

All maximal congruences on an inverse semigroup of fence-preserving transformations

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ABSTRACT. We study the inverse semigroup of all partial injections on an n -element set $\{1, \dots, n\}$, which preserve the zig-zag order $1 \prec 2 \succ 3 \prec 4 \succ \dots n$ and the parity for some positive integer n . We characterize all maximal congruences on this inverse semigroup, which are entirely congruences of index two.

Introduction

For $n \in \mathbb{N}$, let $\bar{n} = \{1, 2, \dots, n\}$ be a finite set with n elements and let $F = (\bar{n}, \prec)$ be a fence, also called zig-zag poset, i.e. a partially ordered set, in which the order relation \prec forms a path with alternating orientation:

$$1 \prec 2 \succ 3 \prec 4 \succ \dots n \quad \text{or} \quad 1 \succ 2 \prec 3 \succ 4 \prec \dots n.$$

Each element in F is either maximal or minimal. The fence F is called up-fence (respectively, down-fence) if $1 \prec 2$ (respectively, $1 \succ 2$). In this paper, we consider (without loss of generality) an up-fence (see Figure 1).

For a set $A \subseteq \bar{n}$, a mapping $\alpha : A \rightarrow \bar{n}$ is called partial transformation on \bar{n} . The set A is called domain of α (in symbols: $A = \text{dom}(\alpha)$) and $\text{im}(\alpha) = \{x\alpha : x \in \text{dom}(\alpha)\}$ is the image (range) of α . The cardinality of $\text{im}(\alpha)$ is called the rank of α , denoted by $\text{rank}(\alpha) = |\text{im}(\alpha)|$. For $A \subseteq \bar{n}$,

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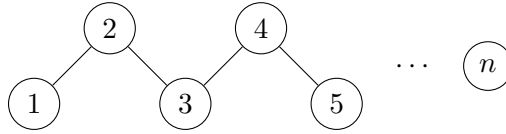


Figure 1: Up fence

we denote by id_A the identity mapping on A . Clearly, if $A = \bar{n}$, then $id_A = id$, the identity mapping on \bar{n} . The empty transformation will be denoted by ϕ , i.e. $\text{dom}(\phi) = \emptyset$. The set of all partial transformations on \bar{n} forms a monoid, denoted by \mathcal{PT}_n with the identity mapping id and the zero ϕ (with respect of the composition of mappings). This monoid is well studied, see for example [1–4].

An $\alpha \in \mathcal{PT}_n$ is called fence-preserving if for all $x, y \in \text{dom}(\alpha)$ the following implication holds: $x \prec y$ implies $x\alpha \prec y\alpha$. The set of all fence-preserving partial (full) transformations on \bar{n} forms a submonoid of \mathcal{PT}_n , denoted by \mathcal{PTF}_n (\mathcal{TF}_n). The monoid \mathcal{TF}_n was first studied by Currie and Visentin [5] in 1991 as well as by Rutkowski [6] in 1992. The main results of these papers are concerning the cardinality of \mathcal{TF}_n for even as well as odd n , by different techniques. In the last 10 years, the monoid \mathcal{PTF}_n and particular submonoids of \mathcal{PTF}_n were studied, see for example [7–11, 16].

The set of all injective partial transformations on \bar{n} forms a submonoid of \mathcal{PT}_n , which is denoted by \mathcal{I}_n , the symmetric inverse semigroup. A semigroup S is called inverse if for all $a \in S$, there is unique $a^{-1} \in S$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. The set

$$\mathcal{IF}_n = \{\alpha \in \mathcal{PTF}_n \cap \mathcal{I}_n : \alpha^{-1} \in \mathcal{PTF}_n\}$$

forms a submonoid of \mathcal{I}_n , which contains all regular elements of \mathcal{PTF}_n . In particular, \mathcal{IF}_n is an inverse semigroup. The Green’s relations in \mathcal{IF}_n are determined by Dimitrova and Koppitz in [9]. Moreover, the authors show that \mathcal{IF}_n is generated by its transformations with rank $\geq n-2$, whenever n is even. For the even case, a generating set of minimal size is determined in [9], which is unique. Recall that the minimal size of a generating set of a semigroup S is called the rank of S and is denoted by $\text{rank}(S)$. In particular, $\text{rank}(\mathcal{IF}_n) = n+1$ (if n is even). In the case n is odd, the situation is quite more complex. The rank of \mathcal{IF}_n , whenever n is odd, is determined by Fernandes, Koppitz, and Musunthia in [11].

Further, let $<$ be the canonical order on \bar{n} . A transformation $\alpha \in \mathcal{PT}_n$ is called order-preserving if for all $x, y \in \text{dom}(\alpha)$ the following implication holds: $x < y$ implies $x\alpha \leq y\alpha$. Let \mathcal{OPPT}_n denote the set of all order-preserving partial transformations on \bar{n} . The set $\mathcal{IOPF}_n^{\text{par}} = \{\alpha \in \mathcal{OPPT}_n \cap \mathcal{IF}_n : x \text{ and } x\alpha \text{ have the same parity for all } x \in \text{dom}(\alpha)\}$ forms a submonoid of \mathcal{IF}_n and is an inverse semigroup. In [7], Sareeto and Koppitz characterize the elements in $\mathcal{IOPF}_n^{\text{par}}$ and show that $\text{rank}(\mathcal{IOPF}_n^{\text{par}}) = 3n - 6$. The same authors give a presentation for $\mathcal{IOPF}_n^{\text{par}}$ [7]. In [8], the Green's relations on $\mathcal{IOPF}_n^{\text{par}}$ are described. In the same paper, Koppitz and Sareeto characterize the ideals of $\mathcal{IOPF}_n^{\text{par}}$ and their maximal subsemigroups.

In the present paper, we consider another submonoid of \mathcal{IF}_n , namely the monoid $\mathcal{IF}_n^{\text{par}} = \{\alpha \in \mathcal{IF}_n : x \text{ and } x\alpha \text{ have the same parity for all } x \in \text{dom}(\alpha)\}$. In fact, any $\alpha \in \mathcal{IF}_n^{\text{par}}$ is a partial injection, which maps any (maximal) consecutive chain I of $\text{dom}(\alpha)$ to a (maximal) consecutive chain of \bar{n} (under the canonical order $<$) such that the maximal element of I , denoted by b , and $b\alpha$ have the same parity.

The aim of the present paper is the study of the congruences on $\mathcal{IF}_n^{\text{par}}$. If σ is a congruence on a semigroup S , then we shall denote by S/σ the partition of S induced by σ (as equivalence relation), i.e. $S/\sigma = \{[a]_\sigma : a \in S\}$, and $[a]_\sigma$ is the σ -class containing the element a . The cardinality of S/σ is called the index of σ . A non-universal congruence σ on S , i.e. $\sigma \neq S \times S$, is called maximal if the following implication holds: if ρ is a non-universal congruence on S with $\sigma \subseteq \rho$, then $\sigma = \rho$. The importance of the study of (maximal) congruences on semigroups was shown by Howie [13]. In particular, he has shown that a congruence-free semigroup is a simple semigroup if it has at least three elements but no zero element. In [12], for certain semigroups S of numbers and of transformations, Sanwong and Sullivan describe all maximal congruences on S . In particular, they describe the maximal congruences on the symmetric inverse semigroup \mathcal{I}_n . Maximal congruences on several other semigroups of transformations were determined in the last twenty years, see for example [14, 15, 17].

In the present paper, we determine the maximal congruences on $\mathcal{IF}_n^{\text{par}}$. It will turn out that all maximal congruences on $\mathcal{IF}_n^{\text{par}}$ have index two.

1. A class of subsets of \bar{n}

We define a class \mathcal{A} of subsets of \bar{n} , for which we will show that each set in \mathcal{A} defines a maximal congruence on $\mathcal{S}\mathcal{F}_n^{par}$.

Definition 1. Let \mathcal{A} be the set of all non-empty subsets A of \bar{n} , which satisfy the following condition: For all $x \in \{0, 1, \dots, n - 3\} \setminus A$ and all $k \in \{2, \dots, \lfloor \frac{n}{2} \rfloor - \lfloor \frac{x}{2} \rfloor\}$ with $x + 2k \notin A$, we have

$$x + r \in A \text{ if and only if } x + 2k - r \in A$$

for all $r \in \{1, \dots, k - 1\}$.

In this section, we prove several properties of the elements in \mathcal{A} . In the remaining part of this section, we will use a short version of the application of Definition 1 to any $A \in \mathcal{A}$. If we write $a, b \notin A$ and $c \in A$ imply $d \in A$, then we mean that there is $x \in \bar{n}$ such that $a + x = c \in A$ ($a - x = c \in A$, respectively) implies that $d = b - x \in A$ ($d = b + x \in A$, respectively), where $b - a$ ($a - b$, respectively) is an even positive integer and $|a - b| \geq 4$. A dual interpretation has the phrase $a, b \in A$ and $c \notin A$ imply $d \notin A$.

Proposition 2. Let $A \in \mathcal{A}$. Then $\{1, 2\} \cap A \neq \emptyset$ and $\{n - 1, n\} \cap A \neq \emptyset$.

Proof. We will show that $\{1, 2\} \cap A \neq \emptyset$. The fact that $\{n - 1, n\} \cap A \neq \emptyset$ can be shown dually (if n is odd) and by similar arguments (if n is even), respectively. Assume that $\{1, 2\} \cap A = \emptyset$. Let $k_1 = \lfloor \frac{n}{2} \rfloor$ and $\hat{n} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$ Clearly, $\hat{n}, \hat{n} + 1 \notin A$. If n is even, then $\hat{n} + 2k_1 = 1 + n$. If n is odd, then $\hat{n} + 2k_1 = 0 + n + 1$. So, $\hat{n} + 2k_1 = 1 + n \notin A$. Then $\hat{n} + 1 \notin A$ implies $n = (n + 1) - 1 = \hat{n} + 2k_1 - 1 \notin A$. Let $\tilde{n} = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even} \end{cases}$ and let $k_2 = \lfloor \frac{n}{2} \rfloor - \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$ Clearly, $\tilde{n}, \tilde{n} + 1 \notin A$. If n is odd, then $\tilde{n} + 2k_2 = 1 + n + 1 - 2 = n$. If n is even, then $\tilde{n} + 2k_2 = 0 + n$. So, $\tilde{n} + 2k_2 = n \notin A$. Therefore, $\tilde{n} + 1 \notin A$ implies $n - 1 = \tilde{n} + 2k_2 - 1 = n - 1 \notin A$. Let $k_3 = k_1 - 1 = \lfloor \frac{n}{2} \rfloor - 1$. Then $\hat{n} + 2k_3 = \hat{n} + 2k_1 - 2 = n + 1 - 2 = n - 1 \notin A$. So, $\hat{n} + 1 \notin A$ implies $n - 2 = (n - 1) - 1 = \hat{n} + 2k_3 - 1 \notin A$. If we continue in that way, then we obtain

$$z_i + 2k_i - 1 = n - i + 1 \notin A \text{ for } 1 \leq i \leq 2 \lfloor \frac{n}{2} \rfloor - 3$$

with $z_i = \begin{cases} \widehat{n} & \text{if } i \text{ is odd,} \\ \widetilde{n} & \text{if } i \text{ is even} \end{cases}$ and $k_i = -\left\lfloor \frac{i-1}{2} \right\rfloor + \begin{cases} k_1 & \text{if } i \text{ is odd,} \\ k_2 & \text{if } i \text{ is even.} \end{cases}$

It is easy to verify that $k_i \in \{2, \dots, \left\lceil \frac{n}{2} \right\rceil\}$ for $1 \leq i \leq 2 \left\lceil \frac{n}{2} \right\rceil - 3$. This shows that $A = \emptyset$, a contradiction. \square

As usually, we denote by $2\mathbb{N}$ ($2\mathbb{N}-1$) the set of all even (odd) positive integers.

Proposition 3. *Let $A \in \mathcal{A}$ such that $1 \notin A$ and $2 \in A$. Then $2\mathbb{N} \cap \bar{n} \subseteq A$.*

Proof. Assume that there is $c \in (2\mathbb{N} \cap \bar{n}) \setminus A$. Suppose that n is even. Then $1 + 2 \left\lceil \frac{n}{2} \right\rceil = 1 + n \notin A$. So $1 + 1 = 2 \in A$ implies $n = 1 + 2 \left\lceil \frac{n}{2} \right\rceil - 1 \in A$. Since $c > 2$, there is $k \in \{2, \dots, \left\lceil \frac{n}{2} \right\rceil\}$ with $c = 2k$. So $0 + 1 = 1 \notin A$ implies $c - 1 = 0 + 2k - 1 \notin A$. Since $c < n$, there is $l \in \{2, \dots, \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{c-1}{2} \right\rfloor\}$ such that $(c - 1) + 2l = n + 1 \notin A$. Then $(c - 1) + 2l - 1 = n \in A$ implies $c = (c - 1) + 1 \in A$, a contradiction. Suppose now that n is odd. Then $0 + 2 \left\lceil \frac{n+1}{2} \right\rceil = n + 1 \notin A$ and $0 + 1 = 1 \notin A$ imply $n = 2 \left\lceil \frac{n+1}{2} \right\rceil - 1 \notin A$. Moreover, $0 + 2 = 2 \in A$ implies $n - 1 = 2 \left\lceil \frac{n+1}{2} \right\rceil - 2 \in A$. By the same argument as in the even case, we obtain $c - 1 \notin A$. Since $c < n - 1$, there is $p \in \{2, \dots, \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{c-1}{2} \right\rfloor\}$ such that $(c - 1) + 2p = n \notin A$. Then $(c - 1) + 2p - 1 = n - 1 \in A$ implies $c = (c - 1) + 1 \in A$, a contradiction. \square

If n is odd and $n \notin A$, then $n - 1 \in A$ (by Proposition 2) and we obtain that $2\mathbb{N} \cap \bar{n} \subseteq A$ with dual arguments. For convenience, we define $n_e = \begin{cases} n - 1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even} \end{cases}$ and $n_o = \begin{cases} n & \text{if } n \text{ is odd,} \\ n - 1 & \text{if } n \text{ is even.} \end{cases}$

Proposition 4. *Let $A \in \mathcal{A}$ such that $n_e \notin A$ and $n_o \in A$. Then $(2\mathbb{N} - 1) \cap \bar{n} = A$.*

Proof. If n is even then $n \notin A$ and $n - 1 \in A$. By dual arguments as in the proof of Proposition 3, we can show that all odd integers in \bar{n} belong to A , i.e. $(2\mathbb{N} - 1) \cap \bar{n} \subseteq A$. Suppose now that n is odd. Then $n \in A$ and $n - 1 \notin A$. It is clear that there is $k \in \{2, \dots, \left\lceil \frac{n}{2} \right\rceil\}$ such that $n + 1 = 2k$,

namely $k = \lceil \frac{n}{2} \rceil$. Note that $0, 0 + 2k \notin A$. So, $0 + 1 = 1 \in A$ implies $n = n + 1 - 1 \in A$. Moreover, $0 + 2k - 2 = n - 1 \notin A$ implies $2 = 0 + 2 \notin A$. If $2 \neq n - 1$, then there is $l \in \{2, \dots, \lceil \frac{n}{2} \rceil\}$ such that $0 + 2l = n - 1 \notin A$. So, $0 + 1 = 1 \in A$ implies $n - 2 = (n - 1) - 1 = 0 + 2l - 1 \in A$ and further, $0 + 2 = 2 \notin A$ implies $n - 3 = 0 + 2l - 2 \notin A$. If $3 \neq n - 3$, then $0 + 2k - 3 = n - 2 \in A$ implies $3 = 0 + 3 \in A$ and further, $0 + 2k - 4 = n - 3 \notin A$ implies $4 = 0 + 4 \notin A$. Continuing this procedure, we obtain that $(2\mathbb{N} - 1) \cap \bar{n} = A$. \square

From Propositions 2 and 3 (and the remark after Proposition 3, respectively), we obtain:

Corollary 5. *Let $A \in \mathcal{A}$ such that $(\bar{n} \setminus A) \cap 2\mathbb{N} \neq \emptyset$. Then $1 \in A$ (and $n \in A$, whenever n is odd).*

From Propositions 2 and 4, we can conclude:

Corollary 6. *Let $A \in \mathcal{A}$ such that $(\bar{n} \setminus A) \cap (2\mathbb{N} - 1) \neq \emptyset$. Then $n_e \in A$.*

Let us give a note to the cardinality of any $A \in \mathcal{A}$.

Proposition 7. *Let $A \in \mathcal{A}$. Then $|A| \geq \lfloor \frac{n}{2} \rfloor$.*

Proof. Assume that $|A| < \lfloor \frac{n}{2} \rfloor$. For $i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$, we put $A_i = \{2i - 1, 2i\}$ and $A_{\lfloor \frac{n}{2} \rfloor} = \begin{cases} \{n - 1, n\} & \text{if } n \text{ is even,} \\ \{n - 2, n - 1, n\} & \text{if } n \text{ is odd.} \end{cases}$ We observe that $\bar{n} = \bigcup_{i=1}^{\lfloor \frac{n}{2} \rfloor} A_i$. From $|A| < \lfloor \frac{n}{2} \rfloor$, it follows the existence of an $i \in \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$ such that $A_i \cap A = \emptyset$. We observe that $i \neq 1$ by Proposition 2. Moreover, $A_i \cap A = \emptyset$ implies $(\bar{n} \setminus A) \cap 2\mathbb{N} \neq \emptyset$. Hence, $1 \in A$ by Corollary 5. Since $i > 1$, we get $2i > 2$, where $0, 0 + 2i \notin A$. Then $0 + 1 = 1 \in A$ implies $0 + 2i - 1 \in A$, a contradiction to $A_i \cap A = \emptyset$. \square

Let $A \in \mathcal{A}$. Then there are $d_1, d_2, \dots, d_l \in \bar{n}$ with $d_1 < d_2 < \dots < d_l$ and $l = |A|$ such that $A = \{d_1, \dots, d_l\}$. We observe that the set $\{d_1, d_1 + 1, d_1 + 2, \dots, d_l\} \setminus A$ consists entirely of maximal interval of size one or two.

Proposition 8. *Let $A = \{d_1, d_2, \dots, d_l\} \in \mathcal{A}$ for some $l \in \bar{n}$, where $d_1 < d_2 < \dots < d_l$. Then*

$$d_{i+1} - d_i \leq 3$$

for all $i \in \{1, \dots, l - 1\}$.

Proof. Assume there exists $j \in \{1, \dots, l - 1\}$ such that $d_{j+1} - d_j \geq 4$. Then $d_j + 1, d_j + 2, d_j + 3 \notin A$. If $d_j \in 2\mathbb{N} - 1$, then $d_j + 1 \geq 4$ is even, whenever $d_j > 1$. Then $d_j \in A$ and $0, d_j + 1 \notin A$ imply $1 \in A$, whenever $d_j > 1$. So, $1 \in A$. Moreover, $0, d_j + 3 \notin A$ and $1 \in A$ imply $d_j + 2 \in A$, which is a contradiction. If $d_j \in 2\mathbb{N}$, then $2 \in A$ since $0, d_j + 2 \notin A$ and $d_j \in A$. Moreover, we obtain $1 \notin A$ since $d_j + 1 \notin A$. By Corollary 5, we obtain a contradiction since not all even elements of \bar{n} are contained in A . \square

For a non-empty subset A of \bar{n} , we define a particular partition.

Definition 9. For $\emptyset \neq A \subseteq \bar{n}$, let A_S be the set of all subsets

$$\{x, x + 1, \dots, x + r\} \quad (x \in \bar{n}, 0 \leq r \leq n - x)$$

of A such that $x - 1 \notin A$ and $x + r + 1 \notin A$.

In fact, A_S is the set of all maximal intervals (blocks) in the set $A \in \mathcal{A}$. Let $X, Y \in A_S$. Then we write $X < Y$ if all elements in X are less than any element in Y with respect to the natural order of \mathbb{N} . Further, we write $X \prec Y$ if $X < Y$ and for each $Z \in A_S$, the following implication holds: If $X \leq Z \leq Y$, then $X = Z$ or $Y = Z$. So, any $\emptyset \neq A \subseteq \bar{n}$ can be written in the following form:

$$A = [A_1 \ A_2 \ \dots \ A_k],$$

where $A_1 \prec A_2 \prec \dots \prec A_k \in A_S$ for some $k \in \bar{n}$. For convenience, we define

$$a_i = \min A_i \quad \text{and} \quad b_i = \max A_i$$

for all $i \in \{1, \dots, k\}$. In particular, we have $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$. Moreover, we can say that A has k blocks.

From now on, we consider any $A \in \mathcal{A}$ in the above form. Using this notation for $A \in \mathcal{A}$, Proposition 8 can be represented as follows.

Let $A = [A_1 \ A_2 \ \dots \ A_k] \in \mathcal{A}$ for some $k \in \bar{n}$. Then for all $i \in \{1, \dots, k - 1\}$, we have

$$a_{i+1} - b_i = 2 \quad \text{or} \quad a_{i+1} - b_i = 3.$$

Lemma 10. *Let $A \in \mathcal{A}$ such that $1 \in A$. If there exists $x \in \bar{n}$ such that $x, x + 1 \notin A$, then $x \in 2\mathbb{N}$.*

Proof. Let $x \in \bar{n}$ such that $x, x + 1 \notin A$. Assume that $x \in 2\mathbb{N} - 1$. Then $x + 1 \in 2\mathbb{N}$. Since $0, x + 1 \notin A$ and $1 \in A$, we obtain $x \in A$. It is a contradiction with $x \notin A$. Hence $x \in 2\mathbb{N}$. \square

Lemma 11. *Let $A = [A_1 \ A_2 \ \cdots \ A_k] \in \mathcal{A}$ for some $k \in \bar{n}$. If there exists $x \in \{1, \dots, k - 1\}$ such that $a_{x+1} - b_x = 3$, then $a_{i+1} - b_i = 2$ for all $i \in \{1, \dots, k - 1\} \setminus \{x\}$.*

Proof. Assume that there exists $y \in \{1, \dots, k - 1\} \setminus \{x\}$ such that $a_{y+1} - b_y = 3$. Then $1 \in A$ by Corollary 5. Note that $b_x + 1, b_x + 2, b_y + 1, b_y + 2 \notin A$. By Lemma 10, we obtain $b_x + 1 \in 2\mathbb{N}$ and $b_y + 1 \in 2\mathbb{N}$. Without loss of generality, we suppose $x < y$. By $b_x + 1, b_y + 1 \notin A$ and $(b_x + 1) + 1 = b_x + 2 \notin A$, we have $b_y = (b_y + 1) - 1 \notin A$, a contradiction. \square

Lemma 12. *Let $A = [A_1 \ A_2 \ \cdots \ A_k] \in \mathcal{A}$ for some $k \in \mathbb{N}$. If $|A_x| \in 2\mathbb{N}$ for some $x \in \{1, \dots, k\}$, then*

$$a_{i+1} - b_i = 2$$

for all $i \in \{1, \dots, k - 1\}$.

Proof. Assume that there exists $y \in \{1, \dots, k - 1\}$ such that $a_{y+1} - b_y = 3$. Then $b_y + 1, b_y + 2 \notin A$ implies $1 \in A$ by Corollary 5. Since $1 \in A$ and $b_y + 1, b_y + 2 \notin A$, we obtain $b_y + 1 \in 2\mathbb{N}$ by Lemma 10 and thus $b_y + 2 \in 2\mathbb{N} - 1$.

Suppose that $a_x \in 2\mathbb{N}$. Then $b_x \in 2\mathbb{N} - 1$. If $b_x < b_y + 1$, then $a_x - 1, b_y + 2 \notin A$ and $(a_x - 1) + 1 = a_x \in A$ imply $b_y + 1 = (b_y + 2) - 1 \in A$, which is a contradiction with $b_y + 1 \notin A$. If $b_y + 2 < a_x$, then $b_y + 1, b_x + 1 \notin A$ and $(b_y + 2) - 1 = b_y + 1 \notin A$ imply $b_x = (b_x + 1) - 1 \notin A$, which is a contradiction with $b_x \in A$.

Suppose that $a_x \in 2\mathbb{N} - 1$. Then $b_x \in 2\mathbb{N}$. Assume that $b_x + 2 \notin A$. Since $0 \notin A$ and $1 \in A$, we get $b_x + 1 = b_x + 2 - 1 \in A$, a contradiction. Hence $b_x + 2 \in A$. If $b_x < b_y + 1$, then $b_y + 1 \neq b_x + 1$ since $b_x, b_y + 1 \in 2\mathbb{N}$. By $b_x + 1, b_y + 2 \notin A$ and $(b_x + 1) + 1 = b_x + 2 \in A$, we have $b_y + 1 = (b_y + 2) - 1 \in A$, which is a contradiction with $b_y + 1 \notin A$. If $b_y + 2 < a_x$, then $a_x > 1$ and $b_y + 2 \neq a_x - 1$ since $a_x \in 2\mathbb{N} - 1$ and $b_y + 1 \in 2\mathbb{N}$. Since $b_y + 1, a_x - 1 \notin A$ and $(b_y + 1) + 1 = b_y + 2 \notin A$, we have $a_x - 2 = (a_x - 1) - 1 \notin A$. Then $0, a_x - 1 \notin A$ implies $1 \notin A$, which is a contradiction. \square

Lemma 13. *Let $A = [A_1 \ A_2 \ \cdots \ A_k] \in \mathcal{A}$ for some $k \in \mathbb{N}$ and let $i, j \in \{1, \dots, k\}$ such that $i \neq j$. If $|A_i|, |A_j| \in 2\mathbb{N} - 1$, a_i and a_j have the same parity, then $|A_i| = |A_j|$.*

Proof. Assume $|A_i| \neq |A_j|$. Without loss of generality, we suppose that $a_i < a_j$ and $|A_i| < |A_j|$. It is easy to see that $a_i - 1$ and $b_j + 1$ have the same parity and that $a_i - 1, b_j + 1 \notin A$. Since $|A_i| < |A_j|$, we have $(b_j + 1) - (|A_i| + 1) \in A_j$. Thus $b_i + 1 = a_i - 1 + |A_i| + 1 \in A$, which is a contradiction with $b_i + 1 \notin A$. Hence $|A_i| = |A_j|$. \square

Lemma 14. *Let $A = [A_1 \ A_2 \ \cdots \ A_k] \in \mathcal{A}$ for some $k \in \mathbb{N}$. If $|A_i| \in 2\mathbb{N} - 1$ for all $i \in \{1, \dots, k\}$, then there is $p \in \{0, 1, \dots, k\}$ such that*

- (1) $a_i \in 2\mathbb{N} - 1$ for all $1 \leq i \leq p$ and $a_j \in 2\mathbb{N}$ for all $p + 1 \leq j \leq k$;
- (2) $|A_i| = |A_j|$ for all $i, j \in \{1, \dots, p\}$ and $|A_i| = |A_j|$ for all $i, j \in \{p + 1, \dots, k\}$.

Proof. First, we will show that there exists $p \in \{0, 1, \dots, k\}$ such that $a_i \in 2\mathbb{N} - 1$ for all $1 \leq i \leq p$ and $a_j \in 2\mathbb{N}$ for all $p + 1 \leq j \leq k$. If $a_i \in 2\mathbb{N} - 1$ for all $i \in \{1, \dots, k\}$, then $p = k$. If there exists $c \in \{1, \dots, k\}$ such that

$$a_c = \min\{2\mathbb{N} \cap \{a_i : i \in \{1, \dots, k\}\}\},$$

then we claim $a_i \in 2\mathbb{N}$ for all $i \geq c$. Assume that there exists $x \in \{1, \dots, k\}$ with $x > c$ such that $a_x \in 2\mathbb{N} - 1$. We have $a_c - 2, a_c - 1 \notin A$ because b_{c-1} is odd or $a_c = 2$. Since $a_c - 2, b_x + 1 \notin A$, $(b_x + 1) - 1 = b_x \in A$, we have $a_c - 1 = (a_c - 2) + 1 \in A$, which is a contradiction with $a_c - 1 \notin A$. Thus $p = c - 1$. The rest of the proof is given by Lemma 13. \square

Let us assign, to each $l \in \bar{n} \cup \{0\}$, an element of the set $\{-1, 0, 1\}$ as follows: For $l \in \bar{n} \cup \{0\}$, let $n^{(l)} = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd and } l \neq 0, \\ -1 & \text{if } n \text{ is odd and } l = 0. \end{cases}$

Theorem 15. *Let $A = [A_1 \ A_2 \ \cdots \ A_k] \in \mathcal{A}$ for some $k \in \mathbb{N}$ such that $|A_i| \in 2\mathbb{N} - 1$ for all $i \in \{1, \dots, k\}$. Then $A\alpha = A$ for all $\alpha \in \mathcal{I}\mathcal{F}_n^{par}$ with $A \subseteq \text{dom}\alpha$.*

Proof. Let $\alpha \in \mathcal{I}\mathcal{F}_n^{par}$ such that $A \subseteq \text{dom}\alpha$. By Lemma 14, we can write $A = [A_1 \ \cdots \ A_p \ A_{p+1} \ \cdots \ A_{p+l}]$ for some $p \in \{0, 1, \dots, k\}$ and $l = k - p$, where $|A_i| = r \in 2\mathbb{N} - 1$ for all $1 \leq i \leq p$ and $|A_{p+j}| = s \in 2\mathbb{N} - 1$ for all $1 \leq j \leq l$. Then $\sum_{i=1}^k |A_i| = pr + ls$. Clearly, since $a_1 \in 2\mathbb{N} - 1$ by Lemma 14,

we have $a_1 = 1$. Since $a_{i+1} - b_i = 2$ for all $i \in \{1, \dots, p + l - 1\} \setminus \{p\}$ and $a_{p+1} - b_p = 3$, we have $\sum_{i=1}^k |A_i| + p + l + n^{(l)} = pr + ls + p + l + n^{(l)} = n$. For $i \in \{1, \dots, p + l\}$, we define $B_i = A_i\alpha$. Since $\alpha \in \mathcal{S}\mathcal{F}_n^{par}$, we have $\sum_{i=1}^k |B_i| = \sum_{i=1}^k |A_i| = pr + ls$. Thus $\sum_{i=1}^k |B_i| + p + l + n^{(l)} = n$. We put $C = \bar{n} \setminus A\alpha$. Then $n = \sum_{i=1}^k |B_i| + |C|$.

Suppose that n is even. If $l \neq 0$, then $\bigcup_{i=p+1}^k B_i = \{n, \dots, n - \left(\left|\bigcup_{i=p+1}^k B_i\right| + l\right)\} \setminus \{1 + n - y(s + 1) : 1 \leq y \leq l\}$. Otherwise, it is easy to verify that $|C| \geq p + l + 1$. This provides $n = \sum_{i=1}^k |B_i| + |C| \geq \left(\sum_{i=1}^k |B_i| + p + l\right) + 1 = n + 1$, a contradiction. If $p \neq 0$, then we obtain $\bigcup_{i=1}^p B_i = \{1, \dots, \left|\bigcup_{i=1}^p B_i\right| + p\} \setminus \{y(r + 1) : 1 \leq y \leq l\}$ by similar arguments.

Suppose now that n is odd. Since $0, n + 1 \notin A$, we obtain that $1 = 0 + 1 \in A$ if and only if $n = (n + 1) - 1 \in A$. If $l \neq 0$, then $p = 0$, i.e. $l = k$, and we have $\bigcup_{i=1}^k B_i = \{n - 1, \dots, n - 1 - \left(\left|\bigcup_{i=1}^k B_i\right| + k\right)\} \setminus \{1 + n - y(s + 1) : 1 \leq y \leq k\}$. Otherwise, it is easy to verify that $|C| \geq l + 2$. This provides $n = \sum_{i=1}^k |B_i| + |C| \geq \left(\sum_{i=1}^k |B_i| + p + l + 1\right) + 1 = n + 1$, a contradiction.

If $p \neq 0$, then $l = 0$, i.e. $p = k$, and we have $\bigcup_{i=1}^k B_i = \{1, \dots, \left|\bigcup_{i=1}^k B_i\right| + k\} \setminus \{y(r + 1) : 1 \leq y \leq k\}$. Otherwise, it is easy to verify that $|C| \geq p$. This provides $n = \sum_{i=1}^k |B_i| + |C| \geq \left(\sum_{i=1}^k |B_i| + p + l - 1\right) + 1 = n + 1$, a contradiction.

Now for any n (even as well as odd), we can calculate:

$$\begin{aligned} \bigcup_{i=1}^p B_i &= \{1, \dots, \left|\bigcup_{i=1}^p B_i\right| + p\} \setminus \{y(r + 1) : 1 \leq y \leq p\} \\ &= \{1, \dots, \sum_{i=1}^p |B_i| + p\} \setminus \{y(r + 1) : 1 \leq y \leq p\} \\ &= \{1, \dots, \sum_{i=1}^p |A_i| + p\} \setminus \{y(r + 1) : 1 \leq y \leq p\} \end{aligned}$$

$$\begin{aligned}
 &= \{1, \dots, \left| \bigcup_{i=1}^p A_i \right| + p\} \setminus \{y(r+1) : 1 \leq y \leq p\} \\
 &= \bigcup_{i=1}^p A_i.
 \end{aligned}$$

Similarly, we obtain $\bigcup_{i=p+1}^k B_i = \left\{ n - n^{(l)}, \dots, n - \left(\left| \bigcup_{i=p+1}^k B_i \right| + l + 1 - n^{(l)} \right) \right\} \setminus \{1 + n - n^{(l)} - y(s+1) : 1 \leq y \leq l\} = \bigcup_{i=p+1}^k A_i$. Therefore, $A\alpha = \bigcup_{i=1}^k A_i\alpha = \bigcup_{i=1}^k B_i = \bigcup_{i=1}^p B_i \cup \bigcup_{i=p+1}^k B_i = \bigcup_{i=1}^p A_i \cup \bigcup_{i=p+1}^k A_i = \bigcup_{i=1}^k A_i = A$. \square

Now we consider an $A = [A_1 \ \dots \ A_k] \in \mathcal{A}$ such that there is $p \in \{1, \dots, k\}$ with $|A_p| \in 2\mathbb{N}$.

Lemma 16. *Let $A = [A_1 \ \dots \ A_k] \in \mathcal{A}$ for some $k \in \mathbb{N}$. If there exists $p \in \{1, \dots, k\}$ such that $|A_p| \in 2\mathbb{N}$, then $1, n_e \in A$.*

Proof. Let $p \in \{1, \dots, k\}$ such that $|A_p| \in 2\mathbb{N}$. If $A = \bar{n}$, then it is clear that $1, n_e \in A$. Suppose now that $A \subset \bar{n}$.

If $1 = a_p$, then $b_p + 1 \in 2\mathbb{N} - 1$ and $b_p + 1 \notin A$. This implies $n_e \in A$ by Corollary 6. Hence $1, n_e \in A$.

If $n = b_p$ and $a_p \neq 1$, then $n_e \in A_p \subseteq A$ and $a_p - 1 \in (\bar{n} \setminus A) \cap 2\mathbb{N}$. The latter one implies $1 \in A$ by Corollary 5. Hence $1, n_e \in A$.

If $1, n_e \notin A_p$, then $a_p - 1 \in \bar{n} \setminus A$ and $b_p + 1 \in \bar{n} \setminus A$ have different parity. So, $(\bar{n} \setminus A) \cap 2\mathbb{N} \neq \emptyset$ and $(\bar{n} \setminus A) \cap 2\mathbb{N} - 1 \neq \emptyset$. This implies $1, n_e \in A$ by Corollaries 5 and 6. \square

Lemma 17. *Let $A = [A_1 \ \dots \ A_k] \in \mathcal{A}$ for some $k \in \mathbb{N}$. If there is unique $p \in \{1, \dots, k\}$ such that $|A_p| \in 2\mathbb{N}$, then for $l = k - p$, we have*

- (1) $a_1, \dots, a_p \in 2\mathbb{N} - 1$ and $a_{p+1}, \dots, a_{p+l} \in 2\mathbb{N}$ (if $l \neq 0$);
- (2) $|A_i| = |A_j|$ for all $i, j \in \{1, \dots, p\}$; $|A_i| = |A_j|$ for all $i, j \in \{p+1, \dots, p+l\}$.

Proof. Let p be the unique element in $\{1, \dots, k\}$ such that $|A_p| \in 2\mathbb{N}$. We have $1, n_e \in A$ by Lemma 16. First, we will show that $a_1, \dots, a_p \in 2\mathbb{N} - 1$ and $a_{p+1}, \dots, a_{p+l} \in 2\mathbb{N}$ (if $l \neq 0$). Note that $|A_i| \in 2\mathbb{N} - 1$ for all $i \in \{1, \dots, p+l\} \setminus \{p\}$ and $a_{i+1} - b_i = 2$ for all $i \in \{1, \dots, k-1\}$ by Lemma 12. Moreover, we have $a_1 = 1$ and $a_i = a_{i-1} + |A_{i-1}| + 1$ for all $2 \leq i \leq p$. Since $|A_{i-1}| + 1$ is even, we obtain $a_i \in 2\mathbb{N} - 1$ for all $2 \leq i \leq p$. If $l = 0$,

then all is shown. Suppose now that $l > 0$. Since $a_{p+1} = a_p + |A_p| + 1$ is even, we can show that $a_{p+1}, \dots, a_{p+l} \in 2\mathbb{N}$ as above. Thus, we have (1). By Lemma 13, we obtain (2). This completes the proof. \square

If n is odd, then 0 and $n + 1$ are even and do not belong to A . So, we can conclude that $1 \in A$ if and only if $n \in A$. Lemma 17 implies that $1 \in A$ and $b_k \in 2\mathbb{N}$ and thus, n is even, whenever there is unique $p \in \{1, \dots, k\}$ with $|A_p| \in 2\mathbb{N}$.

Theorem 18. *Let n be even and let $A = [A_1 \ \dots \ A_k] \in \mathcal{A}$ for some $k \in \mathbb{N}$ such that there is unique $p \in \{1, \dots, k\}$ with $|A_p| \in 2\mathbb{N}$. Then $A\alpha = A$ for all $\alpha \in \mathcal{I}\mathcal{F}_n^{par}$ with $A \subseteq \text{dom}(\alpha)$.*

Proof. Let $\alpha \in \mathcal{I}\mathcal{F}_n^{par}$ such that $A \subseteq \text{dom}(\alpha)$. By Lemma 17, there are $r \in 2\mathbb{N} - 1$ (if $p > 1$), $s \in 2\mathbb{N} - 1$ (if $p < k$), and $t \in 2\mathbb{N}$ such that $|A_i| = r$, for all $1 \leq i \leq p - 1$, $|A_{p+j}| = s$, for all $1 \leq j \leq k - p = l$, and $|A_p| = t$.

Then $\sum_{i=1}^k |A_i| = (p - 1)r + ls + t$. By $a_{i+1} - b_i = 2$ for all $1 \leq i \leq k - 1$

and $n = b_p$, we have $\sum_{i=1}^k |A_i| + p + l - 1 = (p - 1)r + ls + t + p + l - 1 = n$.

For $i \in \{1, \dots, p + l\}$, we define $B_i = A_i\alpha$. Since $\alpha \in \mathcal{I}\mathcal{F}_n^{par}$, then $\min B_i - 1, \max B_i + 1 \notin A\alpha$ for all $1 \leq i \leq k$ and $\sum_{i=1}^{p+l} |B_i| = \sum_{i=1}^{p+l} |A_i| =$

$(p - 1)r + ls + t$. Thus $\sum_{i=1}^{p+l} |B_i| + p + l - 1 = n$. We put $C = \bar{n} \setminus A\alpha$.

Clearly, we have $|C| = p + l - 1$, i.e. $n = \sum_{i=1}^{p+l} |B_i| + |C|$. It is easy to verify that $1, n \in A\alpha$. From $1, n \in A\alpha$, $|C| = p + l - 1$, and $\min B_i - 1, \max B_i + 1 \in$

$C \cup \{0, n + 1\}$, we can conclude that $\bigcup_{i=1}^{p-1} B_i \subseteq \{k : 1 \leq k \leq \left| \bigcup_{i=1}^{p-1} B_i \right| + p - 1\}$,

$\bigcup_{i=p+1}^{p+l} B_i \subseteq \left\{ k : n + 1 - \left(\left| \bigcup_{i=p+1}^{p+l} B_i \right| + l \right) \leq k \leq n \right\}$, and $B_p = \bar{n} \setminus \left(\{k :$

$1 \leq k \leq \left| \bigcup_{i=1}^{p-1} B_i \right| + p - 1\} \cup \left\{ k : n + 1 - \left(\left| \bigcup_{i=p+1}^{p+l} B_i \right| + l \right) \leq k \leq n \right\}$). Since

each B_i , where $1 \leq i \leq k$, is a maximal interval in \bar{n} , we can calculate that

$$\begin{aligned} \bigcup_{i=1}^{p-1} B_i &= \{k : 1 \leq k \leq \left| \bigcup_{i=1}^{p-1} B_i \right| + p - 1\} \setminus \{y(r + 1) : 1 \leq y \leq p - 1\} \\ &= \{k : 1 \leq k \leq \sum_{i=1}^{p-1} |B_i| + p - 1\} \setminus \{y(r + 1) : 1 \leq y \leq p - 1\} \end{aligned}$$

$$\begin{aligned}
 &= \{k : 1 \leq k \leq \sum_{i=1}^{p-1} |A_i| + p - 1\} \setminus \{y(r + 1) : 1 \leq y \leq p - 1\} \\
 &= \{k : 1 \leq k \leq \left| \bigcup_{i=1}^{p-1} A_i \right| + p - 1\} \setminus \{y(r + 1) : 1 \leq y \leq p - 1\} \\
 &= \bigcup_{i=1}^{p-1} A_i.
 \end{aligned}$$

By the same arguments, one can calculate that $B_p = \bar{n} \setminus \left(\{k : 1 \leq k \leq \left| \bigcup_{i=1}^{p-1} B_i \right| + p - 1\} \cup \left\{ k : n + 1 - \left(\left| \bigcup_{i=p+1}^{p+l} B_i \right| + l \right) \leq k \leq n \right\} \right) = A_p$ and $\bigcup_{i=p+1}^{p+l} B_i = \left\{ k : n + 1 - \left(\left| \bigcup_{i=p+1}^{p+l} B_i \right| + l \right) \leq k \leq n \right\} \setminus \{1 + n - y(s + 1) : 1 \leq y \leq l\} = \bigcup_{i=p+1}^{p+l} A_i$. Therefore, $A\alpha = \bigcup_{i=1}^{p+l} A_i\alpha = \bigcup_{i=1}^{p+l} B_i = \bigcup_{i=1}^{p-1} B_i \cup B_p \cup \bigcup_{i=p+1}^{p+l} B_i = \bigcup_{i=1}^{p-1} A_i \cup A_p \cup \bigcup_{i=p+1}^{p+l} A_i = \bigcup_{i=1}^{p+l} A_i = A$. \square

Finally, we consider the case that $\text{dom}(\alpha)$ has more than one maximal interval with even size.

Lemma 19. *Let $A = [A_1 \ A_2 \ \dots \ A_k] \in \mathcal{A}$ for some $k \in \mathbb{N}$. If there exist $p, q \in \{1, \dots, k\}$ with $p \neq q$ such that $|A_p|, |A_q| \in 2\mathbb{N}$, then either there are $t \in 2\mathbb{N} - 1$ and $u \in \bar{n}$ with $2t = n + 1 - (k - 2)(u + 1)$ such that $A = \bar{n} \setminus \{t + lu : 0 \leq l \leq k - 2\}$ and n is odd or $|A_i| = |A_j| \in 2\mathbb{N}$ for all $i, j \in \{1, \dots, k\}$.*

Proof. Let $p, q \in \{1, \dots, k\}$ with $p < q$ such that $|A_p|, |A_q| \in 2\mathbb{N}$. Suppose that there exists $r \in \{1, \dots, k\} \setminus \{p, q\}$ such that $|A_r| \in 2\mathbb{N} - 1$.

First, we assume that $r < p$. Without loss of generality, we can assume that $r = p - 1$ and there is no $p + 1 \leq i \leq q - 1$ with $|A_i| \in 2\mathbb{N}$. Then it is easy to see that $a_r - 1, b_q + 1 \notin A$ as well as $a_p - 1$ and $b_q + 1$ have the same parity. If $|A_q| < |A_r|$, then $a_r - 1 + (|A_q| + 1) = a_r + |A_q| \in A$. This implies $a_q - 1 = b_q + 1 - (|A_q| + 1) \in A$, a contradiction. If $|A_r| < |A_q|$, then $b_q + 1 - (|A_r| + 1) = b_q - |A_r| \in A$. This implies $b_r + 1 = a_r - 1 + (|A_r| - 1) \in A$, a contradiction, too. In the case $r > q$, we obtain a contradiction in a dually way.

It is easy to verify that the following case remains: $1 = p < r < q = k$. Since $1 \in A$, we can conclude that $b_k \in 2\mathbb{N} - 1$. Since $(\bar{n} \setminus A) \cap (2\mathbb{N} - 1) \neq \emptyset$, we can conclude that $n_e \in A$ by Corollary 6. Since b_k is odd, we can

conclude that n is odd. This shows that $A = \bar{n} \setminus \{t_1 < t_2 < \dots < t_{k-1}\}$ for some odd $t_1, t_2, \dots, t_{k-1} \in \bar{n} \setminus \{1, n\}$. Assume that $|A_1| \neq |A_k|$. Without loss of generality, we can assume that $|A_1| < |A_k|$. Then we have $0 + (|A_1| + 1) = b_1 + 1 \notin A$ and $(n + 1) - (|A_1| + 1) \in A_k \subset A$ since $|A_1| < |A_k|$. But $0, n + 1 \notin A$ and $0 + (|A_1| + 1) \notin A$ imply $(n + 1) - (|A_1| + 1) \notin A$, a contradiction. In the same matter, we can show that there is $u \in \bar{n}$ with $u = |A_i|$ for $i = 2, \dots, k - 1$. Hence, $A = \bar{n} \setminus \{t_1 + lu : 0 \leq l \leq k - 2\}$ with $2t_1 = n + 1 - (k - 2)(u + 1)$.

Suppose that $|A_i| \in 2\mathbb{N}$ for all $i \in \{1, \dots, k\}$. Assume that there is $i \in \{1, \dots, k - 1\}$ with $|A_i| \neq |A_{i+1}|$. Without loss of generality, we can assume that $|A_i| < |A_{i+1}|$. Since $|A_i|, |A_{i+1}| \in 2\mathbb{N}$, we can conclude that a_i and b_{i+1} have the same parity. So $a_i - 1$ and $b_{i+1} + 1$ have the same parity and do not belong to A . Let $c = |A_i| + 1$. Then $a_i - 1 + c = b_{i+1} \notin A$ and $b_{i+1} + 1 - c \in A_{i+1} \subset A$ since $|A_i| < |A_{i+1}|$, a contradiction. \square

Theorem 20. *Let $A = [A_1 A_2 \dots A_k] \in \mathcal{A}$ for some $k \in \bar{n}$ such that there are $p, q \in \{1, \dots, k\}$ with $p \neq q$ and $|A_p|, |A_q| \in 2\mathbb{N}$. Then $A\alpha = A$ for all $\alpha \in \mathcal{I} \mathcal{F}_n^{par}$ with $dom(\alpha) \in A$.*

Proof. By Lemma 19, we have to consider two cases. Suppose that n is odd and there are $t \in 2\mathbb{N} - 1$ and $u \in \bar{n}$ with $2t = n + 1 - (k - 2)(u + 1)$ such that $A = \bar{n} \setminus \{t + lu : 0 \leq l \leq k - 2\}$. We put $t_i = t + (i - 1)u$ for all $1 \leq i \leq k - 1$. Then $A_1 = \{1, \dots, t_1 - 1\}$, $A_k = \{t_{k-1} + 1, \dots, n\}$, and $A_i = \{t_{i-1} + 1, \dots, t_i - 1\}$ for $i = 2, \dots, k - 1$ (whenever $k \geq 3$). So, $A \cap 2\mathbb{N} = \bar{n} \cap 2\mathbb{N}$ and thus $A\alpha \cap 2\mathbb{N} = \bar{n} \cap 2\mathbb{N}$. Since $1, n \in A$ and both are odd, it is easy to verify that either $A_1\alpha = A_1$ and $A_k\alpha = A_k$ or $A_1\alpha = A_k$ and $A_k\alpha = A_1$. Therefore, $\bigcup_{i=2}^{k-1} A_i\alpha$ as a subset of $\{t_1 + 1, \dots, t_{k-1} - 1\}$, consisting entirely of maximal intervals of length u . This shows that $\bigcup_{i=2}^{k-1} A_i\alpha = \{t_1 + 1, \dots, t_{k-1} - 1\} \setminus \{t_1 + lu : l = 1, \dots, k - 3\} = \bigcup_{i=2}^{k-1} A_i$. Consequently, $A\alpha = A$.

Suppose now that $|A_i| = |A_j| \in 2\mathbb{N}$ for all $i, j \in \{1, \dots, k\}$. We observe that there is $x \in \bar{n}$ such that $|A_i| = x$ and $b_i + 1 \in \overline{n + 1} \setminus A$ for all $i \in \{1, \dots, k\}$, where $\overline{n + 1} = \{1, \dots, n + 1\}$. We have $a_{i+1} - b_i = 2$ for all $i \in \{1, \dots, k - 1\}$ by Lemma 12. This implies $b_i + 2 = a_{i+1}$ for all $i \in \{1, \dots, k - 1\}$. Moreover, we have $1, n \in A$ by Corollary 5. Then we have $\overline{n + 1} = \bigcup_{i=1}^k (A_i \cup \{b_i + 1\})$, which implies $n + 1 = \sum_{i=1}^k |A_i \cup \{b_i + 1\}| = k(x + 1)$. Hence, A consists entirely of $\frac{n + 1}{x + 1}$ maximal blocks of length

x . Similarly, $A\alpha$ consists entirely of $\frac{n+1}{x+1}$ maximal blocks of length x . Clearly, there is only one possibility to have $\frac{n+1}{x+1}$ maximal blocks of length x in an n -element set. Therefore, $A\alpha = A$ for all $\alpha \in \mathcal{I}\mathcal{F}_n^{par}$ with $A \subseteq \text{dom}(\alpha)$. \square

Let us consider any $A = [A_1 \ A_2 \ \dots \ A_k] \in \mathcal{A}$. Then $|A_i| \in 2\mathbb{N} - 1$ for all $i \in \{1, \dots, k\}$ or there is $p \in \{1, \dots, k\}$ with $|A_p| \in 2\mathbb{N}$. In the first case, we have $A\alpha = A$ for all $\alpha \in \mathcal{I}\mathcal{F}_n^{par}$ with $A \subseteq \text{dom}(\alpha)$ by Theorem 15. In the second case, we have to distinguish between p is unique with that property or not. Then we get $A\alpha = A$ by Theorems 18 and 20, respectively. So, we obtain the following corollary.

Corollary 21. *Let $A \in \mathcal{A}$ and let $\alpha \in \mathcal{I}\mathcal{F}_n^{par}$. Then the following statements are equivalent:*

- (1) $A \subseteq \text{dom}(\alpha)$;
- (2) $A\alpha = A$;
- (3) $A \subseteq \text{im}(\alpha)$.

Proof. Suppose that $A \subseteq \text{dom}(\alpha)$. Then $A\alpha = A$ by the previous argumentation. Suppose that $A\alpha = A$. Then it is clear that $A \subseteq \text{im}(\alpha)$. Suppose that $A \subseteq \text{im}(\alpha)$. Then $A \subseteq \text{dom}(\alpha^{-1})$. Recall that $\alpha^{-1} \in \mathcal{I}\mathcal{F}_n^{par}$. This implies $A\alpha^{-1} = A$ by (1) \rightarrow (2), and thus $A \subseteq \text{dom}(\alpha)$. This completes the proof. \square

2. Maximal congruences on $\mathcal{I}\mathcal{F}_n^{par}$

This section provides the main result of this paper, the characterization of the maximal congruences on $\mathcal{I}\mathcal{F}_n^{par}$. Clearly, all congruences with index two are maximal. It will turn out that all maximal congruences on $\mathcal{I}\mathcal{F}_n^{par}$ have index two, that means all of them consisting entirely of two equivalent classes. First, we will determine all congruences on $\mathcal{I}\mathcal{F}_n^{par}$ with index two.

Lemma 22. *Let $\rho = (\Gamma \times \Gamma) \cup (\bar{\Gamma} \times \bar{\Gamma})$ be a congruence on $\mathcal{I}\mathcal{F}_n^{par}$, where $\Gamma \subset \mathcal{I}\mathcal{F}_n^{par}$ with $id \in \Gamma$ and $\bar{\Gamma} = \mathcal{I}\mathcal{F}_n^{par} \setminus \Gamma$. Then the following statements holds:*

- (1) $\phi \notin \Gamma$;

(2) $\alpha \in \Gamma$ if and only if $\alpha^{-1} \in \Gamma$.

Proof. (1) Assume that $\phi \in \Gamma$. Then $(id, \phi) \in \rho$, which implies $(\alpha, \phi) = (\alpha id, \alpha \phi) \in \rho$ for all $\alpha \in \mathcal{I}\mathcal{F}_n^{par}$. So, we have $\rho = \mathcal{I}\mathcal{F}_n^{par} \times \mathcal{I}\mathcal{F}_n^{par}$, which is a contradiction with $\rho = (\Gamma \times \Gamma) \cup (\bar{\Gamma} \times \bar{\Gamma})$. Hence $\phi \notin \Gamma$.

(2) Let $\alpha \in \Gamma$. Assume that $\alpha^{-1} \notin \Gamma$, i.e. $\alpha^{-1} \in \bar{\Gamma}$. By (1), we have $(\alpha^{-1}, \phi) \in \rho$. Since $\alpha = \alpha\alpha^{-1}\alpha$, we have $(\alpha, \phi) = (\alpha\alpha^{-1}\alpha, \alpha\phi\alpha) \in \rho$, i.e. $\alpha \notin \Gamma$, which is a contradiction. Hence $\alpha^{-1} \in \Gamma$. The converse direction becomes clear by $(\alpha^{-1})^{-1} = \alpha$. \square

The following lemma determines the congruences on $\mathcal{I}\mathcal{F}_n^{par}$ with index two.

Lemma 23. *Let $\Gamma \subseteq \mathcal{I}\mathcal{F}_n^{par}$ with $id \in \Gamma$ and let $\rho = (\Gamma \times \Gamma) \cup (\bar{\Gamma} \times \bar{\Gamma})$ with $\bar{\Gamma} = \mathcal{I}\mathcal{F}_n^{par} \setminus \Gamma$. Then ρ is a congruence on $\mathcal{I}\mathcal{F}_n^{par}$ if and only if for all $\alpha, \beta \in \mathcal{I}\mathcal{F}_n^{par}$, the following statements are equivalent:*

(1) $\alpha, \beta \in \Gamma$;

(2) $\alpha\beta \in \Gamma$.

Proof. Suppose that ρ is a congruence on $\mathcal{I}\mathcal{F}_n^{par}$ and let $\alpha, \beta \in \mathcal{I}\mathcal{F}_n^{par}$. If $\alpha, \beta \in \Gamma$, then $(\alpha, id) \in \rho$ and we have $(\alpha\beta, \beta) = (\alpha\beta, id\beta) \in \rho$, which implies $\alpha\beta \in \Gamma$. Next, let $\alpha \in \bar{\Gamma}$. Then $(\alpha, \phi) \in \rho$. For any $\beta \in \mathcal{I}\mathcal{F}_n^{par}$, we have $(\alpha\beta, \phi) = (\alpha\beta, \phi\beta) \in \rho$ and $(\beta\alpha, \phi) = (\beta\alpha, \phi\beta) \in \rho$, which implies $\alpha\beta, \beta\alpha \in \bar{\Gamma}$. This shows that $\alpha, \beta \in \Gamma$ if and only if $\alpha\beta \in \Gamma$.

Suppose that $\alpha, \beta \in \Gamma$ if and only if $\alpha\beta \in \Gamma$ for all $\alpha, \beta \in \mathcal{I}\mathcal{F}_n^{par}$. Since ρ is an equivalence relation, it remains to show that ρ is a congruence. Let $\alpha, \beta, \gamma \in \mathcal{I}\mathcal{F}_n^{par}$ such that $(\alpha, \beta) \in \rho$. If $\alpha, \beta, \gamma \in \Gamma$, then we have $(\gamma\alpha, \gamma\beta), (\alpha\gamma, \beta\gamma) \in \Gamma \times \Gamma \subseteq \rho$. Otherwise, we have $(\gamma\alpha, \gamma\beta), (\alpha\gamma, \beta\gamma) \in \bar{\Gamma} \times \bar{\Gamma} \subseteq \rho$. Hence ρ is a congruence. \square

For any $A \in \mathcal{A}$, we define

$$\Gamma_A = \{\alpha \in \mathcal{I}\mathcal{F}_n^{par} : A \subseteq \text{dom}(\alpha)\}.$$

We observe that $id \in \Gamma_A$. Clearly, for each $A \in \mathcal{A}$, the relation

$$\rho_A = (\Gamma_A \times \Gamma_A) \cup (\bar{\Gamma}_A \times \bar{\Gamma}_A)$$

with $\bar{\Gamma}_A = \mathcal{I}\mathcal{F}_n^{par} \setminus \Gamma_A$ is an equivalence relation on $\mathcal{I}\mathcal{F}_n^{par}$. We aim to show that ρ_A is a maximal congruence on $\mathcal{I}\mathcal{F}_n^{par}$.

Theorem 24. *Let $A \in \mathcal{A}$. Then ρ_A is a maximal congruence on $\mathcal{I}\mathcal{F}_n^{par}$.*

Proof. Clearly, ρ_A is an equivalence relation on $\mathcal{I}\mathcal{F}_n^{par}$. So, we will prove that ρ_A is a congruence on $\mathcal{I}\mathcal{F}_n^{par}$ by Lemma 23. Let $\alpha, \beta \in \Gamma_A$. Then $A \subseteq \text{dom}(\alpha)$ and $A \subseteq \text{dom}\beta$. By Corollary 21, we obtain $A\alpha = A$ and $A\beta = A$. Thus, $A\alpha\beta = A\beta = A$. This implies $A \subseteq \text{dom}(\alpha\beta)$ by Corollary 21. Hence $\alpha\beta \in \Gamma_A$. Conversely, let $\alpha\beta \in \Gamma_A$. Then $A \subseteq \text{dom}(\alpha\beta)$. Moreover, we obtain $A \subseteq \text{im}(\alpha\beta)$ by Corollary 21. So, we have $A \subseteq \text{dom}(\alpha\beta) \subseteq \text{dom}(\alpha)$ and $A \subseteq \text{im}(\alpha\beta) \subseteq \text{im}(\beta)$, i.e. $A \subseteq \text{dom}(\beta)$ by Corollary 21. Hence $\alpha, \beta \in \Gamma_A$. This shows that ρ_A is a congruence on $\mathcal{I}\mathcal{F}_n^{par}$, and since its index is two, we can conclude that ρ_A is a maximal congruence on $\mathcal{I}\mathcal{F}_n^{par}$. \square

Next, we will show that any maximal congruence ρ on $\mathcal{I}\mathcal{F}_n^{par}$ is of the form $\rho = \rho_A$, for a suitable set $A \in \mathcal{A}$.

Lemma 25. *Let ρ be a congruence on $\mathcal{I}\mathcal{F}_n^{par}$ and let $\alpha \in \mathcal{I}\mathcal{F}_n^{par}$. If $\alpha \in [id]_\rho$, then*

$$id|_{\text{dom}(\alpha)}, id|_{\text{dom}(\alpha^{-1})}, \alpha^{-1} \in [id]_\rho.$$

Proof. Let $\alpha \in [id]_\rho$, i.e. $(\alpha, id) \in \rho$. So, we obtain $(id|_{\text{dom}(\alpha)}, \alpha^{-1}) = (\alpha\alpha^{-1}, id\alpha^{-1}) \in \rho$ and $(id|_{\text{dom}(\alpha^{-1})}, \alpha^{-1}) = (\alpha^{-1}\alpha, \alpha^{-1}id) \in \rho$. Thus, $(id|_{\text{dom}(\alpha)}, id|_{\text{dom}(\alpha^{-1})}) \in \rho$ (by transitivity). Together with $(\alpha, id) \in \rho$, we obtain $(\alpha, id|_{\text{dom}(\alpha^{-1})}) = (id|_{\text{dom}(\alpha)}\alpha, id|_{\text{dom}(\alpha^{-1})}id) \in \rho$. Hence, we obtain $id|_{\text{dom}(\alpha)}, id|_{\text{dom}(\alpha^{-1})}, \alpha^{-1} \in [\alpha]_\rho = [id]_\rho$. \square

Lemma 26. *Let ρ be a non-universal congruence on $\mathcal{I}\mathcal{F}_n^{par}$. Then there exists $\beta \in [id]_\rho$ such that $\emptyset \neq \text{dom}(\beta) \subseteq \text{dom}(\alpha)$ for all $\alpha \in [id]_\rho$.*

Proof. Assume that for all $\beta \in [id]_\rho$, there is $\alpha \in [id]_\rho$ such that $\text{dom}(\beta) \not\subseteq \text{dom}(\alpha)$. It is clear that $\phi \notin [id]_\rho$ (see also the proof of Lemma 22(1)). There is $\alpha_2 \in [id]_\rho$ with $|\text{dom}(\alpha_2)| \leq |\text{dom}(\alpha)|$ for all $\alpha \in [id]_\rho$. We observe that $\text{dom}(\alpha_2) \neq \emptyset$ since $\alpha_2 \neq \phi$. Then there is $\alpha_1 \in [id]_\rho$ with $\text{dom}(\alpha_2) \not\subseteq \text{dom}(\alpha_1)$. Now, we have $(\alpha_1, id) \in \rho$, which gives $(id|_{\text{dom}(\alpha_2)}\alpha_1, id|_{\text{dom}(\alpha_2)}) \in \rho$. By Lemma 25, we obtain $id|_{\text{dom}(\alpha_2)} \in [id]_\rho$ since $\alpha_2 \in [id]_\rho$. Then $id|_{\text{dom}(\alpha_2)}\alpha_1 \in [id]_\rho$ with $\text{dom}(id|_{\text{dom}(\alpha_2)}\alpha_1) \subseteq \text{dom}(\alpha_2)$. Since $\text{dom}(\alpha_2) \not\subseteq \text{dom}(\alpha_1)$, we have $|\text{dom}(id|_{\text{dom}(\alpha_2)}\alpha_1)| < |\text{dom}(\alpha_2)|$, which is a contradiction with $|\text{dom}(\alpha_2)| \leq |\text{dom}(\alpha)|$ for all $\alpha \in [id]_\rho$. \square

Lemma 27. *Let ρ be a non-universal congruence on $\mathcal{I}\mathcal{F}_n^{par}$. Then there exists $\beta \in [id]_\rho$ with $dom(\beta) \in \mathcal{A}$ such that $dom(\beta) \subseteq dom(\alpha)$ for all $\alpha \in [id]_\rho$.*

Proof. By Lemma 26, there is $\beta \in [id]_\rho$ such that $\emptyset \neq dom(\beta) \subseteq dom(\alpha)$ for all $\alpha \in [id]_\rho$. Let $A = dom(\beta)$. We claim that $\beta \in \mathcal{A}$. Assume that there exist $x \in \{0, \dots, n-3\}$ and $k \in \{2, \dots, \lfloor \frac{n}{2} \rfloor - \lfloor \frac{x}{2} \rfloor\}$ with $x, x+2k \notin A$ such that $|\{x+r, x+2k-r\} \cap A| = 1$ for some $r \in \{1, \dots, k-1\}$. Then let

$$\gamma = \begin{pmatrix} 1 & \cdots & x-1 & x+1 & \cdots & x+2k-1 & x+2k+1 & \cdots & n \\ 1 & \cdots & x-1 & x+2k-1 & \cdots & x+1 & x+2k+1 & \cdots & n \end{pmatrix}$$

and it is easy to verify that $\gamma \in \mathcal{I}\mathcal{F}_n^{par}$. We know that $id|_{dom(\beta)} \in [id]_\rho$ by Lemma 25, i.e. $(id|_{dom(\beta)}, id) \in \rho$. It implies $(\gamma id|_{dom(\beta)}, \gamma) = (\gamma id|_{dom(\beta)}, \gamma id) \in \rho$. Thus $(\gamma id|_{dom(\beta)} \gamma id|_{dom(\beta)}, \gamma \gamma) \in \rho$. We put $\delta = \gamma id|_{dom(\beta)} \gamma id|_{dom(\beta)}$ for convenience. We can see that $dom(\beta) \not\subseteq dom(\delta)$ since $(x+r)\gamma = x+2k-r$, and $(x+2k-r)\gamma = x+r$ implies $\{x+r, x+2k-r\} \cap dom(id|_{dom(\beta)} \gamma id|_{dom(\beta)}) = \emptyset$. Note that $dom(\beta) \subseteq dom(\gamma)$ since $x, x+2k \notin dom(\beta)$ and $dom(\gamma) = \bar{n} \setminus \{x, x+2k\}$. Moreover, we have $\gamma^2 = \gamma \gamma = id|_{dom(\gamma)}$ and $(id|_{dom(\beta)}, id|_{dom(\gamma)}) = (id|_{dom(\beta)} id|_{dom(\gamma)}, id id|_{dom(\gamma)}) \in \rho$. This implies $\gamma^2 = id|_{dom(\gamma)} \in [id]_\rho$. Then $\delta \in [id]_\rho$ since $(\delta, \gamma^2) \in \rho$, which is a contradiction with $dom(\beta) \not\subseteq dom(\delta)$. \square

Now we can state the main result of the paper.

Theorem 28. *Let ρ be a non-universal congruence on $\mathcal{I}\mathcal{F}_n^{par}$. Then ρ is maximal if and only if there is a set $A \in \mathcal{A}$ such that $\rho = \rho_A$.*

Proof. Suppose that ρ is maximal. By Lemma 27, there is $\beta \in [id]_\rho$ with $dom(\beta) \in \mathcal{A}$ such that $dom(\beta) \subseteq dom(\alpha)$ for all $\alpha \in [id]_\rho$. Assume that there is $(\gamma, \delta) \in \rho$ with $dom(\beta) \subseteq dom(\gamma)$ and $dom(\beta) \not\subseteq dom(\delta)$, i.e. $\gamma \in \Gamma_{dom(\beta)}$ and $\delta \notin \Gamma_{dom(\beta)}$. Then $\beta = \gamma \gamma^{-1} \beta$ but $dom(\beta) \not\subseteq dom(\delta) \gamma^{-1} \beta$, i.e. $\delta \gamma^{-1} \beta \notin [id]_\rho$. Thus $(\gamma \gamma^{-1} \beta, \delta \gamma^{-1} \beta) = (\beta, \delta \gamma^{-1} \beta) \notin \rho$, a contradiction. Because of the symmetry of ρ , we can conclude that $\rho \subseteq (\Gamma_A \times \Gamma_A) \cup (\bar{\Gamma}_A \times \bar{\Gamma}_A) = \rho_A$ with $A = dom(\beta) \in \mathcal{A}$. By Theorem 24, ρ_A is a maximal congruence on $\mathcal{I}\mathcal{F}_n^{par}$ and since ρ is maximal, we get $\rho = \rho_A$. The converse direction is given by Theorem 24. \square

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