

# When is a Bogolyubov automorphism inner?

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**ABSTRACT.** Let  $V$  be an infinite-dimensional vector space over a field of characteristic not equal to 2. Given a nondegenerate quadratic form  $f$  on  $V$ , we consider the Clifford algebra  $\text{Cl}(V, f)$ . Any orthogonal linear transformation of  $V$  extends to a Bogolyubov automorphism of  $\text{Cl}(V, f)$ . We obtain necessary and sufficient conditions for a Bogolyubov automorphism to be inner.

## Introduction

Let  $\mathbb{F}$  be a field of characteristic not equal to 2, and let  $V$  be a vector space over  $\mathbb{F}$ . A mapping  $f: V \rightarrow \mathbb{F}$  is called a *quadratic form* if the following conditions are satisfied:

- (1)  $f(\lambda v) = \lambda^2 f(v)$ ,  $\lambda \in \mathbb{F}$ ,  $v \in V$ ;
- (2)  $(v | w) = f(v + w) - f(v) - f(w)$  is a bilinear form on  $V$ .

A quadratic form is said to be *nondegenerate* if the associated bilinear form  $(v | w)$  is nondegenerate, that is,  $(v | V) = (0)$  implies  $v = 0$ .

The *Clifford algebra*  $\text{Cl}(V, f)$  is the associative algebra generated by the vector space  $V$  and the identity element 1 with defining relations

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$v^2 = f(v) \cdot 1$  for all  $v \in V$ . If the quadratic form  $f$  is fixed, we simply write  $\text{Cl}(V)$ .

If  $\{v_i\}_{i \in I}$  is a basis of  $V$  and the index set  $I$  is totally ordered, then the set consisting of the identity 1 and all ordered products

$$v_{i_1} v_{i_2} \cdots v_{i_k}, \quad i_1 < i_2 < \cdots < i_k,$$

forms a basis of the Clifford algebra  $\text{Cl}(V)$ .

The Clifford algebra  $\text{Cl}(V)$  is graded by the cyclic group of order 2:

$$\text{Cl}(V) = \text{Cl}(V)_{\bar{0}} + \text{Cl}(V)_{\bar{1}},$$

where

$$\text{Cl}(V)_{\bar{0}} = \mathbb{F} \cdot 1 + \sum_{n=1}^{\infty} \underbrace{V \cdots V}_{2n}, \quad \text{Cl}(V)_{\bar{1}} = \sum_{n=1}^{\infty} \underbrace{V \cdots V}_{2n-1}.$$

Let  $\varphi: V \rightarrow V$  be an orthogonal linear transformation, that is,  $f(\varphi(v)) = f(v)$  for all  $v \in V$ . It is well-known that  $\varphi$  admits a unique extension to an automorphism  $[\varphi]$  of the Clifford algebra  $\text{Cl}(V)$ . Such automorphisms are called *Bogolyubov automorphisms*.

There is extensive literature on automorphisms of Clifford algebras over Hilbert and pre-Hilbert spaces, as well as on  $C^*$ -completions of these algebras (see, for example, [1, 7, 9]). In particular, H. Araki [1] described Bogolyubov automorphisms of Clifford  $C^*$ -algebras that are inner automorphisms (see also [9]).

Let  $\varphi: V \rightarrow V$  be a linear transformation. For an eigenvalue  $\alpha$  of  $\varphi$ , we denote by

$$V(\alpha) = \ker(\varphi - \alpha \cdot \text{Id})$$

the corresponding eigenspace, where  $\text{Id}$  denotes the identity operator on  $V$ .

A linear transformation  $\varphi$  is called *finitary* if the subspace  $V(1)$  has finite codimension in  $V$ .

Let  $W \subset V$  be a  $\varphi$ -invariant subspace of finite codimension. Then  $\varphi$  gives rise to the linear transformation  $\varphi|_{V/W}$  on the finite-dimensional quotient space  $V/W$ .

In this paper, we describe inner Bogolyubov automorphisms of the Clifford algebra  $\text{Cl}(V)$  in a purely algebraic setting.

**Theorem 1.** *Let  $\mathbb{F}$  be an algebraically closed field of characteristic  $\neq 2$ , and let  $V$  be an infinite-dimensional vector space over  $\mathbb{F}$  with a nondegenerate quadratic form. Let  $\varphi$  be an orthogonal linear transformation.*

The Bogolyubov automorphism  $[\varphi]$  of the Clifford algebra  $\text{Cl}(V)$  is inner if and only if

(1)  $\varphi$  is finitary and either  $\varphi = \text{Id}$  or  $\det(\varphi|_{V/V(1)}) = 1$ ,

or

(2)  $-\varphi$  is finitary,  $\varphi \neq -\text{Id}$ ,  $\dim_{\mathbb{F}}(V/V(-1)) = k \geq 1$ , and

$$(-1)^k \cdot \det(\varphi|_{V/V(-1)}) = -1.$$

**Example 1.** Let  $V$  be countable-dimensional and let  $v_1, v_2, \dots$  be an orthonormal basis of  $V$ . The linear transformation  $\varphi$ , defined by

$$\varphi(v_1) = v_1, \quad \varphi(v_i) = -v_i \quad \text{for } i \geq 2,$$

is orthogonal. We have  $\dim_{\mathbb{F}}(V/V(-1)) = 1$  and  $\det(\varphi|_{V/V(-1)}) = 1$ . For an arbitrary element  $a \in \text{Cl}(V)$ , we have

$$[\varphi](a) = v_1^{-1} a v_1.$$

In [3], we described Bogolyubov derivations of  $\text{Cl}(V)$  that are inner.

The Clifford algebra  $\text{Cl}(V)$  is a unital locally matrix algebra corresponding to the Steinitz number  $2^\infty$  (see [4–6]). Automorphisms and derivations of arbitrary countable-dimensional unital locally matrix algebras were described in [2].

## 1. Proof of the Lemmas

We call a finite-dimensional subspace  $U \subset V$  *nondegenerate* if the restriction of the quadratic form to  $U$  is nondegenerate.

**Lemma 1.** (1) *Suppose that  $V(1)$  has finite codimension in  $V$ . Then there exists an even-dimensional, nondegenerate,  $\varphi$ -invariant subspace  $W \subset V$  such that*

$$W^\perp = \{v \in V \mid (v \mid W) = (0)\} \subseteq V(1).$$

(2) *Suppose that  $V(-1)$  has finite codimension in  $V$ . Then there exists an even-dimensional, nondegenerate,  $\varphi$ -invariant subspace  $W \subset V$  such that*

$$W^\perp = \{v \in V \mid (v \mid W) = (0)\} \subseteq V(-1).$$

*Proof.* We prove only part (1). Part (2) is proved similarly. The subspace  $(\text{Id} - \varphi)(V)$  is finite-dimensional. Let  $\chi(t)$  be the characteristic polynomial of the restriction of  $\varphi$  to the subspace  $(\text{Id} - \varphi)(V)$ . Then  $\varphi$  is a root of the polynomial  $(1 - t)\chi(t)$ .

There exists a finite-dimensional  $\varphi$ -invariant subspace  $U \subseteq V$  such that  $V = U + V(1)$ . For arbitrary elements  $v_1, v_2 \in V$ , we have

$$(\varphi(v_1) \mid v_2) = (v_1 \mid \varphi^{-1}(v_2)).$$

Hence,

$$((\varphi - \text{Id})(v_1) \mid v_2) = (v_1 \mid (\varphi^{-1} - \text{Id})(v_2)) = (v_1 \mid \varphi^{-1}(\text{Id} - \varphi)(v_2)).$$

This implies that  $((\varphi - \text{Id})(V) \mid V(1)) = (0)$ .

Let  $v \in U^\perp$  and suppose that  $(\varphi - \text{Id})(v) \neq 0$ . Then  $(\varphi - \text{Id})(v)$  is orthogonal both to  $U$  and to  $V(1)$ . This contradicts the nondegeneracy of the quadratic form on  $V$ . Therefore, we have proved that  $U^\perp \subseteq V(1)$ .

Suppose that the subspace  $U$  is nondegenerate. If  $\dim_{\mathbb{F}} U$  is even, then we are done. Assume that  $\dim_{\mathbb{F}} U$  is odd. Since the restriction of the quadratic form to  $U^\perp$  is nondegenerate, there exists an element  $v \in U^\perp$  such that  $(v \mid v) \neq 0$ . The subspace  $U + \mathbb{F}v$  satisfies all the required assumptions.

Now suppose that the subspace  $U$  is not nondegenerate. Let  $0 \neq u \in U \cap U^\perp \subseteq U \cap V(1)$ . Since the quadratic form on  $V$  is nondegenerate, there exists an element  $v \in V(1)$  such that  $(u \mid v) \neq 0$ . The subspace  $\mathbb{F}u + \mathbb{F}v$  is nondegenerate, and we obtain the orthogonal decomposition  $V = (\mathbb{F}u + \mathbb{F}v) \oplus (\mathbb{F}u + \mathbb{F}v)^\perp$ .

Consider the projection  $\pi: V \rightarrow V' = (\mathbb{F}u + \mathbb{F}v)^\perp$ . We have  $V' = \pi(U) + V'(1)$ , where the subspace  $\pi(U)$  is  $\varphi$ -invariant and  $\dim_{\mathbb{F}} \pi(U) < \dim_{\mathbb{F}} U$ . Using induction on  $\dim_{\mathbb{F}} U$ , we find an even-dimensional  $\varphi$ -invariant nondegenerate subspace  $W' \subset V'$  such that  $(W')^\perp \cap V' \subseteq V'(1)$ . It remains to choose  $W = \mathbb{F}u + \mathbb{F}v + W'$ .

This completes the proof of the lemma.  $\square$

For a subset  $S$  of an algebra  $A$ , let

$$Z_A(S) = \{a \in A \mid ax = xa \text{ for each } x \in S\}$$

be the centralizer of the set  $S$  in  $A$ .

**Lemma 2.** *Let  $U$  be a nondegenerate even-dimensional subspace of  $V$ . Let  $v_1, \dots, v_n$  be an orthonormal basis of  $U$ . Then*

$$Z_{\text{Cl}(V)}(\text{Cl}(U)) = \text{Cl}(U^\perp)_{\overline{0}} + v_1 \cdots v_n \text{Cl}(U^\perp)_{\overline{1}}.$$

*Proof.* Let  $\{u_i\}_{i \in I}$  be an ordered basis of  $U^\perp$ . Consider an element

$$x = v_{i_1} \cdots v_{i_k} u_{j_1} \cdots u_{j_l},$$

where  $1 \leq i_1 < \cdots < i_k \leq n$ ,  $j_1 < \cdots < j_l$ . For each  $i \notin \{i_1, \dots, i_k\}$ ,

$$v_i x = (-1)^{k+l} x v_i.$$

For each  $t \in \{1, \dots, k\}$ ,

$$v_{i_t} x = (-1)^{k+l-1} x v_{i_t}.$$

Hence, the element  $x$  will commute with each of the elements  $v_1, \dots, v_n$  if and only if either  $k = 0$  and  $l$  is even, or  $k = n$  and  $l$  is odd. Otherwise,  $x$  will anticommute with  $v_i$  for some  $i$ . This completes the proof of the lemma.  $\square$

**Lemma 3.** *Let  $\varphi: V \rightarrow V$  be an orthogonal linear transformation, and let  $[\varphi]$  be the corresponding Bogolyubov automorphism. Let  $v_1, v_2 \in V$  be linearly independent. If  $[\varphi](v_1 v_2) = v_1 v_2$ , then the subspace  $\mathbb{F}v_1 + \mathbb{F}v_2$  is  $\varphi$ -invariant.*

*Proof.* Suppose that  $\varphi(v_1) \notin \mathbb{F}v_1 + \mathbb{F}v_2$ . Extend the system  $v_1, v_2$  to a basis  $v_1, v_2, v_3, \dots$  of  $V$ . Let

$$\varphi(v_1) = \sum_{i=1}^n \alpha_i v_i \quad \text{and} \quad \varphi(v_2) = \sum_{i=1}^n \beta_i v_i.$$

Without loss of generality, let  $\alpha_3 \neq 0$ . We have that

$$\begin{aligned} v_1 v_2 = [\varphi](v_1 v_2) &= (\alpha_1 \beta_2 - \alpha_2 \beta_1) v_1 v_2 + \sum_{i \leq j} \alpha_i \beta_j f(v_i, v_j) 1 + \\ &+ \sum_{\substack{i < j \\ (i,j) \neq (1,2)}} (\alpha_i \beta_j - \alpha_j \beta_i) v_i v_j. \end{aligned}$$

Since the system of vectors  $\{1; v_i v_j, i < j\}$  is linearly independent, it follows that  $\alpha_1 \beta_2 - \alpha_2 \beta_1 = 1$ ,  $\alpha_1 \beta_3 = \alpha_3 \beta_1$  and  $\alpha_2 \beta_3 = \alpha_3 \beta_2$ . However, the last two equalities contradict the first one. Hence,  $\varphi(v_1) \in \mathbb{F}v_1 + \mathbb{F}v_2$ . Analogously,  $\varphi(v_2) \in \mathbb{F}v_1 + \mathbb{F}v_2$ , which completes the proof of the Lemma.  $\square$

## 2. Proof of Theorem 1

*Proof of Theorem 1. Necessity.* Let  $[\varphi]$  be a conjugation by an invertible element  $x$ . There exists a nondegenerate even-dimensional subspace  $V' \subseteq V$  such that  $x \in \text{Cl}(V')$ .

Choose an arbitrary vector  $0 \neq v \in V'^{\perp}$ . Choose arbitrary elements  $v_1, v_2 \in V'^{\perp}$  such that the system  $v, v_1, v_2$  is linearly independent. The elements  $vv_1$  and  $vv_2$  lie in  $Z_{\text{Cl}(V)}(\text{Cl}(V'))$ . Hence,

$$[\varphi](vv_1) = vv_1, \quad [\varphi](vv_2) = vv_2.$$

By Lemma 3, the subspaces  $\mathbb{F}v + \mathbb{F}v_1$  and  $\mathbb{F}v + \mathbb{F}v_2$  are  $\varphi$ -invariant. Hence,

$$\varphi(v) \in (\mathbb{F}v + \mathbb{F}v_1) \cap (\mathbb{F}v + \mathbb{F}v_2) = \mathbb{F}v.$$

We have shown that any element  $v \in V'^{\perp}$  is an eigenvector of  $\varphi$ . Since the linear transformation  $\varphi$  is orthogonal, it follows that

$$\varphi(v) = v \quad \text{or} \quad \varphi(v) = -v.$$

Hence,  $V'^{\perp} \subseteq V(1)$  or  $V'^{\perp} \subseteq V(-1)$ . Consider the first case. The second case is considered similarly.

Let  $v_1, \dots, v_n$  be an orthonormal basis of  $V'$ . It follows from Lemma 2 that for each  $u \in V'^{\perp}$ ,

$$[\varphi](v_1 \cdots v_n u) = v_1 \cdots v_n u.$$

We have that

$$[\varphi](v_1 \cdots v_n u) = \det \varphi|_{V/V(1)} v_1 \cdots v_n u.$$

Thus,  $\det \varphi|_{V/V(1)} = 1$ .

*Sufficiency.* Suppose condition (1) of Theorem 1 is satisfied. Case (2) is considered similarly. By Lemma 1, there exists a nondegenerate, even-dimensional,  $\varphi$ -invariant subspace  $V'$  of  $V$  such that  $V'^{\perp} \subseteq V(1)$ . We have that  $\text{Cl}(V) = \text{Cl}(V') \otimes Z_{\text{Cl}(V)}(\text{Cl}(V'))$ ; see [10].

Let  $v_1, \dots, v_n$  be an orthonormal basis of  $V'$ . For each linearly independent  $u_1, u_2 \in V'^{\perp}$ ,

$$[\varphi](v_1 \cdots v_n u_1) = \det \varphi|_{V/V(1)} v_1 \cdots v_n u_1 = v_1 \cdots v_n u_1 \quad \text{and}$$

$$[\varphi](u_1 u_2) = u_1 u_2.$$

It follows from Lemma 2 that  $[\varphi]$  acts identically on  $Z_{\text{Cl}(V)}(\text{Cl}(V'))$ . Thus  $[\varphi]$  must be a conjugation with an element from  $\text{Cl}(V')$ .

This completes the proof of Theorem 1. □

## References

- [1] Araki, H.: On quasifree states of CAR and Bogoliubov automorphisms. *Publ. Res. Inst. Math. Sci. Kyoto* **6**, 385–442 (1971). <https://doi.org/10.2977/prims/1195193913>
- [2] Bezushchak, O.: Derivations and automorphisms of locally matrix algebras. *J. Algebra* **576**, 1–26 (2021). <https://doi.org/10.1016/j.jalgebra.2021.02.013>
- [3] Bezushchak, O.: On Clifford algebras of infinite dimensional vector spaces. *Contemp. Math. Modern Algebra Volume 2: Groups and Algebras (Proceedings of the 14th Ukraine Algebraic Conference)*, vol. **830**, pp. 53–60. Amer. Math. Soc. (2025). <https://doi.org/10.1090/conm/830/16558>
- [4] Bezushchak, O., Oliynyk, B.: Morita equivalent unital locally matrix algebras. *Algebra Discrete Math.* **29**(2), 173–179 (2020). <http://dx.doi.org/10.12958/adm1545>
- [5] Bezushchak, O., Oliynyk, B.: Unital locally matrix algebras and Steinitz numbers. *J. Algebra Appl.* **19**(09), 2050180 (2020). <https://doi.org/10.1142/S0219498820501807>
- [6] Bezushchak, O.O., Sushchansky, V.I.: Groups of periodically defined linear transformations of an infinite-dimensional vector space. *Ukr. Math. J.* **67**, 1457–1468 (2016). <https://doi.org/10.1007/s11253-016-1165-x>
- [7] Bratteli, O., Robinson, D.W.: *Operator Algebras and Quantum Statistical Mechanics II. Theoretical and Mathematical Physics*. Springer Berlin, Heidelberg (1981). <https://doi.org/10.1007/978-3-662-09089-3>
- [8] Drozd, Yu.A., Kirichenko, V.V.: *Finite Dimensional Algebras*. Springer Berlin, Heidelberg (1994). <https://doi.org/10.1007/978-3-642-76244-4>
- [9] De La Harpe, P.: The Clifford algebra and the Spinor group of a Hilbert space. *Compos. Math.* **25**(3), 245–261 (1972)
- [10] Herstein, I.N.: *Noncommutative Rings*. The Carus Mathematical Monographs, vol. **15**. The Mathematical Association of America (1968)

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