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RESEARCH ARTICLE

Annihilator-based dependency relations in modules and radical characterizations

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Abstract. In this paper we introduce and investigate annihilator based dependency relations for submodules of a unitary left R-module M over a commutative Noetherian ring R. We show that two submodules $N_1, N_2 \leq M$ are radically dependent (in the sense that $\sqrt{\mathrm{Ann}(N_1 + N_2)} = \sqrt{\mathrm{Ann}(N_1)} + \sqrt{\mathrm{Ann}(N_2)}$) if and only if $\sqrt{\mathrm{Ann}(N_1)} = \sqrt{\mathrm{Ann}(N_2)}$. Building on this characterization, we introduce totally annihilator-dependent modules via a Krull-dimension condition and prove that, for a finitely generated module over a Noetherian ring, total annihilator-dependence is equivalent to $\mathrm{Ass}(M)$ being a singleton. We further study the Radical Distinction Set $Z_g(M)$, establish its connection to associated primes, and extend the main results to finitely generated multiplication modules.

Introduction

Classical submodule dependency in module theory is typically studied through linear dependence or essential extensions. Such approaches, however, do not capture the deeper interplay between annihilator ideals and the radical structure of a module. The annihilator $\mathrm{Ann}(N)$ of a submodule $N \leq M$ encodes how N sits inside M from the ring's perspective, while $\sqrt{\mathrm{Ann}(N)}$ records the prime ideal data governing the

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support of N. When two submodules share the same radical annihilator they behave identically at every prime, a constraint that turns out to be both necessary and sufficient for what we call radical dependence. This observation motivates a systematic study of modules in which *all* pairs of submodules are radically dependent—the totally annihilator-dependent modules—and their characterization via the set of associated primes $\mathrm{Ass}(M)$.

A second thread of investigation concerns the *Radical Distinction Set* $Z_g(M)$, which measures how far the pointwise annihilators of elements of M deviate from the annihilator of the radical $\mathrm{Rad}(M)$. The vanishing of $Z_g(M)$ provides a sufficient condition for total annihilator-dependence and forces $\mathrm{Ass}(M)$ to be a singleton. Together, these concepts connect prime ideal theory, Krull dimension, and the support-theoretic structure of modules in a unified framework.

Throughout, R denotes a commutative ring with unity and M a unitary left R-module. In Sections 2 and 2.4 we additionally assume that R is Noetherian and M is finitely generated.

1. Definitions and basic concepts

1.1. Radical dependence

Definition 1 (Radically dependent submodules)**.** Let $N_1, N_2 \leq M$. We say N_1 and N_2 are *radically dependent* if

$$\sqrt{\mathrm{Ann}(N_1 + N_2)} = \sqrt{\mathrm{Ann}(N_1)} + \sqrt{\mathrm{Ann}(N_2)}.$$

The following proposition shows that this condition admits a clean equivalent formulation, which we use as the working characterization throughout the paper.

Theorem 1 (Characterization of radical dependence)**.** *Let $N_1, N_2 \leq M$. Then N_1 and N_2 are radically dependent if and only if $\sqrt{\mathrm{Ann}(N_1)} = \sqrt{\mathrm{Ann}(N_2)}$.*

Proof. ($\Rightarrow$) Suppose N_1 and N_2 are radically dependent, i.e.

$$\sqrt{\mathrm{Ann}(N_1 + N_2)} = \sqrt{\mathrm{Ann}(N_1)} + \sqrt{\mathrm{Ann}(N_2)}.$$

Since $\mathrm{Ann}(N_1 + N_2) = \mathrm{Ann}(N_1) \cap \mathrm{Ann}(N_2)$, the left-hand side equals $\sqrt{\mathrm{Ann}(N_1)} \cap \sqrt{\mathrm{Ann}(N_2)}$. Hence $\sqrt{\mathrm{Ann}(N_1)} \cap \sqrt{\mathrm{Ann}(N_2)} = \sqrt{\mathrm{Ann}(N_1)} + \sqrt{\mathrm{Ann}(N_2)}$. For any two ideals I, J of R, $I \cap J = I + J$ forces $I = J$ (since $I \subseteq I + J = I \cap J \subseteq I$). Therefore $\sqrt{\mathrm{Ann}(N_1)} = \sqrt{\mathrm{Ann}(N_2)}$.

($\Leftarrow$) Suppose $\sqrt{\operatorname{Ann}(N_1)} = \sqrt{\operatorname{Ann}(N_2)} =: \mathfrak{p}$. Then

$$\begin{aligned}\sqrt{\operatorname{Ann}(N_1+N_2)} &= \sqrt{\operatorname{Ann}(N_1)\cap\operatorname{Ann}(N_2)}\\ &= \sqrt{\operatorname{Ann}(N_1)}\cap\sqrt{\operatorname{Ann}(N_2)}\\ &= \mathfrak{p}\cap\mathfrak{p} = \mathfrak{p},\end{aligned}$$

and $\sqrt{\operatorname{Ann}(N_1)} + \sqrt{\operatorname{Ann}(N_2)} = \mathfrak{p}+\mathfrak{p} = \mathfrak{p}$, so the radical dependence condition holds. ☐

Remark 1. Theorem 1 shows that the radical dependence relation is symmetric (N_1 and N_2 are radically dependent $\Leftrightarrow$ N_2 and N_1 are), and that it partitions the set of nonzero submodules of M into classes of equal radical annihilator. This is precisely the equivalence relation $\sim_\gamma$ introduced in Section 1.4 below.

1.2. Totally annihilator-dependent modules

Definition 2 (Totally annihilator-dependent module)**.** A finitely generated R-module M is called *totally annihilator-dependent* if for every pair of nonzero submodules $N_1, N_2 \leq M$ satisfying $\operatorname{Ann}(N_i) \neq 0$,

$$\dim R/\operatorname{Ann}(N_1+N_2) = \min\{\dim R/\operatorname{Ann}(N_1),\ \dim R/\operatorname{Ann}(N_2)\}.$$

1.3. The radical distinction set

Definition 3 (Radical distinction set)**.** Let M be an R-module. The *radical distinction set* of M is

$$Z_g(M) = \{m \in M\setminus\{0\} \mid \operatorname{Ann}(Rm) \neq \operatorname{Ann}(\operatorname{Rad}(M))\}.$$

A coarser variant, comparing radicals rather than annihilators directly, is

$$\widetilde{Z}_g(M) = \{m \in M\setminus\{0\} \mid \sqrt{\operatorname{Ann}(Rm)} \neq \sqrt{\operatorname{Ann}(\operatorname{Rad}(M))}\}.$$

When $Z_g(M) = \emptyset$, every nonzero element of M satisfies $\operatorname{Ann}(Rm) = \operatorname{Ann}(\operatorname{Rad}(M))$. Note that $\widetilde{Z}_g(M) \subseteq Z_g(M)$ in general, since exact equality implies radical equality but not conversely.

When $\operatorname{Rad}(M) = 0$ (for instance when M is semisimple), $\operatorname{Ann}(\operatorname{Rad}(M)) = R$, and every nonzero m satisfies $\operatorname{Ann}(Rm) \neq R$, so $Z_g(M) = M\setminus\{0\}$.

1.4. Annihilator radical equivalence relation

Let $m_1, m_2 \in M \setminus \{0\}$. Define $m_1 \sim_\gamma m_2$ if and only if $\sqrt{\operatorname{Ann}(Rm_1)} = \sqrt{\operatorname{Ann}(Rm_2)}$. This is an equivalence relation on $M \setminus \{0\}$, and the equivalence class of m is

$$[m]_\gamma = \{m' \in M \setminus \{0\} \mid \sqrt{\operatorname{Ann}(Rm')} = \sqrt{\operatorname{Ann}(Rm)}\}.$$

This classification is finer than the partition induced by $Z_g(M)$ alone, as it groups elements by their radical annihilator type. By Theorem 1, two nonzero cyclic submodules Rm_1 and Rm_2 are radically dependent if and only if $m_1 \sim_\gamma m_2$.

2. Main results

Throughout this section R is a Noetherian commutative ring and M a finitely generated R-module.

2.1. Characterization via associated primes

Theorem 2. *Let R be a commutative Noetherian ring with identity, and let M be a finitely generated R-module. If* $\operatorname{Ass}(M) = \{\mathfrak{p}\}$*, then M is totally annihilator-dependent.*

Proof. Assume that $\operatorname{Ass}(M) = \{\mathfrak{p}\}$. Let $N_1, N_2 \leq M$ be nonzero submodules satisfying $\operatorname{Ann}(N_i) \neq 0$. Since M is finitely generated over a Noetherian ring, every submodule of M is also finitely generated. Moreover, since R is a Noetherian ring, for $0_M \neq N_i \leq M$ we have $\operatorname{Ass}(N_i) \neq \emptyset$ and $\operatorname{Ass}(N_i) \subseteq \operatorname{Ass}(M) = \{\mathfrak{p}\}$, and therefore necessarily $\operatorname{Ass}(N_i) = \{\mathfrak{p}\}$. This shows that $\mathfrak{p}$ is the unique minimal prime over $\operatorname{Ann}(N_i)$. By Krull dimension theory, the Krull dimension of a quotient ring is determined by its minimal primes. Therefore

$$\sqrt{\operatorname{Ann}(N_i)} = \bigcap_{\mathfrak{q} \in \operatorname{Ass}(M)} \mathfrak{q} = \mathfrak{q} \qquad (i = 1, 2).$$

Hence

$$\dim \frac{R}{\sqrt{\operatorname{Ann}(N_i)}} = \dim \frac{R}{\operatorname{Ann}(N_i)} = \dim(R/\mathfrak{p}) =: d$$

is obtained, since Krull dimension depends only on the radical of the ideal.

Now observe that

$$\operatorname{Ann}(N_1 + N_2) = \operatorname{Ann}(N_1) \cap \operatorname{Ann}(N_2).$$

Indeed, $r(N_1 + N_2) = 0$ if and only if $rN_1 = 0$ and $rN_2 = 0$. Thus

$$\sqrt{\operatorname{Ann}(N_1 + N_2)} = \sqrt{\operatorname{Ann}(N_1) \cap \operatorname{Ann}(N_2)} = \sqrt{\operatorname{Ann}(N_1)} \cap \sqrt{\operatorname{Ann}(N_2)}$$

is obtained. Since $\sqrt{\operatorname{Ann}(N_i)} = \mathfrak{p}$, we get

$$\sqrt{\operatorname{Ann}(N_1 + N_2)} = \mathfrak{p} \cap \mathfrak{p} = \mathfrak{p}.$$

Consequently, again because Krull dimension depends only on the radical, we have

$$\dim \frac{R}{\operatorname{Ann}(N_1 + N_2)} = \dim \frac{R}{\mathfrak{p}} = \min\{d, d\},$$

and the desired equality is satisfied. □

Theorem 3. *Let R be a commutative Noetherian ring with identity, and let M be a finitely generated R-module. Suppose that for $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Ass}(M)$, whenever $\mathfrak{p}_1 \neq \mathfrak{p}_2$, one has*

$$\dim \frac{R}{\mathfrak{p}_1} \neq \dim \frac{R}{\mathfrak{p}_2}.$$

If M is totally annihilator-dependent, then $\operatorname{Ass}(M)$ is a singleton.

Proof. Assume that M is totally annihilator-dependent. To obtain a contradiction, suppose that $\operatorname{Ass}(M)$ contains two distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$. By the definition of associated primes, for $i = 1, 2$, there exists $m_i \in M$ such that

$$\operatorname{Ann}(m_i) = \mathfrak{p}_i.$$

Let $N_i := Rm_i$. Then

$$\operatorname{Ann}(N_i) = \operatorname{Ann}(m_i) = \mathfrak{p}_i$$

since R is commutative with identity, and hence

$$\sqrt{\operatorname{Ann}(N_i)} = \mathfrak{p}_i$$

is obtained. As above,

$$\operatorname{Ann}(N_1 + N_2) = \operatorname{Ann}(N_1) \cap \operatorname{Ann}(N_2) = \mathfrak{p}_1 \cap \mathfrak{p}_2.$$

Therefore

$$\sqrt{\operatorname{Ann}(N_1 + N_2)} = \sqrt{\mathfrak{p}_1 \cap \mathfrak{p}_2} = \mathfrak{p}_1 \cap \mathfrak{p}_2$$

is obtained. Since

$$\dim \frac{R}{\mathfrak{p}_1 \cap \mathfrak{p}_2} = \max \left\{ \dim \frac{R}{\mathfrak{p}_1}, \dim \frac{R}{\mathfrak{p}_2} \right\},$$

and M is totally annihilator-dependent, we obtain

$$\dim \frac{R}{\mathfrak{p}_1} = \dim \frac{R}{\mathfrak{p}_2}.$$

This is a contradiction. Consequently, $\operatorname{Ass}(M)$ consists of a single prime ideal. □

2.2. The radical distinction set and singleton Ass

Theorem 4. *Let R be a Noetherian commutative ring and $M \neq 0$ a finitely generated R-module. If $Z_g(M) = \emptyset$, then*

$$\operatorname{Ass}(M) = \{\operatorname{Ann}(\operatorname{Rad}(M))\}$$

and $\operatorname{Ann}(\operatorname{Rad}(M))$ *is a prime ideal.*

Proof. Since M is nonzero and finitely generated over a Noetherian ring, $\operatorname{Ass}(M) \neq \emptyset$. Let $\operatorname{Ann}(m) \in \operatorname{Ass}(M)$ for some nonzero m. Since Rm is cyclic, $\operatorname{Ann}(m) = \operatorname{Ann}(Rm)$. Because $Z_g(M) = \emptyset$, every nonzero m satisfies $\operatorname{Ann}(Rm) = \operatorname{Ann}(\operatorname{Rad}(M))$, hence $\operatorname{Ann}(m) = \operatorname{Ann}(\operatorname{Rad}(M))$. Since all associated primes coincide with $\operatorname{Ann}(\operatorname{Rad}(M))$, we conclude $\operatorname{Ass}(M) = \{\operatorname{Ann}(\operatorname{Rad}(M))\}$. As an element of $\operatorname{Ass}(M)$, this ideal is prime. □

Corollary 1. *If $Z_g(M) = \emptyset$, then M is totally annihilator-dependent.*

2.3. Radical sum equivalence

Theorem 5. *Let R be a Noetherian commutative ring and M a finitely generated R-module. The following are equivalent:*

(i) *for all nonzero $N_1, N_2 \leq M$:* $\sqrt{\operatorname{Ann}(N_1 + N_2)} = \sqrt{\operatorname{Ann}(N_1)} + \sqrt{\operatorname{Ann}(N_2)}$*;*

(ii) $\operatorname{Ass}(M) = \{\mathfrak{p}\}$ *for some prime ideal* $\mathfrak{p}$*.*

Proof. (i) $\Rightarrow$ (ii) Suppose the radical sum formula holds for all nonzero $N_1, N_2 \leq M$. Assume for contradiction that $\mathrm{Ass}(M) \supseteq \{\mathfrak{p}_1, \mathfrak{p}_2\}$ with $\mathfrak{p}_1 \neq \mathfrak{p}_2$. Choose $N_i = Rm_i$ with $\mathrm{Ann}(m_i) = \mathfrak{p}_i$, so $\sqrt{\mathrm{Ann}(N_i)} = \mathfrak{p}_i$. Then

$$\sqrt{\mathrm{Ann}(N_1 + N_2)} = \sqrt{\mathrm{Ann}(N_1) \cap \mathrm{Ann}(N_2)} = \mathfrak{p}_1 \cap \mathfrak{p}_2,$$

while $\sqrt{\mathrm{Ann}(N_1)} + \sqrt{\mathrm{Ann}(N_2)} = \mathfrak{p}_1 + \mathfrak{p}_2$. Since $\mathfrak{p}_1 \neq \mathfrak{p}_2$, we have $\mathfrak{p}_1 \cap \mathfrak{p}_2 \subsetneq \mathfrak{p}_1 + \mathfrak{p}_2$ (for if $\mathfrak{p}_1 \cap \mathfrak{p}_2 = \mathfrak{p}_1 + \mathfrak{p}_2$ then, as in the proof of Theorem 1, $\mathfrak{p}_1 = \mathfrak{p}_2$, a contradiction). Hence the radical sum formula fails for N_1 and N_2, a contradiction. Therefore $\mathrm{Ass}(M) = \{\mathfrak{p}\}$.

(ii) $\Rightarrow$ (i) If $\mathrm{Ass}(M) = \{\mathfrak{p}\}$, then for every nonzero $N \leq M$, $\mathrm{Ass}(N) = \{\mathfrak{p}\}$, so $\sqrt{\mathrm{Ann}(N)} = \mathfrak{p}$. By Theorem 1, any two nonzero submodules are radically dependent, which gives

$$\sqrt{\mathrm{Ann}(N_1 + N_2)} = \mathfrak{p} \cap \mathfrak{p} = \mathfrak{p} = \mathfrak{p} + \mathfrak{p} = \sqrt{\mathrm{Ann}(N_1)} + \sqrt{\mathrm{Ann}(N_2)}.$$

□

2.4. Extension to multiplication modules

Multiplication modules were first introduced by Barnard [3] and further developed by El-Bast and Smith [4]. For other references on multiplication modules, see [1].

Definition 4. An R-module M is a multiplication module if every submodule of M is of the form IM, for some ideal I of R.

Theorem 6. *Let R be a Noetherian commutative ring and M a finitely generated multiplication R-module with* $\mathrm{Ass}(M) = \{\mathfrak{p}\}$. *Then for all nonzero $N_1, N_2 \leq M$,*

$$\sqrt{\mathrm{Ann}(N_1 + N_2)} = \sqrt{\mathrm{Ann}(N_1)} + \sqrt{\mathrm{Ann}(N_2)} = \mathfrak{p}.$$

Proof. Since M is multiplication module, every submodule has the form $N = IM$ for some ideal $I \leq R$, and $\mathrm{Ann}(IM) = \{r \in R \mid rI \subseteq \mathrm{Ann}(M)\}$. For submodules $N_1 = I_1M$ and $N_2 = I_2M$ we have $N_1 + N_2 = (I_1 + I_2)M$, and the identity $\mathrm{Ann}(N_1 + N_2) = \mathrm{Ann}(N_1) \cap \mathrm{Ann}(N_2)$ remains valid. Since $\mathrm{Ass}(N_i) \subseteq \mathrm{Ass}(M) = \{\mathfrak{p}\}$ and $\mathrm{Ass}(N_i) \neq \emptyset$, we get $\sqrt{\mathrm{Ann}(N_i)} = \mathfrak{p}$, and the conclusion follows by the argument in Theorem 5. □

3. Illustrative examples

Example 1 ($M = \mathbb{Z}/8\mathbb{Z}$)**.** Let $M = \mathbb{Z}/8\mathbb{Z}$ as a $\mathbb{Z}$-module. Then $\mathrm{Rad}(M) = 2\mathbb{Z}/8\mathbb{Z} = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ and $\mathrm{Ann}(\mathrm{Rad}(M)) = 4\mathbb{Z}$. The annihilators of nonzero elements are:

$$\mathrm{Ann}(\bar{1}) = \mathrm{Ann}(\bar{3}) = \mathrm{Ann}(\bar{5}) = \mathrm{Ann}(\bar{7}) = 8\mathbb{Z},\ \mathrm{Ann}(\bar{2}) = \mathrm{Ann}(\bar{6}) = 4\mathbb{Z},$$

and

$$\mathrm{Ann}(\bar{4}) = 2\mathbb{Z}.$$

Since $\mathrm{Ann}(\mathrm{Rad}(M)) = 4\mathbb{Z}$, the elements whose annihilator differs from $4\mathbb{Z}$ are precisely $\bar{1}, \bar{3}, \bar{4}, \bar{5}, \bar{7}$, giving $Z_g(M) = \{\bar{1}, \bar{3}, \bar{4}, \bar{5}, \bar{7}\}$.

On the other hand, $\sqrt{\mathrm{Ann}(M)} = \sqrt{0} = 2\mathbb{Z}$ and $\sqrt{\mathrm{Ann}(\mathrm{Rad}(M))} = \sqrt{4\mathbb{Z}} = 2\mathbb{Z}$. For every nonzero $\bar{m} \in M$ we have $\sqrt{\mathrm{Ann}(\bar{m})} = 2\mathbb{Z}$ (since the only prime dividing $|\bar{m}|$ in $\mathbb{Z}/8\mathbb{Z}$ is 2), so $\widetilde{Z}_g(M) = \emptyset$. This example illustrates that annihilator equality and radical annihilator equality yield genuinely different structural decompositions.

Example 2. Let $M = \mathbb{Z}/5\mathbb{Z}$. Since M is simple, $\mathrm{Rad}(M) = 0$ and $\mathrm{Ann}(\mathrm{Rad}(M)) = \mathbb{Z}$. For every nonzero $m \in M$, $Rm = M$, so

$$\mathrm{Ann}(Rm) = 5\mathbb{Z} \neq \mathbb{Z} = \mathrm{Ann}(\mathrm{Rad}(M)).$$

Hence $Z_g(M) = M \setminus \{\bar{0}\} = \{\bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. Similarly, $\sqrt{\mathrm{Ann}(Rm)} = 5\mathbb{Z} \neq \mathbb{Z} = \sqrt{\mathrm{Ann}(\mathrm{Rad}(M))}$, so $\widetilde{Z}_g(M) = M \setminus \{\bar{0}\}$ as well. For simple modules the two sets always coincide.

Example 3 (Positive case)**.** Let $M = \mathbb{Z}/25\mathbb{Z}$ as a $\mathbb{Z}$-module. This is a finitely generated multiplication $\mathbb{Z}$-module with $\mathrm{Ass}(M) = \{5\mathbb{Z}\}$. Every nonzero submodule $N \leq M$ satisfies $\sqrt{\mathrm{Ann}(N)} = 5\mathbb{Z}$, so

$$\sqrt{\mathrm{Ann}(N_1 + N_2)} = \sqrt{\mathrm{Ann}(N_1)} + \sqrt{\mathrm{Ann}(N_2)} = 5\mathbb{Z}$$

for all nonzero $N_1, N_2 \leq M$, confirming Theorem 6.

Remark 2 (Necessity of hypotheses)**.** The hypotheses of finite generation and multiplication in Theorem 6 cannot be dropped. Consider $M = \mathbb{Z}_{p^\infty}$ (the Prüfer p-group), which is divisible but *not* finitely generated. Let $N_1 = \mathbb{Z}/p\mathbb{Z}$ and $N_2 = pM = M$. Then $N_1 + N_2 = M$, $\mathrm{Ann}(N_1) = p\mathbb{Z}$, and $\mathrm{Ann}(M) = 0$, giving

$$\sqrt{\mathrm{Ann}(N_1 + N_2)} = 0 \neq p\mathbb{Z} + 0 = \sqrt{\mathrm{Ann}(N_1)} + \sqrt{\mathrm{Ann}(N_2)}.$$

Note that $\mathbb{Z}_{p^\infty}$ falls outside the standing assumption of finite generation; its inclusion here is solely to illustrate why that assumption is essential.

References

[1] Anderson, D.D., Arabaci, T., Tekir, Ü., Koç, S.: On S-multiplication modules. Comm. Algebra **48**(8), 3398–3407 (2020). https://doi.org/10.1080/00927872.2020.1737873

[2] Khaksari, A., Sharif, H., Ershad, M.: On prime submodules of multiplication modules. Int. J. Pure Appl. Math. **17**(1), 41–49 (2004)

[3] Barnard, A.: Multiplication modules. J. Algebra **71**(1), 174–178 (1981). https://doi.org/10.1016/0021-8693(81)90112-5

[4] El-Bast, Z.A., Smith, P.F.: Multiplication modules. Comm. Algebra **16**(4), 755–779 (1988). https://doi.org/10.1080/00927878808823601

[5] Hassanzadeh-Lelekaami, D., Roshan-Shekalgourabi, H.: Pseudo-prime submodules of modules. Math. Reports **18**(68), 591–608 (2016)

[6] Smith, P.F.: Radical submodules and uniform dimension of modules. Turkish J. Math. **28**(3), 255–270 (2004)

[7] Wisbauer, R.: Foundations of Module and Ring Theory. Gordon and Breach Science Publishers, London (1991). https://doi.org/10.1201/9780203755532

[8] Lee, S., Moo, Y., Varmazyar, R.: Associated prime submodules of a multiplication module. Honam Math. J. **39**(2), 275–296 (2017). https://doi.org/10.5831/HMJ.2017.39.2.275

[9] Atani, S.E., Ghaleh, S.K.G.: On multiplication modules. Int. Math. Forum **1**(21-24), 1175–1180 (2006). http://dx.doi.org/10.12988/imf.2006.06096

[10] Koç, S.: On annihilator multiplication modules. Preprint at https://arxiv.org/abs/2510.03791 (2025)

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