sqrt{J(RG)} \cap R = \sqrt{J(R)}$. □

Lemma 7. *Given a ring R and a locally finite group G, $\sqrt{J(R)}G \subseteq \sqrt{J(RG)}$.*

Theorem 9. *Suppose G is a locally finite abelian group and R is any ring. Then RG has a $C\sqrt{J}$ decomposition iff R has a $C\sqrt{J}$ decomposition.*

Proof. Let RG be a $C\sqrt{J}$ ring. As $RG/\omega G \cong R$, R is a $C\sqrt{J}$ ring. Conversely, if R is a $C\sqrt{J}$ ring and $\alpha = \sum_{g \in G} r_g g \in RG$, then we have $r_g = c_g + z_g$, where $c_g \in C(R)$ and $z_g \in \sqrt{J(R)}$. Hence, $\alpha = \sum_{g \in G} (c_g + z_g) g$ $\Rightarrow \alpha = \sum_{g \in G} c_g g + \sum_{g \in G} z_g g$. As $C(RG) = C(R)G$, $\sum_{g \in G} c_g g \in C(RG)$ and by Lemma 7, $\sum_{g \in G} z_g g \in \sqrt{J(RG)}$. This shows that RG is a $C\sqrt{J}$ ring. □

Conclusion

In this research, we introduce $C\sqrt{J}$ rings. The class of $C\sqrt{J}$ rings includes the class of CJ and CN rings properly, and itself is a proper subclass of CU rings. In this context, we discuss various properties of $C\sqrt{J}$ rings, the matrix extensions and group rings.

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Contact information

D. Udar,
S. Saini

Delhi Technological University, Shahbad Daulatpur, Main Bawana Road, Delhi – 110042, India
E-Mail: dineshudar@yahoo.com, shiksha96saini@gmail.com

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