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RESEARCH ARTICLE

Nil-quasi-clean companion matrices

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Abstract. Let R be a ring with identity. An element e in R is called a quasi-idempotent element if $e^2 = ke$ for some central unit k in R. For an element b in R, if there is a positive integer m such that $b^m = 0$, then b is called a nilpotent element of R. An element r in R is called a nil-quasi-clean element if r is a sum of a quasi-idempotent and a nilpotent. If every element of R is nil-quasi-clean, then R is called a nil-quasi-clean ring. This paper completely determines nil-quasi-clean companion matrices over a field.

Introduction

Matrix ring is an important class of rings and it has many applications in operation theory and others. This paper concerns the square matrices over a field. Let us recall some definitions and notations in ring theory. All rings we consider in this paper are associative with identity. Let R be a ring. An element e in R is called an idempotent element if $e^2 = e$. For an element b in R, if there is a positive integer m such that $b^m = 0$, then b is called a nilpotent element of R. An element r in R is

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called a nil-clean element if r is a sum of an idempotent and a nilpotent. We use $Id(R)$, $Nil(R)$, $U(R)$, $NC(R)$ to denote the set of idempotents, the set of nilpotent elements, the set of units, and the set of nil-clean elements of R, respectively. If every element of R is nil-clean, that is, $NC(R) = R$, then R is called a nil-clean ring. As usual, $J(R)$, $C(R)$ denote the Jacobson radical and the center of a ring R, respectively. The set of central units of R is $U_c(R) = U(R) \cap C(R)$. We also use $char(R)$ to denote the characteristic of R and $R^* = R\backslash\{0\}$. Recently, Tang et al. introduced the definition of a quasi idempotent element in [10], which is a generalization of idempotent element. Recall that an element a of R is a quasi-idempotent if $a^2 = ka$ for some $k \in U_c(R)$. We use $QId(R)$ to denote the set of all quasi-idempotents in R. An element r in R is called a nil-quasi-clean element if $r = a + b$ with $a \in QId(R)$ and $b \in Nil(R)$. If every element of R is nil-quasi-clean, then R is called a nil-quasi-clean ring.

The concept of nil-clean ring first appeared in [7], Diesl showed that every nil-clean ring is clean. Many papers are devoted to the study of clean rings and nil-clean rings, especially for the cleanness and nil-cleanness of matrix ring, for examples, [1, 2, 6, 8]. Diesl in [7] posed an open question that whether the matrix ring over a nil-clean is nil-clean. Breaz et al. in [4] showed that the $n \times n$ matrix ring over a field F is nil-clean ring if and only if $F \cong F_2$. For a division ring D, Kosan et al. proved in [9] that $M_n(D)$ is nil-clean if and only if $D \cong F_2$. In [5], Călugăreanu proved that an invertible matrix in $M_2(\mathbb{Z})$ with trace 1 is nil-clean. Concerning the nil-quasi-cleanness, Tang and Zhou in [11] showed that the 2×2 matrix ring over a division ring D is quasi-nil-clean if and only if D is a perfect field with characteristic 2. They asked a question that when the matrix ring $M_n(R)$ over a division ring R is nil-quasi-clean. Motivated by [3], in which the authors studied the nil-cleanness of companion matrices, we study nil-quasi-cleanness of companion matrices in this paper. Hoping it can be helpful and inspiring for characterizing the nil-quasi-cleanness of general matrices.

1. Nil-quasi-clean companion matrices

In this section, we characterize the nil-quasi-cleanness of companion matrices over a field. Most results are the generalization of those in [3]. Let F be a field and n a positive integer. We denote

$$C_{c_0,c_1,\ldots,c_{n-1}} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & -c_{n-2} \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix}$$

the $n \times n$ companion matrix, and the characteristic polynomial of the companion matrix $C_{c_0,c_1,\ldots,c_{n-1}}$ is represented by

$$p(x) = X^n + c_{n-1}X^{n-1} + \cdots + c_1X + c_0.$$

As we all known, any matrix can be put into Frobenius normal form, that is, every square matrix A over a commutative ring is similar to a rational canonical matrix. Since a matrix similar to a nilpotent(or idempotent) matrix is still nilpotent(or idempotent), when we consider the nilpotency or idempotency, it just consider the above companion matrix. Note that a matrix similar to a quasi-idempotent matrix is also a quasi-idempotent matrix.

For easy understanding, we first consider the low order matrices. Firstly, consider a companion matrix of order 2.

Proposition 1. *A companion matrix* $C = \begin{pmatrix} 0 & -c_0 \\ 1 & -c_1 \end{pmatrix}$ *over a finite field* F *with* $char(F) > 2$ *is nil-quasi-clean if and only if one of following holds:*

(1) $c_0 = c_1 = 0$*;*

(2) $c_0 = u^2, c_1 = -2u$ *for some* $u \in F^*$*;*

(3) $c_1 = -u$ *for some* $u \in F^*$*.*

Proof. $\Leftarrow$ If $c_0 = c_1 = 0$, then C is clearly nilpotent and thus nil-quasi-clean. If $c_0 = u^2, c_1 = -2u$ for some $u \in F^*$, then we set $C = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} + \begin{pmatrix} -u & -u^2 \\ 1 & u \end{pmatrix} = E + B$, which is a nil-quasi-clean decomposition, as $E^2 = uE \in QId(M_2(F))$ and $B \in Nil(M_2(F))$. If $c_1 = -u$ for some $u \in F^*$, then we set $C = \begin{pmatrix} 0 & -c_0 \\ 1 & u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -c_0 \\ 0 & u \end{pmatrix} = B + E$, which is a nil-quasi-clean decomposition, as $E^2 = uE \in QId(M_2(F))$ and $B \in Nil(M_2(F))$.

$\Rightarrow$ We may assume that $C = E+B$ with $E^2 = uE$ and B is nilpotent. Then we have $-c_1 = trace(C) = trace(E) + trace(B) = trace(E) = u \cdot rank(E)$. If $rank(E) = 0$, then $E = \mathbf{0}$ and $-c_1 = 0$. In this case, C is nilpotent. So, we get $c_2 = 0$ and hence (1) holds. If $rank(E) = 1$, then we have $c_1 = -u$ and (3) happens. If $rank(E) = 2$, then $E = uI_2$. As C is nil-quasi-clean, then $c_1 = -2u$ and $c_0 = u^2$, which shows that (2) happens. $\square$

Next, we consider the companion matrix of order 3.

Proposition 2. *Let C_{c_0,c_1,c_2} be a companion matrix over a field F and $char(F) = p$. Then C is nil-quasi-clean if and only if one of following holds:*

(1) $c_2 \neq 0$;

(2) $c_2 = 0$ *and* $p = 2$;

(3) $c_2 = 0$, $p = 3$ *and* $C = aI_3 + B$ *with* B *nilpotent;*

(4) $c_2 = 0$, $p \geq 5$ *and* C *is nilpotent.*

Proof. $\Leftarrow$ For the case (1), we let $c_2 = -u$. Then we may decompose C into

$$C = \begin{pmatrix} -u & -u^2 + c_1 & -u^3 + 2c_1u \\ 1 & 0 & -c_1 \\ 0 & 1 & u \end{pmatrix} + \begin{pmatrix} u & u^2 - c_1 & -c_0 + u^3 - 2c_1u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$= B + E$. Note that $B^3 = 0$ and $E^2 = uE$. Thus C is nil-quasi-clean.

For the case (2), we have

$$C = \begin{pmatrix} 0 & 0 & -c_0 \\ 1 & 0 & -c_1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 & u^3 \\ 1 & u & u^2 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} u & 0 & -c_0 - u^3 \\ 0 & u & -c_1 - u^2 \\ 0 & 0 & 0 \end{pmatrix} = B + E.$$

As $B \in Nil(M_3(F))$ and $E^2 = uE \in QId(M_3(F))$, we know that C is nil-quasi-clean. The cases (3) and (4) are clearly nil-quasi-clean.

$\Rightarrow$ Suppose that $C = E + B$, where $E^2 = uE$ and B is nilpotent. Then we have $-c_2 = trace(C) = trace(E) = u{\cdot}rank(E)$. If $rank(E) \neq 0$, then $c_2 = -u \cdot rank(E)$, so (1) is correct. If $rank(E) = 0$, we proceed with the characteristic of the field. When $p = 2$, C with nil-quasi-clean decomposition, (2) is true. When $p = 3$, and C is nil-quasi-clean, we have $rank(E) = 0$ or $rank(E) = 3$. Hence $E = \mathbf{0}$ or $E = uI_3$ i.e., $C = aI_3 + B, a \in F$, so (3) is true. When $p \geqslant 5$ and C is nil-quasi-clean, we have $E = 0$. Hence $C \in Nil(M_3(F))$, so (4) is also true. □

Lemma 1 ([3, Lemma 1]). *Let* $f = X^n + f_{n-1}X^{n-1} + \cdots + f_1X + f_0 \in F[X]$ *be a monic polynomial. For every* $(c_1, \cdots, c_{n-1}) \in F^{n-1}$ *there exists a unique tuple* $(\alpha_0, \cdots, \alpha_{n-1}) \in F^n$ *such that the matrix*

$$M = \begin{pmatrix} -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -c_{n-2} \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \in M_n(F)$$

is similar to the companion matrix $C_{f_0,f_1,\ldots,f_{n-1}}$ *of* f.

Next result is a generalization of [3, Proposition 3].

Proposition 3. *Let* n, m, k *be three positive integers and* $n = m + k$. *Fix* $c_0, c_1, \ldots, c_{n-1} \in F$ *and companion matrix* $C_{c_0,c_1,\ldots,c_{n-1}}$. *For each polynomial* $g \in F[X]$ *with* $deg(g) \leqslant n-2$, *there are two matrixes* $E, M \in M_n(F)$ *such that* $C_{c_0,c_1,\ldots,c_{n-1}} = E+M$, *where* $E^2 = uE$ *for some* $u \in F^*$ *and the characteristic polynomial of* M *is* $X^n + (k\cdot u + c_{n-1})X^{n-1} + g$.

Proof. Let $g = f_{n-2}X^{n-2} + \cdots + f_1X + f_0 \in F[X]$. Consider the block matrix

$$E = \begin{pmatrix} uI_k & E_{12} \\ 0 & 0 \end{pmatrix} \in M_n(F),$$

where $$E_{12} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_0 - c_0 \\ 0 & 0 & \cdots & 0 & \alpha_1 - c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \alpha_{k-2} - c_{k-2} \\ \alpha_{n-2} & \alpha_{n-3} & \cdots & \alpha_k & \alpha_{k-1} - c_{k-1} \end{pmatrix} \in M_{k\times m}(F),$$ and $\alpha_0, \alpha_1, \cdots, \alpha_{n-2} \in F$.

By direct computation, we know that $E^2 = uE \in QId(M_n(F))$. So E is a quasi-idempotent matrix. To complete the proof, we show by induction on $k \geqslant 1$ that there are uniquely determined $\alpha_0, \alpha_1, \cdots, \alpha_{n-2}$ such that $M = C - E$ has the characteristic polynomial $f = X^n + (k\cdot u + c_{n-1})X^{n-1} + g$.

The step $k = 1$ is Lemma 1 (note that $f_{n-1} = \alpha_{n-1} + c_{n-1}$), let $\alpha_{n-1} = u$ and $f_{n-1} = u + c_{n-1}$. Suppose the claim is true for $k \geqslant 2$, and let $M = C - E$. Expanding by the first row we get

$$|XI_n - M| = \left|\begin{array}{cccc|cccc}
X+u & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha_0 \\
-1 & X+u & \cdots & 0 & 0 & \cdots & 0 & \alpha_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & X+u & \alpha_{n-2} & \cdots & \alpha_k & \alpha_{k-1} \\
\hline
 & & & -1 & X & \cdots & 0 & c_k \\
 & & & & -1 & \ddots & & \vdots \\
 & & & & & \ddots & X & c_{n-2} \\
 & & & & & & -1 & X+c_{n-1}
\end{array}\right|$$

$$= (X+u)\left|\begin{array}{cccc|cccc}
X+u & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha_1 \\
-1 & X+u & \cdots & 0 & 0 & \cdots & 0 & \alpha_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & X+u & \alpha_{n-2} & \cdots & \alpha_k & \alpha_{k-1} \\
\hline
 & & & -1 & X & \cdots & 0 & c_k \\
 & & & & -1 & \ddots & & \vdots \\
 & & & & & \ddots & X & c_{n-2} \\
 & & & & & & -1 & X+c_{n-1}
\end{array}\right| + \alpha_0.$$

It is easy to find that the coefficient of X^{n-1} in $|XI_n - M|$ is $k \cdot u + c_{n-1}$.

By division algorithm, we obtain $f = (X+u)q + \alpha_0$ and $|XI_n - M| = f$ if and only if

$$q = \left|\begin{array}{cccc|cccc}
X+u & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha_1 \\
-1 & X+u & \cdots & 0 & 0 & \cdots & 0 & \alpha_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & X+u & \alpha_{n-2} & \cdots & \alpha_k & \alpha_{k-1} \\
\hline
 & & & -1 & X & \cdots & 0 & c_k \\
 & & & & -1 & \ddots & & \vdots \\
 & & & & & \ddots & X & c_{n-1} \\
 & & & & & & -1 & X+c_{n-1}
\end{array}\right|.$$

Again, the coefficient of X^{n-2} in q is $(k-1) \cdot u + c_{n-1}$. We apply induction hypothesis so that determine (uniquely) $\alpha_0, \alpha_1, \cdots, \alpha_{n-2}$. This completes our proof. □

Lemma 2. *Let $A \in M_n(F)$ and $A = E + B$ be a nil-quasi-clean decomposition, where $E^2 = uE$ and $B^m = 0$ for some positive integer m. Then there exists a positive integer k such that $trace(A) = k \cdot u$. Moreover,*

1. *If $char(F) = 0$ and $trace(A) = k \cdot u$, then*
   - *(a) $k \leqslant n$;*
   - *(b) $k = 0$ if and only if A is nilpotent;*
   - *(c) $k = n$ if and only if $A = uI_n + B$, where B is nilpotent.*
2. *If $char(F) = p > 0$, then*
   - *(a) there exists $k \in \{1, 2, \cdots, p\}$ such that $trace(A) = k \cdot u$, and $k \leqslant n$ or $k = p$;*
   - *(b) if $n < k = p$, then A is nilpotent;*
   - *(c) if $k = n < p$, then $A = uI_n + B$, where B is nilpotent;*
   - *(d) if $k = n = p$, then A is nilpotent or $A = uI_n + B$, where B is nilpotent.*

Proof. Since $A = E + B$, we have $trace(A) = trace(E) + trace(B) = trace(E)$. Because $E^2 = uE$, we know that $trace(E) = u \cdot rank(E)$, there is $k \in \mathbb{N}$ such that $trace(A) = k \cdot u$. Note that k is unique if $char(F) = 0$ and $k \leqslant n$, so the statement (1)(a) is obvious. Note k is unique only modulo p if $char(F) = p$, hence (2)(a) is also hold.

(1)(b) If $k = 0$, then $rank(E) = 0$, so A is nilpotent. The converse is obvious.

(1)(c) If $k = n$, then $rank(E) = n$, so $A = uI_n + B$, where B is nilpotent. The converse is obvious.

(2)(b) If $k = p$, then $trace(A) = trace(E) = u \cdot rank(E) = 0$, hence $p \mid rank(E)$. We know $rank(E) \leqslant n < p$, so $rank(E) = 0$, i.e., $E = 0$, so A is nilpotent.

(2)(c) If $k = n$, then $rank(E) \equiv n \ (mod\, p)$. Since $rank(E) \leqslant n < p$, so $rank(E) = n$ and it follows that $A = uI_n + B$ with B is nilpotent.

(2)(d) As $A = E + B$, where $E^2 = uE \in QId(M_n(F))$ and $B \in Nil(M_n(F))$, we have $trace(A) = u \cdot rank(E) = 0$, so $rank(E) \in \{0, p\}$. This implies $E \in \{0, uI_n\}$. This completes the proof. ☐

Theorem 1. *Let $C = C_{c_0,c_1,\ldots,c_{n-1}}$ be a companion matrix over a field F. Then C is nil-quasi-clean if and only if one of the following conditions holds:*

(1) *C is nilpotent matrix;*

(2) *$C = uI_n + B$ with B is nilpotent matrix; (i.e. $c_i = (-u)^i\binom{n}{n-i}$ for all $i \in \{0, 1, \cdots, n-1\}$);*

(3) *$char(F) = 0$ and there exists a positive integer k such that $-c_{n-1} = k \cdot u$ and $n > k$;*

(4) *$char(F) = p$ and there exists $k \in \{1, 2, \cdots, p\}$ such that $-c_{n-1} = k\cdot u$ and $n > k$.*

Proof. $\Rightarrow$ Suppose that $C = E + B$ with $E^2 = uE \in QId(M_n(F))$ and $B \in Nil(M_n(F))$. We may assume that C is not a nilpotent and $C \neq uI_n + B$.

If $char(F) = 0$, by Lemma 2(1), we have that there exists a unique $k \leqslant n$ such that $-c_{n-1} = k\cdot u$. Since C is neither nilpotent nor $uI_n + B$, we have $0 < k < n$.

If $char(F) = p$, by Lemma 2(2), there is a unique $k \in \{1, 2, \cdots, p\}$ such that $-c_{n-1} = k \cdot u$. If $0 \neq E \neq uI_n$ is a quasi-idempotent such that $C - E$ is nilpotent matrix, then $rank(E) \equiv k \ (mod\ p)$. As $0 \neq rank(E) \neq n$, we have $k \leqslant rank(E) < n$ and hence $n > k$.

$\Leftarrow$ If C is nilpotent matrix or $C = uI_n + B$, then C is obviously nil-quasi-clean. If we are in one of the cases (3) or (4), we can apply Proposition 3 for $g = 0$ to obtain a nil-quasi-clean decomposition for C. □

Theorem 2. *Let $n \geqslant 3$ be an integer. The following statements are equivalent for a field F:*

(1) *every companion matrix $C \in M_n(F)$ is nil-quasi-clean;*

(2) *$char(F) = p < n$;*

(3) *if $C \in M_n(F)$ is a companion matrix then for every polynomial $g \in F[X]$ of degree at most $n-2$ there exist two matrices $E, M \in M_n(F)$ such that $C = E + M$, $E^2 = uE$ with $u \in F^*$ and $|XI_n - M| = X^n + g$.*

Proof. (2)$\Rightarrow$(3) For every companion matrix C, we have $trace(C) = -c_{n-1} = k\cdot u$ with $k \in \{1, 2, \cdots, p\}$ and we use Proposition 3 and the result follows.

(3)$\Rightarrow$(1) It is obvious.

(1)⇒(2) Since every element of the field F can be the trace of a companion matrix, by Lemma 2, every element from F has the form $k \cdot u$, $k \in \mathbb{N}$. This implies that there exists a prime p such that $char(F) = p$. Moreover, suppose $p \geqslant n$, then use Theorem 1 to observe that the companion matrix $C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & 0 & 0 \\ & & & & 1 & 0 \end{pmatrix}$ is not nil-quasi-clean. Since $trace(C) = 0$, so $p | rank(E)$ and $rank(E) \leqslant n \leqslant p$. Hence $rank(E) = 0$, which means C is nilpotent, contradiction. Or $rank(E) = p = n$, which means $E = uI_n$ and $C - E$ is nilpotent, $C - E = \begin{pmatrix} -u & 0 & 0 & \cdots & 0 & 0 \\ 1 & -u & 0 & \cdots & 0 & -1 \\ 0 & 1 & -u & \cdots & 0 & 0 \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & -u & 0 \\ & & & & 1 & -u \end{pmatrix}$ is not nilpotent matrix, a contradiction. Therefore, $p < n$, and the proof is complete. □

Finally, we illustrate above theorem by an example of companion matrix of order 4. Note that for $n = 4$, the characteristics of a field is only 2 or 3.

Example 1. First, for $n = 4$ and $char(F) = 2$:

(i) for $c_3 = 0$ we have

$$C = \begin{pmatrix} u & 0 & 0 & u^4 \\ 1 & u & c_2 - u^2 & uc_2 - u^3 \\ 0 & 1 & 0 & -c_2 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} -u & 0 & 0 & -u^4 - c_0 \\ 0 & -u & u^2 - c_2 & u^3 - c_1 - uc_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$= N + E$, where N is nilpotent and $E^2 = uE$ for some $u \in F^*$.

(ii) for $c_3 \neq 0$ we have

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -c_0 \\ 0 & 0 & 0 & -c_1 \\ 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & -c_3 \end{pmatrix} = N + E,$$

where $N \in Nil(M_4(F))$ and $E^2 = -c_3E \in QId(M_4(F))$.

Secondly, for $n = 4$ and $char(F) = 3$:

(i) for $c_3 = 0$ we have

$$C = \begin{pmatrix} u & 0 & 0 & -u^4 \\ 1 & u & 0 & -u^3 \\ 0 & 1 & u & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} -u & 0 & 0 & u^4 - c_0 \\ 0 & -u & 0 & u^3 - c_1 \\ 0 & 0 & -u & -c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} = N + E,$$

where N is nilpotent and $E^2 = -uE$ for some $u \in F^*$.

(ii) for $c_3 \neq 0$ we have

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -c_0 \\ 0 & 0 & 0 & -c_1 \\ 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & -c_3 \end{pmatrix} = N + E,$$

where $N \in Nil(M_4(F))$ is nilpotent and $E^2 = -c_3E \in QId(M_4(F))$.

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