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An application of the Zhou radical to the e-reversibility of rings

Burcu Ungor, Handan Kose, and Abdullah Harmanci

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Abstract. Let R be a ring and e be an idempotent element of R. The Zhou radical of R denoted by $\delta(R)$ is the intersection of maximal essential right ideals of R. In the literature, e-reversible rings were studied regarding the question of how idempotent elements affect the reversible property of rings. In this paper, we provide an application of the Zhou radical of a ring to the reversibility depending on idempotents. Accordingly, we study rings R in which $ab = 0$ implies $bae \in \delta(R)$ (or $eba \in \delta(R)$), where $a, b \in R$, called *Zhou right* (or *left*) *e-reversible rings.* Besides studying the structure of Zhou e-reversible rings, we investigate relations between Zhou e-reversible rings and some known rings, such as Zhou e-reduced rings, central reversible rings, NI-rings, semiperfect rings and matrix rings. In addition to these, we determine the Zhou radical of some certain rings, such as Morita context rings and special subrings of a direct product of rings.

Introduction

Throughout this paper, all rings are associative with identity. For a ring R, we use $N(R)$, $J(R)$, $U(R)$, $C(R)$ and $\mathrm{Id}(R)$ to denote the set of all nilpotent elements, the Jacobson radical, the set of all invertible elements, the center and the set of all idempotent elements of R, respectively. Also,

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$\delta(R)$ stands for the intersection of maximal essential right ideals of a ring R. The ideal $\delta(R)$ is defined in [20] and named as *Zhou radical* in [5]. The $n \times n$ full (resp., upper triangular) matrix ring over R is denoted by $M_n(R)$ (resp., $U_n(R)$), and $D_n(R)$ denotes the subring of $U_n(R)$ having all diagonal entries are equal, and $V_n(R) = \{(a_{ij}) \in D_n(R) \mid a_{ij} = a_{(i+1)(j+1)}$ for $i = 1, \dots, n-2$ and $j = 2, \dots, n-1\}$ is a subring of $D_n(R)$. The ring of integers and the ring of integers modulo n are denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively.

The notion of reduced ring and its various generalizations have been comprehensively studied in the literature. A ring is called *reduced* if it has no nonzero nilpotent elements. Reduced rings are extended to e-reduced rings in [14]. Let R be a ring and $e \in \mathrm{Id}(R)$. Then R is called *left* (or *right*) *e-reduced* if $eN(R) = 0$ (or $N(R)e = 0$). As a generalization of the notion of e-reduced ring, in [11], a ring R is said to be *Zhou right* (resp., *left*) *e-reduced* provided that $N(R)e \subseteq \delta(R)$ (resp., $eN(R) \subseteq \delta(R)$). A ring R is called *Zhou e-reduced* if it is both Zhou right e-reduced and Zhou left e-reduced.

Reversible rings, as a natural common generalization of commutative rings, integral domains and reduced rings, were studied by Cohn in [1]. There are many papers to investigate reversible rings and their generalizations. For instance, in [7], as a generalization of reversible rings, central reversible rings were investigated. A ring R is called *central reversible* if for any $a, b \in R$, having $ab = 0$ implies that ba is central in R. A version of reversibility depending on idempotents was studied in [9] as another generalization of reversible rings. In this direction, a ring R is said to be *right* (resp., *left*) *e-reversible* if for any $a, b \in R$, having $ab = 0$ implies $bae = 0$ (resp., $eba = 0$). The ring R is called *e-reversible* if it is both left and right e-reversible. In [10], e-reversibility of rings was discussed from the perspective of quasinilpotents as a generalization of e-reversible rings.

In ring theory, the Zhou radical, idempotent elements, the reversibility and related notions have important roles and generated wide interest. With this motivation, in this paper, we relate these concepts and give an application of the Zhou radical to the e-reversibility of rings. In this perspective, we consider "Zhou e-reversibility". The contents of the paper is as follows: In Section 2, we deal with the Zhou radical of certain rings to use in the sequel of the paper. Within this scope, we determine the Zhou radicals of the Morita context and the ring $R[A, B]$ which is a subring of a direct product of copies of A, where A is a ring and B is a subring of A. In Section 3, we concentrate on the Zhou e-reversible rings, and exhibit some sources for Zhou e-reversible rings. Also, some results related to the structure of Zhou right e-reversible rings are observed. In Section 4, we focus on some ring extensions in terms of the Zhou e-reversibility. Finally, in Section 5, certain matrix rings are investigated related to the Zhou e-reversible property.

1. Notes on the Zhou radical

As a generalization of small submodules, δ-small submodules introduced by Zhou in [20] to study δ-semiperfect modules. Let M be a module and N a submodule of M. Then N is called *δ-small* in M if whenever $M = N + L$ and M/L is singular, then $M = L$. The sum of δ-small submodules is denoted by $\delta(M)$. Considering the ring R as a right R-module over itself, the ideal $\delta(R)$ is introduced as the sum of δ-small right ideals of R. We begin with some properties of $\delta(R)$ that will be used in the sequel.

Let S_r denote the right socle of the ring R, that is, S_r is the sum of minimal right ideals of R. Then $J(R/S_r) = \delta(R)/S_r$ by [20, Corollary 1.7]. It is clear by definitions that $J(R) \subseteq \delta(R)$. The next example shows that this inclusion is strict. Also, in case $N(R)$ is an ideal of R, we have $N(R) \subseteq \delta(R)$.

Examples 1.1. (1) Consider the ring $R = U_2(\mathbb{Z}_2)$. Then

$$J(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \delta(R) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}.$$

This shows $J(R) \subsetneqq \delta(R)$.
(2) Let $R = M_2(\mathbb{Z})$. Then $J(R) = \delta(R) = 0$.

The following properties relating to the Zhou radical are obvious.

Lemma 1.2. Let R be a ring and n a positive integer.

(1) Let I_i denote the i^{th}-row of $U_n(R)$. Then $\delta(U_n(R)) = \sum_{i=1}^{n} \delta(I_i)$.

(2) $\delta(D_n(R)) = \{(a_{ij}) \in D_n(R) \mid a_{ii} \in \delta(R)\}$.

(3) $\delta(V_n(R)) = \{(a_{ij}) \in V_n(R) \mid a_{ii} \in \delta(R)\}$.

(4) $\delta(M_n(R)) = M_n(\delta(R))$.

A *Morita context* is a 6-tuple $\mathcal{M} = (R, V, W, S, \phi, \psi)$, where R, S are rings, ${}_RV_S$ and ${}_SW_R$ are bimodules with context products $\phi\colon V \times W \to R$ and $\psi\colon W \times V \to S$ written multiplicatively as $(v, w) \mapsto vw$ and $(w, v) \mapsto wv$ such that $\mathcal{T}(\mathcal{M}) = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$ is an associative ring with the obvious matrix operations. The ring $\mathcal{T}(\mathcal{M})$ is *the Morita context ring associated with* $\mathcal{M}$. The ring $\mathcal{T}(\mathcal{M})$ is called *trivial* if the context products are trivial, i.e., $VW = 0$ and $WV = 0$ (see for detail [16] and [19]). This is also called *null context*. In [18], maximal ideals of $\mathcal{T}(\mathcal{M})$ are determined as follows.

Lemma 1.3 ([18]). *Let $\mathcal{M} = (R, V, W, S)$ be a Morita context and $\mathcal{T}(\mathcal{M}) = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$ be the Morita context ring. Then the following hold.*

(1) *Let I be a maximal right ideal of R and $V_I = \{v \in V \mid vW \subseteq I\}$ and $M_I = \begin{bmatrix} I & V_I \\ W & S \end{bmatrix}$. Then V_I is a right S-submodule of V and M_I is a maximal right ideal of $\mathcal{T}(\mathcal{M})$.*

(2) *Let J be a maximal right ideal of S and $W_J = \{w \in W \mid wV \subseteq J\}$ and $M_J = \begin{bmatrix} R & V \\ W_J & J \end{bmatrix}$. Then W_J is a right R-submodule of W and M_J is a maximal right ideal of $\mathcal{T}(\mathcal{M})$.*

(3) *Let K be a maximal right ideal of $\mathcal{T}(\mathcal{M})$ which is a different type from ones in* (1) *and* (2)*. Then $\begin{bmatrix} J(R) & V_0 \\ W_0 & J(S) \end{bmatrix} \subseteq K$, where $V_0 = \{v \in V \mid vW \subseteq J(R)\}$ and $W_0 = \{w \in W \mid wV \subseteq J(S)\}$.*

In the following, we investigate the Zhou radical of Morita context rings. First, we start with the essential maximal right ideals of Morita context rings.

Lemma 1.4. *Let $\mathcal{M} = (R, V, W, S)$ be a Morita context with the Morita context ring $\mathcal{T}(\mathcal{M})$. Let I be an essential maximal right ideal of R and J be an essential maximal right ideal of S. Then the following hold.*

(1) *Let $V_I = \{v \in V \mid vW \subseteq I\}$ and $M_I = \begin{bmatrix} I & V_I \\ W & S \end{bmatrix}$. Then V_I is a right S-submodule of V and M_I is an essential maximal right ideal of $\mathcal{T}(\mathcal{M})$.*

(2) *Let* $W_J = \{w \in W \mid wV \subseteq J\}$ *and* $M_J = \begin{bmatrix} R & V \\ W_J & J \end{bmatrix}$. *Then* W_J *is a right* R*-submodule of* W *and* M_J *is an essential maximal right ideal of* $\mathcal{T}(\mathcal{M})$.

Proof. The maximality of M_I and M_J are known by Lemma 1.3. As for M_I being an essential right ideal of $\mathcal{T}(\mathcal{M})$, we assume otherwise. So there exists a right ideal K of $\mathcal{T}(M)$ such that $M_I \oplus K = \mathcal{T}(M)$. Multiplying from both sides by $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we have $eM_Ie \oplus eKe = e\mathcal{T}(M)e$. Hence $I \oplus eKe = R$. It entails that I is not essential. This is a contradiction. Thus M_I is an essential right ideal of $\mathcal{T}(\mathcal{M})$. On the other hand, M_J being an essential right ideal of $\mathcal{T}(\mathcal{M})$ is treated similarly. □

Theorem 1.5. *Let* $\mathcal{M} = (R, V, W, S)$ *be a Morita context with the Morita context ring* $\mathcal{T}(\mathcal{M})$. *Then* $\delta(\mathcal{T}(\mathcal{M})) \subseteq \begin{bmatrix} \delta(R) & V_1 \\ W_1 & \delta(S) \end{bmatrix}$, *where* $V_1 = \{v \in V \mid vW \subseteq \delta(R)\}$ *and* $W_1 = \{w \in W \mid wV \subseteq \delta(S)\}$.

Proof. Since $\delta(\mathcal{T}(\mathcal{M}))$ is an ideal of $\mathcal{T}(\mathcal{M})$, by [19, Lemma 2.1(3)], $\delta(\mathcal{T}(\mathcal{M})) = \begin{bmatrix} A & K \\ L & B \end{bmatrix}$, where A is an ideal of R, B is an ideal of S, K is a submodule of ${}_RV_S$ and L is a submodule of ${}_SW_R$ with $KW \subseteq A$, $WK \subseteq B$, $LV \subseteq B$, $VL \subseteq A$, $AV \subseteq K$, $WA \subseteq L$, $BW \subseteq L$ and $VB \subseteq K$. We claim that $A \subseteq \delta(R)$ and $B \subseteq \delta(S)$. Assume contrary that $A \not\subseteq \delta(R)$. So there exists an essential maximal right ideal I of R such that $A \not\subseteq I$. Consider the right ideal $M_I = \begin{bmatrix} I & V_I \\ W & S \end{bmatrix}$ of $\mathcal{T}(\mathcal{M})$, where $V_I = \{v \in V \mid vW \subseteq I\}$. By Lemma 1.4(1), M_I is an essential maximal right ideal of $\mathcal{T}(\mathcal{M})$. It entails $\delta(\mathcal{T}(\mathcal{M})) \subseteq M_I$, and so $A \subseteq I$, a contradiction. Hence $A \subseteq \delta(R)$. By a similar discussion, we obtain $B \subseteq \delta(S)$. Consider the sets $V_1 = \{v \in V \mid vW \subseteq \delta(R)\}$ and $W_1 = \{w \in W \mid wV \subseteq \delta(S)\}$. Having $KW \subseteq A$ and $LV \subseteq B$ yield $K \subseteq V_1$ and $L \subseteq W_1$, respectively. It follows

$$\delta(\mathcal{T}(\mathcal{M})) = \begin{bmatrix} A & K \\ L & B \end{bmatrix} \subseteq \begin{bmatrix} \delta(R) & V_1 \\ W_1 & \delta(S) \end{bmatrix}.$$

□

Theorem 1.6. *Let* $\mathcal{M} = (R, V, W, S)$ *be a trivial Morita context with the Morita context ring* $\mathcal{T}(\mathcal{M})$. *Then* $\delta(\mathcal{T}(M)) = \begin{bmatrix} \delta(R) & V \\ W & \delta(S) \end{bmatrix}$.

Proof. In a trivial Morita context, $V_1 = V$ and $W_1 = W$, and so $\delta(\mathcal{T}(M)) \subseteq \begin{bmatrix} \delta(R) & V \\ W & \delta(S) \end{bmatrix}$ by Theorem 1.5. For the reverse inclusion, consider the right ideals $X = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ W & 0 \end{bmatrix}$ of $\mathcal{T}(\mathcal{M})$. Note that X and Y are nilpotent right ideals. It follows that $X, Y \subseteq \delta(\mathcal{T}(\mathcal{M}))$. Hence $X + Y \subseteq \delta(\mathcal{T}(\mathcal{M}))$. By [20, Theorem 1.6(2) and Lemma 1.3(1)], $X+Y$ is δ-small in $\mathcal{T}(\mathcal{M})$. By the fact that $\delta(R)$ and $\delta(S)$ are δ-small in R and S, respectively, $\begin{bmatrix} \delta(R) & V \\ W & \delta(S) \end{bmatrix}/(X+Y)$ is δ-small in $\begin{bmatrix} R & V \\ W & S \end{bmatrix}/(X+Y)$, and so $\begin{bmatrix} \delta(R) & V \\ W & \delta(S) \end{bmatrix}$ is δ-small in $\begin{bmatrix} R & V \\ W & S \end{bmatrix}$ by [20, Lemma 1.3(1)]. Thus $\begin{bmatrix} \delta(R) & V \\ W & \delta(S) \end{bmatrix} \subseteq \delta(\mathcal{T}(\mathcal{M}))$. Therefore $\delta(\mathcal{T}(M)) = \begin{bmatrix} \delta(R) & V \\ W & \delta(S) \end{bmatrix}$. □

We illustrate Theorem 1.6 by the following example.

Example 1.7. Let $\mathcal{M}$ denote the Morita context $\mathcal{M} = (\mathbb{Z}_6, \bar{2}\mathbb{Z}_6, \bar{3}\mathbb{Z}_6, \mathbb{Z}_6)$ and $\mathcal{T}(\mathcal{M})$ be a Morita context ring $\mathcal{T}(\mathcal{M}) = \begin{bmatrix} \mathbb{Z}_6 & \bar{2}\mathbb{Z}_6 \\ \bar{3}\mathbb{Z}_6 & \mathbb{Z}_6 \end{bmatrix}$. Since $\mathcal{M}$ is a trivial Morita context, we have $\delta(\mathcal{T}(\mathcal{M})) = \begin{bmatrix} \delta(\mathbb{Z}_6) & \bar{2}\mathbb{Z}_6 \\ \bar{3}\mathbb{Z}_6 & \delta(\mathbb{Z}_6) \end{bmatrix}$ by Theorem 1.6. The ring $\mathbb{Z}_6$ being semisimple entails that $\delta(\mathbb{Z}_6) = \mathbb{Z}_6$. It follows $\delta(\mathcal{T}(\mathcal{M})) = \mathcal{T}(\mathcal{M})$.

In the next result, we investigate under what conditions the reverse inclusion in Theorem 1.5 holds.

Theorem 1.8. *Let $\mathcal{M} = (R, V, W, S)$ be a Morita context with the Morita context ring $\mathcal{T}(\mathcal{M})$. If $\delta(R)$ and $\delta(S)$ are nil ideals of R and S, respectively, then $\delta(\mathcal{T}(\mathcal{M})) = \begin{bmatrix} \delta(R) & V_1 \\ W_1 & \delta(S) \end{bmatrix}$, where $V_1 = \{v \in V \mid vW \subseteq \delta(R)\}$ and $W_1 = \{w \in W \mid wV \subseteq \delta(S)\}$.*

Proof. By Theorem 1.5, $\delta(\mathcal{T}(\mathcal{M})) \subseteq \begin{bmatrix} \delta(R) & V_1 \\ W_1 & \delta(S) \end{bmatrix}$. For the reverse inclusion, consider the sets $X = \begin{bmatrix} \delta(R) & V_1 \\ 0 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 0 \\ W_1 & \delta(S) \end{bmatrix}$. On the one hand, X and Y are right ideals of $\mathcal{T}(\mathcal{M})$. On the other hand, since $\delta(R)$ and $\delta(S)$ are nil ideals, X and Y are also nil. It follows that $X, Y \subseteq \delta(\mathcal{T}(\mathcal{M}))$. Hence $X + Y \subseteq \delta(\mathcal{T}(\mathcal{M}))$, that is, $\begin{bmatrix} \delta(R) & V_1 \\ W_1 & \delta(S) \end{bmatrix} \subseteq \delta(\mathcal{T}(\mathcal{M}))$. This completes the proof. □

Lemma 1.9. *Let $\mathcal{M} = (R, V, W, S)$ be a Morita context with the Morita context ring $\mathcal{T}(M)$ and $A = \begin{bmatrix} r & v \\ w & s \end{bmatrix}$. Then the following hold.*

(1) *If $A \in Id(\mathcal{T}(M))$, then $r \in Id(R)$ and $s \in Id(S)$.*

(2) *If $A \in N(\mathcal{T}(M))$, then $r \in N(R)$ and $s \in N(S)$.*

Proof. Clear by definitions. □

In [2], the Dorroh extension of a ring R was introduced by Dorroh as a way to embed a ring R without an identity into a ring with an identity $\mathbb{Z} \oplus R$, and it is one of the important methods of constructing new rings and analyzing some properties of rings. Let R be a ring and S be an associative ring that may not possess an identity element and an (R, R)-bimodule obeying multiplication in S, that is, for any $a \in R$ and $s, t \in S$, $a(ts) = (at)s$, $t(as) = (ta)s$ and $(ts)a = t(sa)$. The *Dorroh extension* (in other words, *ideal extension*) of S by R, denoted by $D(R, S)$, is the abelian group $R \times S$ with multiplication defined by $(a_1, t_1)(a_2, t_2) = (a_1a_2, a_1t_2 + t_1a_2 + t_1t_2)$ for a_1, $a_2 \in R$ and t_1, $t_2 \in S$. Then $(1, 0)$ is the identity of $D(R, S)$. Maximal ideals and right (or left) ideals of Dorroh extensions were characterized by Mesyan in [15, Proposition 5]. The Zhou radical $\delta(D(R, S))$, idempotents and nilpotents of $D(R, S)$ are characterized in [11, Lemma 2.8] as the following.

Lemma 1.10. *Let S be an algebra over a ring R and consider the Dorroh extension $D(R, S)$ of S by R. Let $(r, s) \in D(R, S)$. Then we have the following.*

(1) *$\delta(D(R, S)) = \delta(R) \oplus S$.*

(2) *$(r, s) \in Id(D(R, S))$ if and only if $r \in Id(R)$ and $(r + s)^2 = r + s$.*

(3) *$(r, s) \in N(D(R, S))$ with $(r, s)^n = 0$ if and only if $r^n = 0$ and $(r + s)^n = 0$.*

Let A be a ring and B a subring of A and $R[A, B]$ denote the set

$$R[A, B] = \{(a_1, a_2, \ldots, a_n, b, b, \ldots) \mid a_i \in A, b \in B, 1 \leq i \leq n, n \in \mathbb{Z}^+\}.$$

Then $R[A,B]$ is a ring under the componentwise addition and multiplication. In the following, we determine the right socle and the Zhou radical of this ring.

Theorem 1.11. *Let A be a ring and B a subring of A. Then*

$$Soc(R[A,B]) = R[Soc(A),0].$$

Proof. Let I be a minimal right ideal of A. We claim that $R[I,0]$ is a minimal right ideal of $R[A,B]$. It is clear that $R[I,0]$ is a right ideal of $R[A,B]$. For the minimality, let Y be a right ideal of $R[A,B]$ with $Y \subseteq R[I,0]$. Set

$$I_Y = \{a \in A \mid \text{there exists } (a_1, a_2, \dots, a_n, 0, 0, \dots) \in Y, \\ \text{for some } i, a = a_i\}.$$

Let $< I_Y >$ denote the right ideal generated by I_Y. Then $< I_Y > \subseteq I$ and $R[< I_Y >, 0] \subseteq Y \subseteq R[I,0]$. Since I is minimal, $I =< I_Y >$. Thus $Y = R[I,0]$, i.e., $R[I,0]$ is minimal. So $R[\mathrm{Soc}(A),0] \subseteq \mathrm{Soc}(R[A,B])$.

Let X be a minimal right ideal of $R[A,B]$ and

$$x = (a_1, a_2, \dots, a_n, b, b, \dots) \in X.$$

Multiplying x from the right by $t = (\underbrace{1,1,\dots,1}_{n \text{ times}}, 0, 0, \dots) \in R[A,B]$, we get $xt = (a_1, a_2, \dots, a_n, 0, 0, \dots) \in X$. Set

$$S = \{(a_1, a_2, \dots, a_n, 0, 0, \dots) \in R[A,B] \mid \text{there exists} \\ (a_1, a_2, \dots, a_n, b, b, \dots) \in X\}.$$

Then $S \subseteq X \subseteq R[A,B]$. Since S is a right ideal of $R[A,B]$ and X is simple, $S = X$. We now say that $\mathrm{Soc}(R[A,B]) \subseteq R[A,0]$.

Let I_S denote the right ideal of A generated by the entries of the elements of S. We may conclude that $R[I_s,0] = S$ since S is minimal. Next we claim that I_S is a minimal right ideal of A. Let J be a right ideal of A with $J \subseteq I_s$. Then $R[J,0] \subseteq R[I_s,0]$. Since $R[J,0]$ is a right ideal and $R[I_s,0]$ is a minimal right ideal, $R[J,0] = R[I_s,0]$. Thus $J = I_S$. It entails that $\mathrm{Soc}(R[A,B]) \subseteq R[\mathrm{Soc}(A),0]$. Therefore $\mathrm{Soc}(R[A,B]) = R[\mathrm{Soc}(A),0]$. □

Theorem 1.12. *Let A be a ring and B a subring of A. Then the following hold:*

(1) $J(R[A,B]) = R[J(A), J(A) \cap J(B)]$;

(2) $U(R[A,B]) = R[U(A), U(B)]$;

(3) $\delta(R[A,B]) = R[\delta(A), \delta(A) \cap \delta(B)]$.

Proof. (1) It is proved in [4, Lemma 3.11].
(2) Let $X = (a_1, a_2, \dots, a_n, b, b, \dots) \in U(R[A,B])$, and let $Y = (y_1, y_2, \dots, y_n, x, x, \dots) \in U(R[A,B])$ be with $XY = YX = (1,1,1,\dots)$. Then $a_i y_i = y_i a_i = 1$ $(i = 1,2,3,\dots,n)$, $bx = xb = 1$. Hence $a_i \in U(A)$, $b \in U(B)$, and so $X \in R[U(A), U(B)]$. Let $X = (a_1, a_2, \dots, a_n, b, b, \dots) \in R[U(A), U(B)])$. Then $a_i^{-1} \in U(A)$ $(i = 1,2,3,\dots,n)$, $b^{-1} \in U(B)$. Write $X^{-1} = (a_1^{-1}, a_2^{-1}, \dots, a_n^{-1}, b^{-1}, b^{-1}, b^{-1}, \dots) \in R[A,B]$. Then $XX^{-1} = X^{-1}X = 1 \in R[A,B]$. Hence $X \in U(R[A,B])$.
(3) It is known that

$$J(R[A,B]/\mathrm{Soc}(R[A,B])) = \delta(R[A,B])/\mathrm{Soc}(R[A,B]).$$

By (1) and Theorem 1.11, we have

$$\begin{aligned}
&J(R[A,B]/Soc(R[A,B])) \\
&\quad = J(R[A,B]/R[Soc(A),0]) \\
&\quad = J(R[A/Soc(A),(B+Soc(A))/Soc(A)]) \\
&\quad = R[J(A/Soc(A)), J(A/Soc(A)) \cap J((B+Soc(A))/Soc(A))] \\
&\quad = R[\delta(A)/Soc(A), (\delta(A)/Soc(A)) \cap (\delta(B+Soc(A))/Soc(A))] \\
&\quad = R[\delta(A)/Soc(A), (\delta(A)/Soc(A)) \cap ((\delta(B)+Soc(A))/Soc(A))] \\
&\quad = R[\delta(A)/Soc(A), (\delta(A) \cap (\delta(B)+Soc(A)))/Soc(A)] \\
&\quad = R[\delta(A)/Soc(A), ((\delta(A) \cap \delta(B))+Soc(A))/Soc(A)].
\end{aligned}$$

It follows that $\delta(R[A,B]) = R[\delta(A), \delta(A) \cap \delta(B)]$. □

2. Application of the Zhou radical to the e-reversibility

In this section, we present an application of the Zhou radical to the e-reversibility of rings. In this direction, we give our main definition as follows.

Definition 2.1. Let R be a ring, $e \in \mathrm{Id}(R)$ and $a \in R$. Then a is called *Zhou right e-reversible* if whenever $ab = 0$ for any $b \in R$, $bae \in \delta(R)$. The ring R is called *Zhou right e-reversible* if every element of R is Zhou right e-reversible. Zhou left e-reversible rings are defined similarly, i.e., having $ab = 0$ implies $eba \in \delta(R)$. The ring R is said to be *Zhou e-reversible* if every element of R is both Zhou right e-reversible and Zhou left e-reversible.

In the following, we study the Zhou e-reversibility as an element-wise notion, and present an example to exhibit that the Zhou e-reversibility of elements in rings is not left-right symmetric. In all of the cases in the context, however, we discuss Zhou e-reversibility only as a right side condition.

Example 2.2. Consider the ring $R = M_2(\mathbb{Z})$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \in R$ and $E = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in \mathrm{Id}(R)$. Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R$ with $AB = 0$. Then $c = d = 0$, and so $BAE = 0 \in \delta(R)$. Hence A is Zhou right E-reversible. On the other hand, let $C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \in R$. Thus $AC = 0$, but $ECA = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \notin \delta(R)$. Therefore A is not Zhou left E-reversible.

It is clear that every ring is Zhou 0-reversible. Also, a ring R is Zhou right (equivalently, left) 1-reversible, then R is Zhou e-reversible for every $e \in \mathrm{Id}(R)$. In the sequel, we assume that $e \in \mathrm{Id}(R)\backslash\{0\}$. Obviously, every right e-reversible ring is Zhou right e-reversible. Also, if $N(R) \subseteq \delta(R)$ for a ring R, then R is Zhou e-reversible for every $e \in \mathrm{Id}(R)$. We illustrate the notion of Zhou e-reversible rings in the following.

Example 2.3. Let $R = A/I$ denote the ring in [11, Example 2.4] defined by the ring $A = \mathbb{Z}_2 < a, b >$ be the free algebra with noncommuting indeterminates a, b over $\mathbb{Z}_2$, and I stand for the ideal generated by aAb, $a^2 - a$ and $b^2 - b$. We identify the elements in A with their images in R for simplicity. Then

$$R = \{0, 1, a, b, ba, a + b, a + ba, b + ba, a + b + ba, 1 + a, 1 + b, 1 + ba, 1 + a + b, 1 + a + ba, 1 + b + ba, 1 + a + b + ba\}$$

and $aR = \{0, a\}$, $(ba)R = \{0, ba\}$, $(1+a+b+ba)R = \{0, 1+a+b+ba\}$ are minimal right ideals of R. Hence $\mathrm{Soc}(R_R) = aR \oplus (ba)R \oplus (1+a+b+ba)R$ and $\delta(R) = \{0, a, ba, a + ba, 1 + b, 1 + b + ba, 1 + b + a, 1 + b + a + ba\}$. Also, $\mathrm{Id}(R) = \{0, 1, a, b, 1 + a, 1 + b, b + ba, 1 + a + ba, a + b + ba, 1 + a + b + ba, a + ba, 1 + b + ba\}$ and $N(R) = \{0, ba\}$. Thus having $N(R) \subseteq \delta(R)$ yields that R is Zhou e-reversible for every $e \in \mathrm{Id}(R)$. On the other hand, $\delta(U_2(R)) = \begin{bmatrix} \delta(R) & R \\ 0 & \delta(R) \end{bmatrix}$. Since $N(U_2(R)) \subseteq \delta(U_2(R))$, $U_2(R)$ is Zhou E-reversible for each $E \in \mathrm{Id}(U_2(R))$.

In [8], a ring R is called *right* (resp., *left*) *e-semicommutative* if for any $a, b \in R$, having $ab = 0$ implies $aRbe = 0$ (resp., $eaRb = 0$).

The ring R is called *e-semicommutative* in case R is both right and left e-semicommutative.

Examples 2.4. The following are some sources for Zhou e-reversible rings.

(1) Every semisimple ring is Zhou e-reversible.

(2) Every local ring is Zhou e-reversible.

(3) Every Zhou e-reduced ring is Zhou e-reversible.

(4) Every e-semicommutative ring is Zhou e-reversible.

(5) Every NI-ring is Zhou e-reversible.

(6) Every central reversible ring is Zhou e-reversible.

Proof. We only prove the statements for the right case, a similar proof applies to the left case as well.
(1) Let R be a semisimple ring. Then $\delta(R) = R$ and so R is Zhou e-reversible for each $e \in \mathrm{Id}(R)$.
(2) Let R be a local ring and $a, b \in R$ with $ab = 0$. Then $a \in J(R)$ or $b \in J(R)$. Hence $bae \in J(R) \subseteq \delta(R)$ for each $e \in \mathrm{Id}(R)$ in either case.
(3) Let R be a Zhou e-reduced ring with $e \in \mathrm{Id}(R)$. For any $a, b \in R$ such that $ab = 0$, we have $ba \in N(R)$. Therefore $bae \in \delta(R)$.
(4) Assume that R is an e-semicommutative ring with $e \in \mathrm{Id}(R)$. Let $a, b \in R$ with $ab = 0$. Then $(ba)(ba) = 0$. Hence $baRbae = 0$. Having $baeR \subseteq baR$ yields $(bae)R(bae) = 0$. Thus $R(bae)$ is a nilpotent left ideal of R. It entails that $R(bae) \subseteq \delta(R)$ or $bae \in \delta(R)$.
(5) Assume that R is an NI-ring, i.e., $N(R)$ is an ideal of R. Then $N(R) \subseteq J(R) \subseteq \delta(R)$. Hence R is Zhou e-reversible for each $e \in \mathrm{Id}(R)$.
(6) It is a consequence of [7, Theorem 2.19] and (6). □

Recall that a ring R is called *right (quasi-)duo* if every (maximal) right ideal of R is two-sided. The next result gives another source for Zhou e-reversible rings.

Theorem 2.5. *Every right quasi-duo ring is Zhou right e-reversible for every idempotent e.*

Proof. Let R be a right quasi-duo ring and $a, b \in R$ such that $ab = 0$. Assume that $bae \notin \delta(R)$ and we get a contradiction. There exists an essential maximal right ideal I of R such that $bae \notin I$. By hypothesis,

I is an ideal and $baeR + I = R$. There exist $r \in R$ and $s \in I$ such that $baer + s = 1$. Multiplying the latter from the left by a, we get $as = a \in I$. Since I is an ideal, $bae \in I$. This is a contradiction. Thus $bae \in \delta(R)$. Therefore R is Zhou right e-reversible. □

In the sequel of this section, we observe some results related to the structure of Zhou right e-reversible rings. In a ring R, any $e \in \mathrm{Id}(R)$ is called *right* (resp., *left) semicentral* if $er = ere$ (resp., $re = ere$) for all $r \in R$.

Proposition 2.6. *If R is a Zhou right e-reversible ring, then $\overline{e} = e + \delta(R)$ is a left semicentral idempotent in $R/\delta(R)$. The converse holds if $\overline{e}(R/\delta(R))\overline{e}$ is reversible.*

Proof. Since R is Zhou right e-reversible, having $e(1-e) = 0$ yields $(1-e)Re \subseteq \delta(R)$. Therefore $re - ere \in \delta(R)$ for each $r \in R$, i.e., $\overline{e} \in R/\delta(R)$ is left semicentral. For the converse statement, assume that $\overline{e}(R/\delta(R))\overline{e}$ is reversible. Then $R/\delta(R)$ is right $\overline{e}$-reversible by [9, Proposition 2.9]. If $a, b \in R$ such that $ab = 0$, then having $\overline{a}\overline{b} = 0$ yields $\overline{b}\overline{a}\overline{e} = 0$. This implies that $bae \in \delta(R)$. □

We now give a characterization of Zhou right e-reversibility in terms of nilpotents with nilpotency index 2.

Theorem 2.7. *The following are equivalent for a ring R.*

(1) *R is Zhou right e-reversible.*

(2) *For any $a \in R$, if whenever $a^2 = 0$, then $ae \in \delta(R)$.*

Proof. (1) ⇒ (2) Let $a \in R$ with $a^2 = 0$. Assume that $ae \notin \delta(R)$ and we reach a contradiction. Let M be an essential maximal right ideal of M such that $ae \notin M$. Then $a \notin M$. Hence there exist $r \in R$ and $m \in M$ such that $ar + m = 1$. Multiplying the latter equality from the left by a, we get $a(1-m) = 0$. By (1), $(1-m)ae \in \delta(R)$. Since $(1-m)ae \in M$ and $mae \in M$, we have $ae \in M$. This contradicts the assumption. Thus $ae \in \delta(R)$.
(2) ⇒ (1) Let $a, b \in R$ with $ab = 0$. So $(ba)^2 = 0$. By (2), $bae \in \delta(R)$. □

To illustrate Theorem 2.7, we see the following examples.

Examples 2.8. (1) Let R be a reduced ring and n an integer with $n \geq 2$. Then $U_n(R)$ is Zhou right E-reversible for each $E \in \mathrm{Id}(U_n(R))$. Indeed,

let $A = (a_{ij}), B = (b_{ij}) \in U_n(R)$ with $AB = 0$. Then $a_{ii}b_{ii} = 0$. By assumption, $b_{ii}a_{ii} = 0$. Then the diagonal entries of BAE are zero for any $E \in \mathrm{Id}(U_n(R))$. Since $N(U_n(R)) = \{(a_{ij}) \in U_n(R) \mid a_{ii} = 0\}$, we have $N(U_n(R)) \subseteq \delta(U_n(R))$. Therefore $U_n(R)$ is Zhou right E-reversible for each $E \in \mathrm{Id}(U_n(R))$.
(2) Let F be a field. Then $M_n(F)$ is Zhou right E-reversible for each $E \in \mathrm{Id}(M_n(F))$ by the fact that $\delta(M_n(F)) = M_n(F)$.
(3) The ring $M_2(\mathbb{Z})$ is not Zhou right E-reversible for $E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Indeed, consider $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \in N(M_2(\mathbb{Z}))$. Then $A^2 = 0$. But $AE = \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix} \notin \delta(M_2(\mathbb{Z}))$ since $\delta(M_2(\mathbb{Z})) = 0$. Hence by Theorem 2.7, $M_2(\mathbb{Z})$ is not Zhou right E-reversible.

Proposition 2.9. *Let R be a Zhou right e-reversible ring. Then $ab = 0$ implies $beae$, $aeb \in \delta(R)$ for any $a, b \in R$.*

Proof. Let $a, b \in R$ such that $ab = 0$. Since R is Zhou right e-reversible, $bae \in \delta(R)$. By Proposition 2.6, $ae - eae \in \delta(R)$. So $bae - beae \in \delta(R)$. It follows $beae \in \delta(R)$. On the other hand, $ab = 0$ implies $abr = 0$ and so $brae \in \delta(R)$. Hence $bRae \subseteq \delta(R)$. Thus $(aeb)R(aeb) \subseteq \delta(R)$. Since the Zhou radical is a semiprime ideal by [11, Proposition 2.6.], $aeb \in \delta(R)$. □

In what follows, we consider a condition $(*)$ under which $bea \in \delta(R)$ in a Zhou right e-reversible ring R. This condition is compared to Proposition 2.9.

$$\text{For any } a, b \in R, \text{ having } ab = 0 \text{ implies } bea \in \delta(R) \qquad (*)$$

Note that all reversible rings satisfy the condition $(*)$.

Theorem 2.10. *Let R be a Zhou right e-reversible ring. Then the following are equivalent:*

(1) *R satisfies the $(*)$ condition;*

(2) *$ex - exe \in \delta(R)$ for any $x \in R$;*

(3) *$\overline{e}$ is central in $R/\delta(R)$.*

Proof. (1) ⇒ (2) Let $x \in R$ and $a = ex - exe$, $b = e$. Then $ab = 0$. Since R satisfies the $(*)$ condition, $bea \in \delta(R)$. But $bea = ex - exe$. Thus $ex - exe \in \delta(R)$.
(2) ⇒ (1) Let $a, b \in R$ with $ab = 0$. By hypothesis, $ea - eae \in \delta(R)$. Multiplying $ea - eae \in \delta(R)$ from the left by b and use the fact that $\delta(R)$ is an ideal of R, we have $bea - beae \in \delta(R)$. On the other hand, Zhou right e-reversibility of R entails that $beae \in \delta(R)$ from Proposition 2.9. Combining $bea - beae \in \delta(R)$ with $beae \in \delta(R)$, we conclude that $bea \in \delta(R)$.
(2) ⇒ (3) Note that $\overline{e} \in \mathrm{Id}(R/\delta(R))$. Let $x \in R$. On the one hand, we have $ex - exe \in \delta(R)$ by (2). On the other hand, $xe - exe \in \delta(R)$ by Proposition 2.6. It follows that $ex - xe \in \delta(R)$. It entails $\overline{ex} = \overline{xe}$, as desired.
(3) ⇒ (2) Let $x \in R$. Then $\overline{ex} = \overline{xe}$, and so $ex - xe \in \delta(R)$ by (3). Hence $(ex - exe) - (xe - exe) \in \delta(R)$. Since $xe - exe \in \delta(R)$ by Proposition 2.6, we have $ex - exe \in \delta(R)$. □

The condition "being reduced of the ring" is not superfluous in Examples 2.8(1) by the following example.

Example 2.11. Let D be a division ring with $\mathrm{char}(D) \neq 2$. Consider the rings $M_2(D)$ and $U_2(M_2(D))$. We first note

$$\delta(M_2(D)) = M_2(\delta(D)) = M_2(D) \text{ and } \delta(U_2(M_2(D))) = \begin{bmatrix} 0 & M_2(D) \\ 0 & M_2(D) \end{bmatrix}.$$

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \in M_2(D)$, $X = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} \in U_2(M_2(D))$, $A' = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \in M_2(D)$, $Y = \begin{bmatrix} A' & A' \\ 0 & 0 \end{bmatrix} \in U_2(M_2(D))$, $Z = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \in M_2(D)$, $E = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} \in \mathrm{Id}(U_2(M_2(D)))$, $T = \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix} \in M_2(D)$, $U = \begin{bmatrix} 4 & 0 \\ -4 & 0 \end{bmatrix} \in M_2(D)$. Set $F = \begin{bmatrix} T & U \\ 0 & 0 \end{bmatrix} \in U_2(M_2(D))$. Then $XY = 0$ and $YXE = F \notin \delta(U_2(M_2(D)))$. Note that $M_2(D)$ is not reduced since $AA' = 0$ and $A'A = A' \neq 0$.

Theorem 2.12. *Let R be a ring with $e \in Id(R)$ and $\overline{R} = R/\delta(R)$. If $\overline{R}$ is right $\overline{e}$-reversible, then R is Zhou right e-reversible.*

Proof. Let $a, b \in R$ with $ab = 0$. Then $\overline{a}\overline{b} = 0$ in $\overline{R}$. The ring $\overline{R}$ being right $\overline{e}$-reversible yields $\overline{b}\overline{a}\overline{e} = 0$ entailing that $bae \in \delta(R)$. □

Corollary 2.13. *Let R be a ring with $e \in Id(R)$ and $\overline{R} = R/\delta(R)$. If $\overline{R}$ is reversible, then R is Zhou right e-reversible.*

Theorem 2.14. *Let $\{R_i\}_{i\in I}$ be a family of rings for a finite index set I, $R = \prod_{i\in I} R_i$ and $e_i^2 = e_i \in R_i$ for each $i \in I$ and set $e = (e_i) \in R$. Then R_i is Zhou right e_i-reversible for each $i \in I$ if and only if R is Zhou right e-reversible.*

Proof. By definition, note that $\delta(R) = \prod_{i\in I} \delta(R_i)$. Assume that R_i is Zhou right e_i-reversible for each $i \in I$. Let $a = (a_i)$, $b = (b_i) \in R$ with $ab = 0$. Then $a_ib_i = 0$. By assumption, $b_ia_ie_i \in \delta(R_i)$ for each $i \in I$. Hence $bae \in \delta(R)$. Conversely, suppose that R is Zhou right e-reversible. Let $a_i, b_i \in R_i$ with $a_ib_i = 0$, where $i \in I$. Consider $a = (\dots, a_i, \dots)$, $b = (\dots, b_i, \dots) \in R$, where i^{th} entries are a_i and b_i, respectively, and other entries are zero. Then $ab = 0$. By supposition, $bae \in \delta(R)$. Componentwise equality implies $b_ia_ie_i \in \delta(R_i)$. So R_i is Zhou right e_i-reversible for each $i \in I$. □

It is worth considering that the Zhou radical commute with infinite direct products of rings, but the next example shows that this is not true.

Example 2.15. Let F be a field and $R = \prod F$ denote an infinite direct product of F. Clearly, $\delta(F) = F$. On the other hand, since R is not semisimple, $\delta(R) \neq R$. This means that $\delta(\prod F) \neq \prod \delta(F)$.

Recall that a ring R is called *semiperfect* if $R/J(R)$ is semisimple and idempotents of $R/J(R)$ can be lifted to R. As a consequence of Theorem 2.14, we obtain under what conditions semiperfect rings are Zhou e-reversible.

Theorem 2.16. *Let R be a semiperfect ring. If R satisfies one of the following conditions, then it is Zhou right e-reversible for each $e \in Id(R)$.*

(1) *R is commutative.*

(2) *R is left morphic left quasi-duo.*

Proof. If R is commutative semiperfect, then it is a finite direct product of local rings by [13, Theorem 23.11]. If R is left morphic left quasi-duo semiperfect, then it is a finite direct product of local rings by [6, Proposition 2.15]. In both cases, R is Zhou right e-reversible by Examples 2.4(2) and Theorem 2.14. □

Let $e, f \in \mathrm{Id}(R)$. Then e and f are called *isomorphic* if Re and Rf are isomorphic as left R-modules, equivalently, eR and fR are isomorphic as right R-modules.

Theorem 2.17. *Let R be a ring and $e, f \in Id(R)$. If R is Zhou right e-reversible and e and f are isomorphic, then R is Zhou right f-reversible.*

Proof. Let $g\colon Re \to Rf$ denote the isomorphism of the left R-modules Re and Rf. Assume that R is a Zhou right e-reversible ring. Then g being an isomorphism implies that there exists $r \in R$ such that $f = g(re)$. Hence $ef = eg(re) = g(ere)$. Thus $ef - f = g(ere) - g(re) = g(ere - re) \in \delta(R)$ due to $ere - re \in \delta(R)$ by Proposition 2.6. So $ef - f \in \delta(R)$. Let a, $b \in R$ with $ab = 0$. By assumption, $bae \in \delta(R)$. Since $\delta(R)$ is an ideal in R, $baef \in \delta(R)$. Therefore $baf \in \delta(R)$. □

We close this section by investigating the Zhou right e-reversibility of corner rings.

Proposition 2.18. *Let R be a ring, $e \in Id(R)$ with $ReR = R$ and $f \in Id(eRe)$. If R is Zhou right f-reversible, then eRe is Zhou right f-reversible.*

Proof. Assume that R is Zhou right f-reversible. Let $a, b \in eRe$ with $ab = 0$. By assumption, $baf \in \delta(R)$. Since $baf = ebafe \in e\delta(R)e$ and $e\delta(R)e = \delta(eRe)$ by [17, Theorem 3.9], we have $baf \in \delta(eRe)$. □

3. Some ring extensions

In this section, we focus on the Zhou e-reversibility of some ring extensions such as Morita contexts, Dorroh extensions, skew formal power series rings, special matrix rings and special subrings of a direct product of rings.

Proposition 3.1. *Let $\mathcal{M} = (R, V, W, S)$ be a trivial Morita context with the Morita context ring $\mathcal{T}(\mathcal{M}) = \begin{bmatrix} R & V \\ W & S \end{bmatrix}$. Let $E = \begin{bmatrix} e & x \\ y & f \end{bmatrix} \in Id(\mathcal{T}(\mathcal{M}))$ and $e \in Id(R)$, $f \in Id(S)$. Then $\mathcal{T}(M)$ is Zhou right E-reversible if and only if R is Zhou right e-reversible and S is Zhou right f-reversible.*

Proof. By Theorem 1.6, $\delta(\mathcal{T}(\mathcal{M})) = \begin{bmatrix} \delta(R) & V \\ W & \delta(S) \end{bmatrix}$. Assume that $\mathcal{T}(M)$ is Zhou right E-reversible. Let r, $r' \in R$ and s, $s' \in S$ with $rr' = 0$

and $ss' = 0$. Consider $A = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}$, $B = \begin{bmatrix} r' & 0 \\ 0 & s' \end{bmatrix} \in \mathcal{T}(M)$. Then $AB = 0$. By assumption, $BAE \in \delta(\mathcal{T}(\mathcal{M}))$. It implies that $r're \in \delta(R)$ and $s'sf \in \delta(S)$, and so R is Zhou right e-reversible and S is Zhou right f-reversible. Conversely, let $A = \begin{bmatrix} r & v \\ w & s \end{bmatrix}$, $B = \begin{bmatrix} r_1 & v_1 \\ w_1 & s_1 \end{bmatrix} \in \mathcal{T}(M)$ with $AB = 0$. Then $rr_1 = 0$ and $ss_1 = 0$. Since R is Zhou right e-reversible and S is Zhou right f-reversible, $r_1re \in \delta(R)$ and $s_1sf \in \delta(S)$, respectively. Hence $BAE = \begin{bmatrix} r_1re & * \\ * & s_1sf \end{bmatrix} \in \delta(\mathcal{T}(\mathcal{M}))$. Therefore $\mathcal{T}(M)$ is Zhou right E-reversible. □

Now we give two direct consequences of Proposition 3.1. Let R and S be any rings, M an R-S-bimodule and T the formal triangular matrix ring $\begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$. It is known that $\delta(T) = \begin{bmatrix} \delta(R) & M \\ 0 & \delta(S) \end{bmatrix}$ by Theorem 1.6.

Corollary 3.2. *Let $T = \begin{bmatrix} R & M \\ 0 & S \end{bmatrix}$ and $e \in Id(R)$, $f \in Id(S)$. Then for $E = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \in Id(T)$, the ring T is Zhou right E-reversible if and only if R is Zhou right e-reversible and S is Zhou right f-reversible.*

Corollary 3.3. *Let R be a ring, n a positive integer, $e \in Id(R)$, and $E \in Id(U_n(R))$ with all diagonal entries equal to e. Then R is Zhou right e-reversible if and only if $U_n(R)$ is Zhou right E-reversible.*

Proposition 3.4. *Let S be an algebra over a ring R and consider the Dorroh extension $D(R,S)$ of S by R. Let $(e,s) \in Id(D(R,S))$. Then $D(R,S)$ is Zhou right (e,s)-reversible if and only if R is Zhou right e-reversible.*

Proof. For the necessity, let $a, b \in R$ such that $ab = 0$. Having $(e,s) \in \mathrm{Id}(D(R,S))$ yields $e \in \mathrm{Id}(R)$. Consider $(a,0),(b,0) \in D(R,S)$. Then $(a,0)(b,0) = 0$. Since $D(R,S)$ is Zhou right (e,s)-reversible, we have $(b,0)(a,0)(e,s) = (bae, bas) \in \delta(D(R,S))$. By Lemma 1.10(1), $bae \in \delta(R)$. For the sufficiency, let $(a,x),(b,y) \in D(R,S)$ with $(a,x)(b,y) = 0$. Then $ab = 0$. Since R is Zhou right e-reversible, $bae \in \delta(R)$. By Lemma 1.10(1), we have $(b,y)(a,x)(e,t) = (bae,*) \in \delta(D(R,S))$. □

Let R be a ring, $\sigma\colon R \to R$ a ring homomorphism and $R[[x,\sigma]]$ denote the ring of skew formal power series $\{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in R\}$. Addition in $R[[x,\sigma]]$ is usual one and multiplication is defined by $xa = \sigma(a)x$. The ideal $< x >$ is an essential maximal right ideal of $R[[x,\sigma]]$, and $\delta(R[[x,\sigma]]) \subseteq \delta(R)+ < x >$ by [3, Proposition 3.15]. In general, for a ring R, $J(R[[x]]) = J(R)+ < x >\subseteq \delta(R[[x]]) \subseteq \delta(R)+ < x >$. We now prove that the inclusion is strict in some cases.

Proposition 3.5. *For a semisimple ring R, the following inclusion is strict*

$$\delta(R[[x,\sigma]]) \subset \delta(R)+ < x > .$$

Proof. Since R is semisimple, we have $\delta(R) = R$ and $J(R) = 0$. Also, $R[[x,\sigma]]$ is not semisimple. Then $J(R[[x,\sigma]]) =< x >\subseteq \delta(R[[x,\sigma]]) \subsetneq \delta(R)+ < x >= R[[x,\sigma]]$. □

Proposition 3.6. *Let R be an abelian ring, $\sigma\colon R \to R$ a ring homomorphism and $e \in Id(R)$. If $R[[x,\sigma]]$ is Zhou right e-reversible, then R is Zhou right e-reversible.*

Proof. Since R is abelian, all idempotents of $R[[x,\sigma]]$ are contained in R. Let $a, b \in R$ with $ab = 0$. Consider $f(x) = a, g(x) = b \in R[[x,\sigma]]$, and so $f(x)g(x) = 0$. We have $g(x)f(x)e \in \delta(R[[x,\sigma]])$ since $R[[x,\sigma]]$ is Zhou right e-reversible. Having $\delta(R[[x,\sigma]]) \subseteq \delta(R)+ < x >$ entails that $bae \in \delta(R)$. This completes the proof. □

There are rings R and positive integers $n \geq 2$ such that $M_n(R)$ need not be Zhou right E-reversible for some $E \in \mathrm{Id}(M_n(R))$ as shown below.

Remark 3.7. Let R be a ring which is not semisimple and $n \geq 2$ be an integer. Consider $A = e_{11} + e_{12} - e_{21} - e_{22}, B = e_{12} - e_{22} \in M_n(R)$ and $E = e_{11} + e_{12} \in \mathrm{Id}(M_n(R))$. Then $AB = 0$, but $BAE = -e_{11} - e_{12} + e_{21} + e_{22} \notin \delta(M_n(R))$ since $\delta(M_n(R)) = M_n(\delta(R))$. Therefore $M_n(R)$ is not Zhou right E-reversible.

On the contrast to the ring $M_n(R)$, some subrings of $M_n(R)$ are Zhou right E-reversible for each integer $n \geq 2$.

Proposition 3.8. *Let R be a ring with $e \in$Id(R) and $n \geq 2$ an integer. Then the following are equivalent.*

(1) *R is Zhou right e-reversible.*

(2) *$D_n(R)$ is Zhou right E-reversible, where $E = eI_n$.*

(3) *$V_n(R)$ is Zhou right E-reversible, where $E = eI_n$.*

Proof. (1) ⇒ (2) and (3): Note that $N(D_n(R))$ and $N(V_n(R))$ are ideals of the rings $D_n(R)$ and $V_n(R)$, respectively. If R is Zhou right e-reversible, then (2) and (3) hold by Examples 2.4(5).
(2) ⇒ (3) Obvious.
(3) ⇒ (1) Assume that (3) holds. Let $a, b \in R$ with $ab = 0$. Write $A = aI_n$ and $B = bI_n$. Then $AB = 0$, and so $BAE \in \delta(V_n(R))$. By Lemma 1.2, we get $bae \in \delta(R)$. □

Let A be a ring, B a subring of A, and consider the ring

$$R[A,B] = \{(a_1, a_2, \dots, a_n, b, b, \dots) \mid a_i \in A,\ b \in B,\ 1 \leqslant i \leqslant n, n \in \mathbb{Z}^+\}.$$

By Theorem 1.12, it is known that $\delta(R[A,B]) = R[\delta(A), \delta(A) \cap \delta(B)]$. Let $e \in \mathrm{Id}(B)$ and $E = (e, e, e, \dots) \in \mathrm{Id}(R[A,B])$. We investigate the Zhou right e-reversibility of $R[A,B]$ in the following result.

Theorem 3.9. *Let A be a ring, B a subring of A, $e \in Id(B)$ and $E = (e, e, e, \dots) \in Id(R[A,B])$. Then the following are equivalent:*

(1) *A and B are both Zhou right e-reversible.*

(2) *$R[A,B]$ is Zhou right E-reversible.*

Proof. (1) ⇒ (2) Let $C = (a_1, \dots, a_n, b, b, \dots), D = (x_1, \dots, x_m, y, y, \dots)$ in $R[A,B]$ with $CD = 0$. We consider the two cases: $n \geq m$ and $n < m$.
Case I. Assume $n \geq m$. Then $a_i x_i = 0$ for $i = 1, 2, \dots, m$, $a_{m+i} y = 0$ for $i = 1, 2, \dots, n-m$ and $by = 0$. By (1), $x_i a_i e \in \delta(A)$ for $i = 1, 2, \dots, m$, $y a_{m+i} e \in \delta(A)$ for $i = 1, 2, \dots, n-m$ and $ybe \in \delta(A) \cap \delta(B)$. It follows that $DCE \in \delta(R[A,B])$.
Case II. Suppose $n < m$. Then $a_i x_i = 0$ for $i = 1, 2, \dots, n$, $b x_{n+i} = 0$ for $i = 1, 2, \dots, m-n$ and $by = 0$. By (1), $x_i a_i e \in \delta(A)$ for $i = 1, 2, \dots, n$, $x_{n+i} b \in \delta(A)$ for $i = 1, 2, \dots, m-n$ and $ybe \in \delta(A) \cap \delta(B)$.

In both cases, we have $DCE \in \delta(R[A,B])$.
(2) ⇒ (1) Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ with $a_1 a_2 = 0$ and $b_1 b_2 = 0$. Set $Y = (a_1, b_1, b_1, \dots)$ and $Z = (a_2, b_2, b_2, \dots)$. Then $YZ = 0$. By (2), $ZYE \in \delta(R[A,B])$. Hence $a_2 a_1 e \in \delta(A)$ and $b_2 b_1 e \in \delta(A) \cap \delta(B)$, and so $b_2 b_1 e \in \delta(B)$. This completes the proof. □

We now give an example to illustrate Theorem 3.9.

Example 3.10. Let F be a field, $A = M_2(F)$, $B = U_2(F)$ and consider

$$R[A,B] = \{(a_1, a_2, a_3, \dots, a_n, b, b, b, \dots) \mid n \in \mathbb{N}, a_i \in A, b \in B\}.$$

Since $\delta(A) = A$ and $\delta(B) = e_{12}F + e_{22}F$, we see by Theorem 1.12 that $\delta(R[A,B]) = \{(a_1, a_2, \dots, a_n, b, b, \dots) \mid n \in \mathbb{N}, a_i \in A,\ b \in \delta(B)\} = R[A, \delta(B)]$. For $e_{11} \in \mathrm{Id}(B)$ and $E = (e_{11}, e_{11}, e_{11}, \dots) \in \mathrm{Id}(R[A,B])$, in the light of Examples 2.8 and Theorem 3.9, $R[A,B]$ is Zhou right E-reversible.

4. Some Zhou *e*-reversible subrings of matrix rings

The rings $H_3(\mathbb{Z}, R)$: Let R be a ring and consider the ring $H_3(\mathbb{Z}, R) = \left\{ \begin{bmatrix} n & a_1 & a_2 \\ 0 & a_3 & a_4 \\ 0 & 0 & n \end{bmatrix} \mid a_1, a_2, a_3, a_4 \in R, n \in \mathbb{Z} \right\}$ with the usual matrix addition and multiplication. We have the following.

Lemma 4.1. *The following hold for a ring R.*

(1) $N(H_3(\mathbb{Z}, R)) = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{bmatrix} \in H_3(\mathbb{Z}, R) \mid c \in N(R) \right\}$.

(2) $\delta(H_3(\mathbb{Z}, R)) = \begin{bmatrix} 0 & R & R \\ 0 & \delta(R) & R \\ 0 & 0 & 0 \end{bmatrix}$.

Proof. It is routine. □

Theorem 4.2. *A ring R is Zhou right e-reversible for each $e \in Id(R)$ if and only if $H_3(\mathbb{Z}, R)$ is Zhou right E-reversible for $E = e_{11} + e_{22}e + e_{33}$.*

Proof. Clear. □

The rings $H_{(s,t)}(R)$: Let R be a ring and s, $t \in C(R)$ be invertible in R. Let

$$H_{(s,t)}(R) = \left\{ \begin{bmatrix} a & 0 & 0 \\ c & d & e \\ 0 & 0 & f \end{bmatrix} \in M_3(R) \mid a, c, d, e, f \in R, a - d = sc, d - f = te \right\}.$$

Then $H_{(s,t)}(R)$ is a subring of $M_3(R)$.

Lemma 4.3. *Let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & f \\ 0 & 0 & g \end{bmatrix} \in H_{(1,1)}(R)$. Then*

(1) *$A \in \delta(H_{(1,1)}(R))$ if and only if a, d, $g \in \delta(R)$.*

(2) *$A \in Id(H_{(1,1)}(R))$ if and only if a, d, $g \in Id(R)$.*

Proof. It is routine. □

Theorem 4.4. *Let R be a ring, $e \in Id(R)$ and $E = eI_3 \in Id(H_{(1,1)}(R))$. Then R is Zhou right e-reversible if and only if $H_{(1,1)}(R)$ is Zhou right E-reversible.*

Proof. For the necessity, let $A = \begin{bmatrix} a & 0 & 0 \\ c & d & f \\ 0 & 0 & g \end{bmatrix}$ and $B = \begin{bmatrix} x & 0 & 0 \\ y & z & u \\ 0 & 0 & v \end{bmatrix} \in H_{(1,1)}(R)$ with $AB = 0$. Then $ax = 0$, $dz = 0$, $gv = 0$. Since R is Zhou right e-reversible, by Lemma 4.3(1), $\{xae, zde, vge\} \subseteq \delta(R)$. Then $BAE = \begin{bmatrix} xae & 0 & 0 \\ * & zde & * \\ 0 & 0 & vge \end{bmatrix} \in \delta(H_{(1,1)}(R))$. For the sufficiency, let $a, b \in R$ with $ab = 0$. We write $A = aI_3$ and $B = bI_3$. Then $AB = 0$. Since $H_{(1,1)}(R)$ is Zhou right E-reversible, $BAE = \begin{bmatrix} bae & 0 & 0 \\ 0 & bae & 0 \\ 0 & 0 & bae \end{bmatrix} \in \delta(H_{(1,1)}(R))$. So we get $bae \in \delta(R)$ by Lemma 4.3(1). □

Generalized matrix rings: Let R be a ring and s be a central element of R. Then $\begin{bmatrix} R & R \\ R & R \end{bmatrix}$ becomes a ring denoted by $K_s(R)$ with addition defined componentwise and multiplication defined in [12] by

$$\begin{bmatrix} a_1 & x_1 \\ y_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & x_2 \\ y_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + sx_1y_2 & a_1x_2 + x_1b_2 \\ y_1a_2 + b_1y_2 & sy_1x_2 + b_1b_2 \end{bmatrix}.$$

In [12], $K_s(R)$ is called a *generalized matrix ring over R*.

Lemma 4.5. *For a ring R, we have the following:*

(1) *If $E = \begin{bmatrix} e & v \\ w & f \end{bmatrix} \in Id(K_0(R))$, then $e, f \in Id(R)$.*

(2) $\delta(K_0(R)) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, d \in \delta(R), b, c \in R \right\}$.

Proof. (1) Let $E^2 = E = \begin{bmatrix} e & v \\ w & f \end{bmatrix} \in K_0(R)$. Then $e^2 = e$ and $f^2 = f$ obviously.

(2) Let $\delta(K_0(R)) = \begin{bmatrix} A & V \\ W & B \end{bmatrix}$. We claim that $A, B \subseteq \delta(R)$. Firstly, assume that $A \nsubseteq \delta(R)$. Then there exists an essential maximal right ideal M of R such that $A \nsubseteq M$. Consider the right ideal $I = \begin{bmatrix} M & R \\ R & R \end{bmatrix}$ of $K_0(R)$. The maximality of M in R yields the maximality of I in $K_0(R)$. In order to show that I is essential in $K_0(R)$, let $0 \neq \alpha = \begin{bmatrix} r & v \\ w & s \end{bmatrix} \in K_0(R)$. If $r = 0$, then $0 \neq \alpha \in I$. If $r \neq 0$, then there exists $r_1 \in R$ such that $0 \neq rr_1 \in M$ by the essentiality of M in R. Hence $0 \neq \alpha \begin{bmatrix} r_1 & 0 \\ 0 & 0 \end{bmatrix} \in I$, and so I is essential in $K_0(R)$. Thus $\delta(K_0(R)) \subseteq I$, this entails $A \subseteq M$. This contradiction shows $A \subseteq \delta(R)$. By a similar discussion, we obtain $B \subseteq \delta(R)$. It follows $\delta(K_0(R)) = \begin{bmatrix} A & V \\ W & B \end{bmatrix} \subseteq \begin{bmatrix} \delta(R) & R \\ R & \delta(R) \end{bmatrix}$. For the reverse inclusion, consider the subsets $X = e_{12}R$ and $Y = e_{21}R$ of $K_0(R)$. Clearly, X and Y are nilpotent ideals in $K_0(R)$. Consequently, we have $X, Y \subseteq \delta(K_0(R))$, and so $X + Y \subseteq \delta(K_0(R))$. By [20, Theorem 1.6(2) and Lemma 1.3(1)], $X + Y$ is δ-small in $K_0(R)$. By the fact that $\delta(R)$ is δ-small in R, we have that $\begin{bmatrix} \delta(R) & R \\ R & \delta(R) \end{bmatrix} /(X+Y)$ is δ-small in $K_0(R)/(X+Y)$, and so $\begin{bmatrix} \delta(R) & R \\ R & \delta(R) \end{bmatrix}$ is δ-small in $K_0(R)$ by [20, Lemma 1.3(1)]. Thus $\begin{bmatrix} \delta(R) & R \\ R & \delta(R) \end{bmatrix} \subseteq \delta(K_0(R))$. Therefore $\delta(K_0(R)) = \begin{bmatrix} \delta(R) & R \\ R & \delta(R) \end{bmatrix}$. □

Theorem 4.6. *Let R be a ring and $e, f \in Id(R)$ and $E = ee_{11} + fe_{22} \in Id(K_0(R))$. Then R is Zhou right e-reversible and Zhou right f-reversible if and only if $K_0(R)$ is Zhou right E-reversible.*

Proof. For the necessity, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} u & v \\ t & z \end{bmatrix} \in K_0(R)$ with $AB = 0$. Then $au = 0$, $dz = 0$. Consider $E = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \in \mathrm{Id}(K_0(R))$. Since R is Zhou right e-reversible and Zhou right f-reversible, we have $\{uae, zdf\} \subseteq \delta(R)$. By Lemma 4.5(2), we have $BAE = \begin{bmatrix} uae & * \\ * & zdf \end{bmatrix} \in \delta(K_0(R))$. So $K_0(R)$ is Zhou right E-reversible.

For the sufficiency, suppose that $K_0(R)$ is Zhou right E-reversible. Let $a, b \in R$ with $ab = 0$. Set $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} \in K_0(R)$. Having $ab = 0$ yields $AB = 0$. Since $K_0(R)$ is Zhou right E-reversible, $BAE = \begin{bmatrix} bae & 0 \\ 0 & baf \end{bmatrix} \in \delta(K_0(R))$. By Lemma 4.5(2), $bae, baf \in \delta(R)$. Hence R is Zhou right e-reversible and Zhou right f-reversible. This completes the proof. □

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Contact information

B. Ungor
Department of Mathematics, Ankara University, Ankara, Turkey
E-Mail: bungor@science.ankara.edu.tr

H. Kose
Department of Computer Science, Ankara University, Ankara, Turkey
E-Mail: handankose@ankara.edu.tr

A. Harmanci
Department of Mathematics, Hacettepe University, Ankara, Turkey
E-Mail: harmanci@hacettepe.edu.tr

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