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RESEARCH ARTICLE

Locally Artinian supplements related to Krasner hypermodules

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Abstract. We introduce and investigate two new classes of Krasner left R-hypermodules: *locally Artinian supplemented hypermodules* and *RLA−local hypermodules*. A Krasner left R-hypermodule H is called *locally Artinian* if every finitely generated subhypermodule of H is Artinian, and it is called *RLA−local* if it is local and $\mathrm{Rad}(H)$ is a locally Artinian subhypermodule of H. A hypermodule H is called *locally Artinian supplemented* if every subhypermodule U of H has a locally Artinian supplement V in H, meaning that V is a supplement of U in H such that $U \cap V$ is locally Artinian. We investigate fundamental closure properties of these classes, including closure under factor hypermodules and finite direct sums. We show that every RLA−local hypermodule is amply locally Artinian supplemented, and we characterize locally Artinian supplemented hypermodules in terms of their radicals. We further prove that for normal π-projective hypermodules, locally Artinian supplemented and amply locally Artinian supplemented conditions are equivalent. Several new results distinguishing these classes from ss-supplemented hypermodules and Artinian hypermodules are also investigated. Structural results concerning essential subhypermodules are established, and the relationship between essentiality and locally Artinian supplements is investigated, thereby situating the present study within the broader context of smallness and essentiality in Krasner hypermodules.

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Introduction

Hypercompositional structure theory generalizes algebraic structures by replacing binary operations with hyperoperations, which associate each pair of elements with a *set* of elements rather than a single value. A non-empty set equipped with at least one such hyperoperation is called an *hypercompositional structure*. Prominent examples include hypergroups, hyperrings, and hypermodules, which extend groups, rings, and modules, respectively. The concept of hypergroups was introduced by Marty in 1934 [14]. Canonical hypergroups were developed by Mittas [17]. Krasner established hyperrings and hyperfields in 1983 [13], and his student Mittas, followed by Massouros [15], initiated the study of hypermodules. Further contributions to the theory of hyperrings and related structures were made by Ameri, Norouzi, Corsini, Davvaz, Omidi, and Leoreanu-Fotea [2, 7, 8, 10]. In contemporary research, hyperstructure theory is an expanding research field, contributing substantially to modern algebra and its applications in cryptography and information theory [9, 23].

Let H be a non-empty set and $\mathcal{P}(H)$ the set of all subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \to \mathcal{P}(H)$, and the pair $(H, \circ)$ is called a *hypergroupoid*. For non-empty subsets $X, Y \subseteq H$ one sets $X \circ Y = \bigcup_{x \in X,\, y \in Y} x \circ y$, with the conventions $a \circ X = \{a\} \circ X$ and $X \circ a = X \circ \{a\}$. A hypergroupoid is a *semihypergroup* if $\circ$ is associative, a *quasihypergroup* if $x \circ H = H = H \circ x$ for all $x \in H$, and a *hypergroup* if it is both.

A *canonical hypergroup* is a commutative hypergroup $(H, \circ)$ possessing an identity element e (satisfying $\{a\} = (a \circ e) \cap (e \circ a)$ for all a), unique inverses a^{-1}, and the implication: $c \in a \circ b \Rightarrow a \in c \circ b^{-1}$ and $b \in a^{-1} \circ c$.

The triple $(R, +, \cdot)$ is a *Krasner hyperring* if $(R, +)$ is a canonical hypergroup, $(R, \cdot)$ is a semigroup with a bilaterally absorbing zero ($a \cdot 0 = 0 = 0 \cdot a$ for all a), and multiplication distributes over $+$ on both sides. Throughout this paper, R denotes a Krasner hyperring with identity element 1_R.

A *Krasner left R-hypermodule* is a canonical hypergroup $(H, +)$ together with a map $\cdot : R \times H \to H$ satisfying, for all $x, y \in R$ and $a, b \in H$:

(1) $x \cdot (a + b) = x \cdot a + x \cdot b$;

(2) $(x + y) \cdot a = x \cdot a + y \cdot a$;

(3) $(x \cdot y) \cdot a = x \cdot (y \cdot a)$;

(4) $1_R \cdot a = a$;

(5) $x \cdot 0_H = 0_H$.

In this paper all hypermodules are Krasner left R-hypermodules over a Krasner hyperring R with identity, unless otherwise stated.

A non-empty subset N of H is a *subhypermodule* of H, written $N \leq H$, if N is itself a Krasner left R-hypermodule under the induced operations, equivalently if $r \cdot a \in N$ and $a - b \subseteq N$ for all $r \in R$ and $a, b \in N$ [13]. For a subhypermodule $K \leq H$, the *quotient hypermodule* $H/K = \{a + K \mid a \in H\}$ is a Krasner left R-hypermodule with hyperaddition $(a + K) + (a' + K) = \{b + K \mid b \in a + a'\}$ and an external operation $r \cdot (a+K) = r \cdot a + K$. A non-zero hypermodule H is *simple* [22] if its only subhypermodules are $\{0_H\}$ and H. Let N and K be R-hypermodules. A function $\psi : N \to P(K)$ is called an *R-homomorphism* if $\psi(n_1 + n_2) \subseteq \psi(n_1) + \psi(n_2)$ for all $n_1, n_2 \in N$ and $\psi(r \cdot n) = r \cdot \psi(n)$ for all $r \in R$ and $n \in N$. ψ is called a *strong homomorphism* whenever $\psi(n_1 + n_2) = \psi(n_1) + \psi(n_2)$ for all $n_1, n_2 \in N$. A single valued function $\psi : N \to K$ is called a *normal homomorphism* if $\psi(n_1 + n_2) = \psi(n_1) + \psi(n_2)$ for all $n_1, n_2 \in N$ and $\psi(r \cdot n) = r \cdot \psi(n)$ for all $r \in R$ and $n \in N$ in [7].

Two subhypermodules N, N' of H are *independent* if $N \cap N' = \{0_H\}$; their sum is then written $N \oplus N'$ and called a *direct sum*. A subhypermodule N is a *direct summand* of H if $H = N \oplus K$ for some $K \leq H$ [11].

A subhypermodule N of H is *small* in H, written $N \ll H$, if $N + L \neq H$ for every proper subhypermodule L of H. The *radical* $\mathrm{Rad}(H)$ is defined as the sum of all small subhypermodules of H; if H has no small subhypermodules, we set $\mathrm{Rad}(H) = H$. The radical is always a subhypermodule of H [19].

A non-zero hypermodule H is *hollow* if every proper subhypermodule of H is small in H, and *local* if the sum of all proper subhypermodules of H is itself a proper subhypermodule. Every local hypermodule is hollow. The hypermodule H is *local* if and only if it is hollow and $\mathrm{Rad}(H) \neq H$ [19].

The dual notion to smallness is essentiality. Following the hypermodule context developed in [18], a subhypermodule N of H is called *essential* in H, written $N \trianglelefteq H$, if $N \cap K \neq \{0_H\}$ for every non-zero subhypermodule K of H. The interplay between essential subhypermodules and supplement conditions in the hypermodule context was first systematically examined in [18], and we shall build upon that context in Section 2

to develop new results connecting essentiality with locally Artinian supplements.

Let U, V be subhypermodules of H. Then V is a *supplement* of U in H if $U + V = H$ and $U \cap V \ll V$, equivalently if V is minimal in $\{L \leq H \mid U + L = H\}$. The hypermodule H is *supplemented* if every subhypermodule has a supplement in H, and *amply supplemented* if every subhypermodule of H has ample supplements, meaning every V with $U + V = H$ contains a supplement of U [19].

Kaynar and Nişancı Türkmen [12] generalized the socle of a Krasner hypermodule H by the set $\mathrm{Soc}_s(H) = \sum\{T \ll H \mid T \text{ is simple}\}$, so that $\mathrm{Soc}_s(H) \subseteq \mathrm{Rad}(H) \cap \mathrm{Soc}(H)$. Using this, a subhypermodule K of H is an *ss-supplement* of N in H if $H = N + K$ and $N \cap K \subseteq \mathrm{Soc}_s(K)$, and H is *ss-supplemented* if every subhypermodule has an ss-supplement. A Krasner hypermodule H is *strongly local* if it is local and $\mathrm{Rad}(H)$ is semisimple in [12]. We refer to the reader as major contributions to the literature on hyperstructures in [1, 3–5, 11, 16, 20].

The central aim of this paper is to simultaneously weaken the intersection condition in the supplement from $N \cap K \subseteq \mathrm{Soc}_s(K)$ (ss-supplement) to $N \cap K$ being locally Artinian, and to study the resulting classes systematically in the Krasner hypermodules. This parallels and generalizes the module theoretic study of locally Artinian supplemented modules.

The paper is organized as follows. Section 2 introduces locally Artinian hypermodules, RLA−local hypermodules, and locally Artinian supplemented hypermodules, and develops their fundamental structural properties. New theorems on the interaction of essentiality and locally Artinian supplements are also investigated in Section 2. Section 3 concludes the paper and outlines several open problems.

1. Locally Artinian supplements in Krasner hypermodules

In this section we introduce and study the main objects of the paper. Throughout, H denotes a Krasner left R-hypermodule. Subhypermodules are denoted consistently by U, V, K, L.

We begin by extending the notion of a locally Artinian module to the hypermodules. This is the foundational definition upon which the rest of the paper.

Definition 1.1. A Krasner left R-hypermodule H is called *locally Artinian* if every finitely generated subhypermodule of H is Artinian. A

local hypermodule H is called *RLA−local* if $\mathrm{Rad}(H)$ is a locally Artinian subhypermodule of H. A Krasner hyperring R is called an *RLA−local hyperring* if R is an RLA−local R-hypermodule over itself.

Remark 1.2. Every semisimple Krasner hypermodule is locally Artinian, since every finitely generated semisimple hypermodule is a finite direct sum of simple hypermodules, each of which is Artinian. Consequently, every strongly local Krasner hypermodule is RLA−local, since $\mathrm{Rad}(H)$ semisimple implies $\mathrm{Rad}(H)$ locally Artinian. The converse fails, as Example 1.3 below shows.

The implications among the three local conditions are therefore:

$$\text{strongly local} \Longrightarrow \text{RLA}-\text{local} \Longrightarrow \text{local}.$$

Both arrows are strict in general.

The following example shows that each inclusion in the above chain is proper, by exhibiting hypermodules satisfying one condition but failing the stronger one.

Example 1.3. (1) Consider the finitely generated $\mathbb{Z}$-hypermodule $H = \mathbb{Z}_8$. Then $\mathrm{Rad}(\mathbb{Z}_8) = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \cong \mathbb{Z}_4$. Since $\mathbb{Z}_4$ is Artinian, $\mathrm{Rad}(H)$ is locally Artinian, so $\mathbb{Z}_8$ is RLA−local. On the other hand, $\mathrm{Rad}(\mathbb{Z}_8) \cong \mathbb{Z}_4$ is not semisimple (it has the non-trivial subhypermodule $\{\bar{0}, \bar{4}\} \cong \mathbb{Z}_2$), so $\mathbb{Z}_8$ is not strongly local [12].

(2) Consider the $\mathbb{Z}$-hypermodule $H = \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, p \nmid b\}$, where p is a prime. Then $\mathbb{Z}_{(p)}$ is a local hyperring, hence a local hypermodule over itself. However, $\mathrm{Rad}(\mathbb{Z}_{(p)}) = p\mathbb{Z}_{(p)} \cong \mathbb{Z}_{(p)}$ as $\mathbb{Z}$-hypermodules, which is not Artinian (it contains the infinite descending chain $p\mathbb{Z}_{(p)} \supset p^2\mathbb{Z}_{(p)} \supset \cdots$). Thus $\mathbb{Z}_{(p)}$ is local but not RLA−local.

A fundamental problem in the study of any class of hypermodules is whether the class is closed under factor hypermodules. The following proposition presents an implication for RLA−local hypermodules.

Proposition 1.4. *If H is an RLA−local hypermodule, then every factor hypermodule of H is RLA−local.*

Proof. Let $N \leq H$ and $\pi : H \to H/N$ be the canonical normal homomorphism. Since local hypermodules are hollow and homomorphic images of hollow hypermodules are hollow [19], H/N is hollow, and since π is

surjective onto a non-zero module, H/N is local. Since H is hollow, we have $\mathrm{Rad}(H/N) = \pi(\mathrm{Rad}(H)) = \frac{\mathrm{Rad}(H)+N}{N}$ by [19]. As π is a normal homomorphism and $\mathrm{Rad}(H)$ is locally Artinian, $\pi(\mathrm{Rad}(H))$ is locally Artinian since homomorphic images of locally Artinian hypermodules are locally Artinian. Therefore $\mathrm{Rad}(H/N)$ is locally Artinian, and H/N is RLA−local. □

Having introduced RLA−local hypermodules, we now define the supplement condition that is the central object of this paper.

Definition 1.5. Let H be a Krasner left R-hypermodule and U, V subhypermodules of H. We call V a *locally Artinian supplement* of U in H if V is a supplement of U in H (i.e. $U + V = H$ and $U \cap V \ll V$) and $U \cap V$ is locally Artinian.

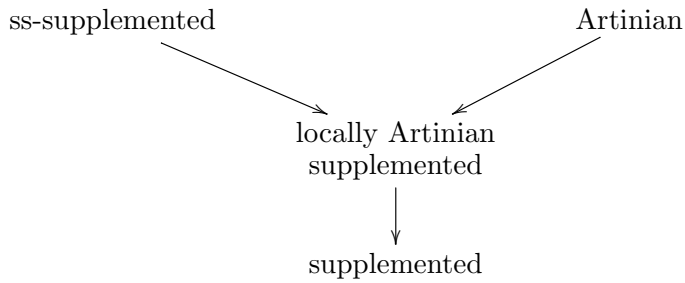
The hypermodule H is called *locally Artinian supplemented* if every subhypermodule of H has a locally Artinian supplement in H. A subhypermodule U of H has *ample locally Artinian supplements* in H if every subhypermodule L with $H = U + L$ contains a locally Artinian supplement of U in H. The hypermodule H is called *amply locally Artinian supplemented* if every subhypermodule of H has ample locally Artinian supplements in H.

Remark 1.6. Since $\mathrm{Soc}_s(V) \subseteq \mathrm{Rad}(V)$ and locally Artinian hypermodules need not be semisimple, the locally Artinian supplement condition is strictly weaker than the ss-supplement condition ($U \cap V \subseteq \mathrm{Soc}_s(V)$). Similarly, every ss-supplemented hypermodule is locally Artinian supplemented, but the converse fails; see Example 1.7 below.

Before investigating properties of locally Artinian supplemented hypermodules, we present counterexamples separating the classes.

Example 1.7. For a prime $p \in \mathbb{Z}$, consider the $\mathbb{Z}$-hypermodule $H = \mathbb{Z}_{p^\infty} = \mathbb{Z}[1/p]/\mathbb{Z}$ as the Prüfer p-group. Since every proper subhypermodule of H is finite cyclic, then H is Artinian. Hence H is locally Artinian supplemented. However, H is not ss-supplemented by [12], since the unique simple subhypermodule $\mathbb{Z}_p \subseteq H$ is not small in H, $\mathrm{Soc}_s(H) = 0$. Hence H is not semisimple.

The implications among the main classes studied in this paper are therefore:



None of the arrows above reverses in general.

The next result shows that locally Artinian supplemented hypermodules arise from supplemented hypermodules whose radicals satisfy a finiteness condition. This gives a criterion for identifying locally Artinian supplemented hypermodules without verifying the supplement condition for every subhypermodule.

Lemma 1.8. *Let H be a supplemented hypermodule such that* $\mathrm{Rad}(H)$ *is locally Artinian. Then H is locally Artinian supplemented.*

Proof. Let $U \leq H$. Since H is supplemented, there exists $K \leq H$ with $H = U + K$ and $U \cap K \ll K$. As $U \cap K \ll K \leq H$, we have $U \cap K \ll H$, hence $U \cap K \leq \mathrm{Rad}(H)$. Since every subhypermodule of a locally Artinian hypermodule is locally Artinian, then any finitely generated subhypermodule of $U \cap K$ is a finitely generated subhypermodule of $\mathrm{Rad}(H)$. Therefore $U \cap K$ is locally Artinian. Therefore K is a locally Artinian supplement of U in H. □

Under the additional hypothesis that $\mathrm{Rad}(H) \ll H$, several conditions on H become equivalent. The following theorem clarifies the relationship between supplementedness, the radical, and the locally Artinian condition. Note that in the statement $\mathrm{Rad}(H) \ll H$ holds, in particular, when H is a normal projective hypermodule [19].

Theorem 1.9. *Let H be a Krasner left R-hypermodule with* $\mathrm{Rad}(H) \ll H$. *Then the following statements are equivalent:*

(1) H is locally Artinian supplemented;

(2) H is supplemented and $\mathrm{Rad}(H)$ *has a locally Artinian supplement in H;*

(3) H is supplemented and $\mathrm{Rad}(H)$ *is locally Artinian.*

Proof. (1) ⇒ (2): Since H is locally Artinian supplemented, it is supplemented, and $\mathrm{Rad}(H)$ has a locally Artinian supplement in H by Definition 1.5.

(2) ⇒ (3): Since $\mathrm{Rad}(H) \ll H$, the hypermodule H itself is a supplement of $\mathrm{Rad}(H)$ in H. So $\mathrm{Rad}(H) + H = H$ and $\mathrm{Rad}(H) \cap H = \mathrm{Rad}(H) \ll H$. Since H is a locally Artinian supplement of $\mathrm{Rad}(H)$, we have $\mathrm{Rad}(H)$ is locally Artinian.

(3) ⇒ (1): This follows from Lemma 1.8. □

Remark 1.10. Theorem 1.9 shows that, when $\mathrm{Rad}(H) \ll H$, the locally Artinian supplemented condition reduces entirely to the supplemented condition together with a mild finiteness assumption on the radical. This is a useful reduction because the condition $\mathrm{Rad}(H) \ll H$ is exist in many important cases, such as for normal projective and finitely generated hypermodules.

The following theorem shows that RLA−local effects not merely the condition of locally Artinian supplements, but the condition of ample locally artinian supplements.

Theorem 1.11. *Every RLA−local hypermodule is amply locally Artinian supplemented.*

Proof. Let H be RLA−local and let $U, V \leq H$ with $H = U + V$. Since H is local it is amply supplemented [19], so there exists $V' \leq V$ with $H = U + V'$ and $U \cap V' \ll V'$. Then $U \cap V' \leq \mathrm{Rad}(V') \leq \mathrm{Rad}(H)$. Since H is RLA−local, $\mathrm{Rad}(H)$ is locally Artinian, and so $U \cap V'$ is locally Artinian as a subhypermodule of $\mathrm{Rad}(H)$. Hence V' is a locally Artinian supplement of U contained in V, and H is amply locally Artinian supplemented. □

When a supplement is formed with respect to a maximal subhypermodule, the structure simplifies: the supplement is necessarily local, and the RLA−local condition characterizes exactly when it is a locally Artinian supplement.

Proposition 1.12. *Let H be a Krasner left R-hypermodule and U a maximal subhypermodule of H. A subhypermodule V of H is a locally Artinian supplement of U in H if and only if $H = U + V$ and V is an RLA−local hypermodule.*

Proof. (⇒): Let V be a locally Artinian supplement of U in H. Then $H = U + V$, $U \cap V \ll V$, and $U \cap V$ is locally Artinian. Since U is maximal

in H and V is a supplement of U, the hypermodule V is local. If U is maximal and $U+V=H$ with $U\cap V\ll V$, then $V/(U\cap V)\cong H/U$ is simple, so $U\cap V$ is a maximal subhypermodule of V, and since $U\cap V\ll V$ this ensures V to be local. Moreover $\mathrm{Rad}(V)=U\cap V$, which is locally Artinian by assumption. Hence V is RLA$-$local.

($\Leftarrow$): Suppose $H=U+V$ and V is RLA$-$local. Then V is local and $\mathrm{Rad}(V)$ is locally Artinian. Since U is a maximal subhypermodule of H and V is local, we have $U\cap V\leq \mathrm{Rad}(V)$. In particular $U\cap V$ is locally Artinian. Moreover $U\cap V\ll V$ since $U\cap V\leq \mathrm{Rad}(V)\ll V$. Hence V is a locally Artinian supplement of U in H. □

We now turn to the stability of locally Artinian supplemented hypermodules under finite sums. The following lemma is the key step; it is an adaptation of the supplement lemma to the locally Artinian supplemented hypermodules.

Lemma 1.13. *Let H be a Krasner left R-hypermodule and U,V subhypermodules of H with U locally Artinian supplemented. If $U+V$ has a locally Artinian supplement K in H, then U also has a locally Artinian supplement in H.*

Proof. Let K be a locally Artinian supplement of $U+V$ in H. Then $H=(U+V)+K$, $(U+V)\cap K\ll K$, and $(U+V)\cap K$ is locally Artinian. Consider the subhypermodule $(K+V)\cap U\leq U$. Since U is locally Artinian supplemented, there exists a locally Artinian supplement L of $(K+V)\cap U$ in U: $U=[(K+V)\cap U]+L$, $(K+V)\cap L\ll L$ and $(K+V)\cap L$ is locally Artinian.

We claim that $K+L$ is a locally Artinian supplement of U in H. First:

$$H=(U+V)+K=[(K+V)\cap U]+L+V+K=V+(K+L),$$

and then $U+(K+L)\supseteq U+K\supseteq H$, so $H=U+(K+L)$. We obtain $U\cap(K+L)\ll K+L$. We have

$$\begin{aligned} U\cap(K+L) &\subseteq [K\cap(U+V)]+[L\cap(K+V)]\\ &\subseteq [K\cap(U+V)]+[L\cap(K+V)]. \end{aligned}$$

Since $(U+V)\cap K\ll K\leq K+L$ and $(K+V)\cap L\ll L\leq K+L$, both summands are small in $K+L$, so $U\cap(K+L)\ll K+L$. Finally, $U\cap(K+L)$ is locally Artinian as a subhypermodule of the sum of two locally Artinian hypermodules $(U+V)\cap K$ and $(K+V)\cap L$, both of

which are locally Artinian. Hence $K+L$ is a locally Artinian supplement of U in H. □

Proposition 1.14. *Let $H = U + V$ where U and V are locally Artinian supplemented subhypermodules of H. Then H is locally Artinian supplemented.*

Proof. Let $W \leq H$. Since $H = U + V + W$, the subhypermodule $\{0_H\}$ is a locally Artinian supplement of H in H. Since V is locally Artinian supplemented by Lemma 1.13, we obtain that $U + W$ has a locally Artinian supplement in H. By Lemma 1.13 again with U as above and V replaced by W, we conclude that W has a locally Artinian supplement in H. Since W was arbitrary, H is locally Artinian supplemented. □

Corollary 1.15. *Every finite sum of locally Artinian supplemented subhypermodules is locally Artinian supplemented.*

Proof. Proof follows from Proposition 1.14. □

The following example illustrates that the locally Artinian supplemented property is not always inherited by the hypermodule.

Example 1.16. The $\mathbb{Z}$-hypermodule $R = \mathbb{Z}$ is not locally Artinian supplemented: indeed $\mathbb{Z}$ is not locally Artinian (the subhypermodule $\mathbb{Z}$ itself is finitely generated but not Artinian), and not every subhypermodule of $\mathbb{Z}$ has a supplement. Consider the canonical projection $\phi : \mathbb{Z} \to \mathbb{Z}_5$. Since the hypermodule $\mathbb{Z}_5$ is an Artinian $\mathbb{Z}$-hypermodule, so it is locally Artinian supplemented. This shows that a factor hypermodule of a non-locally-Artinian-supplemented hypermodule may be locally Artinian supplemented.

We now prove that the locally Artinian supplemented property descends to factor hypermodules. This is an important closure result because it shows the class is compatible with the quotient construction in hypermodule theory.

Proposition 1.17. *If H is (amply) locally Artinian supplemented, then every factor hypermodule of H is (amply) locally Artinian supplemented.*

Proof. Let $N \leq H$ and $\pi : H \to H/N$ the canonical normal homomorphism. Let $\bar{U} \leq H/N$, so $\bar{U} = U/N$ for some $U \leq H$ containing N. Since

H is locally Artinian supplemented, there exists $V \leq H$ with $H = U+V$, $U \cap V \ll V$, and $U \cap V$ locally Artinian. Then

$$\frac{H}{N} = \frac{U}{N} + \frac{V+N}{N}, \quad \frac{U}{N} \cap \frac{V+N}{N} = \frac{(U \cap V)+N}{N} = \pi(U \cap V).$$

Since $U \cap V \ll V$, we have $\pi(U \cap V) \ll \pi(V) = \frac{V+N}{N}$. Since $U \cap V$ is locally Artinian, so is $\pi(U \cap V)$ (for the same reason as in the proof of Proposition 1.4). Therefore $\frac{V+N}{N}$ is a locally Artinian supplement of $\frac{U}{N}$ in H/N, proving that H/N is locally Artinian supplemented. The ample case follows analogously, following the inclusion $\pi(V') \subseteq \pi(V)$ when $V' \subseteq V$ is a locally Artinian supplement of U contained in V. □

We will study the hypercompositional structures of amply locally Artinian supplemented hypermodules in this part.

Proposition 1.18. *Let H be a Krasner left R-hypermodule. If every subhypermodule of H is locally Artinian supplemented, then H is amply locally Artinian supplemented.*

Proof. Let $K, L \leq H$ with $H = K + L$. Since L is locally Artinian supplemented by hypothesis, there exists $L' \leq L$ such that $L = (K \cap L) + L'$, $(K \cap L) \cap L' \ll L'$, and $(K \cap L) \cap L'$ is locally Artinian. By the modular law, $H = K + L = K + (K \cap L) + L' = K + L'$. Furthermore, $K \cap L' \leq (K \cap L) \cap L' \ll L'$, so $K \cap L' \ll L'$ and $K \cap L'$ is locally Artinian. Hence L' is a locally Artinian supplement of K contained in L, showing that K has ample locally Artinian supplements. Since K was arbitrary, H is amply locally Artinian supplemented. □

Lemma 1.19. *Let H be an amply locally Artinian supplemented hypermodule and V a supplement subhypermodule of H. Then V is amply locally Artinian supplemented.*

Proof. Let V be a supplement of some $U \leq H$. Let $X, Y \leq V$ with $V = X + Y$. Then $H = (U + X) + Y$. Since H is amply locally Artinian supplemented, there exists a locally Artinian supplement $Y' \leq Y$ of $U + X$ in H: $H = (U + X) + Y'$, $(U + X) \cap Y' \ll Y'$, and $(U + X) \cap Y'$ is locally Artinian. Since $Y' \leq V$ and $X + Y' \leq V$ while $H = U + (X + Y')$, the minimality of V as a supplement of U yields $V = X + Y'$. Moreover, $X \cap Y' \leq (U+X) \cap Y' \ll Y'$, so $X \cap Y' \ll Y'$, and $X \cap Y'$ is locally Artinian as a subhypermodule of $(U + X) \cap Y'$. Therefore Y' is a locally Artinian supplement of X in V, and V is amply locally Artinian supplemented. □

The next result characterizes amply locally Artinian supplemented hypermodules through an internal decomposition of their subhypermodules. This decomposition is the hypermodule analogue of a result for amply supplemented modules.

Proposition 1.20. *Let H be a Krasner left R-hypermodule. Then H is amply locally Artinian supplemented if and only if every subhypermodule U of H can be written as $U = K + L$, where K is a locally Artinian supplement subhypermodule of H and $L \ll H$ is locally Artinian.*

Proof. ($\Rightarrow$): Let $U \leq H$. Since H is locally Artinian supplemented, U has a locally Artinian supplement V in H: $H = U + V$ and $U \cap V \ll V$. By the ample condition applied to V, there exists a locally Artinian supplement $K \leq U$ of V in H. Set $L = U \cap V$. By the modular law, $U = U \cap H = U \cap (K+V) = K+(U \cap V) = K+L$. The subhypermodule K is a locally Artinian supplement of V in H, and is amply locally Artinian supplemented by Lemma 1.19. Since $L = U \cap V \ll V$, we have $L \ll H$ (for if $L + W = H$ for some $W \leq H$, then $V + W = H + W = H$ and since V is a supplement this effects standard arguments; cf. the proof of Lemma 1.13). So $L \leq U \cap V$ and $(U+V) \cap K \ll K$ was locally Artinian.

($\Leftarrow$): Let $U, S \leq H$ with $H = U + S$. Write $U = K + L$ as in the hypothesis. Since K is a supplement subhypermodule and $L \ll H$, we have $H = U + S = K + L + S = K + S$ by $L \ll H$. Then $K \cap S \leq (K + L) \cap S = U \cap S$, and the locally Artinian supplement of U in H is obtained by refining K inside S using the supplement condition for K. The details follow the argument of Lemma 1.13. □

We now develop new results connecting essential subhypermodules with locally Artinian supplements. The essentiality context for hypermodules was established in [18], and we use it here to obtain results that have no direct analogue in the earlier literature on locally Artinian supplemented hypermodules.

Definition 1.21 ([18]). A subhypermodule N of a Krasner left R-hypermodule H is called *essential* in H, written $N \trianglelefteq H$, if $N \cap K \neq \{0_H\}$ for every non-zero subhypermodule K of H.

The following lemma captures the interaction between smallness and essentiality in the locally Artinian supplemented hypermodules.

Lemma 1.22. *Let H be a locally Artinian supplemented hypermodule and $N \leq H$. If $N \ll H$ and N is locally Artinian, then every essential*

subhypermodule of H meets every supplement subhypermodule of H non-trivially in N.

Proof. Let $E \trianglelefteq H$ be essential in H, and let V be a supplement of some $U \leq H$ in H. We have $U \cap V \ll V$. Suppose $E \cap V = \{0_H\}$. Then since $E \trianglelefteq H$ and $V \leq H$, we have $V = \{0_H\}$ by Definition 1.21. But then $H = U + V = U$, meaning $U = H$, and the supplement condition becomes vacuous. In the non-trivial case $V \neq \{0_H\}$, we necessarily have $E \cap V \neq \{0_H\}$. This intersection lies in V and, since $U \cap V$ is locally Artinian, the finitely generated subhypermodules of $E \cap (U \cap V)$ are Artinian. □

Theorem 1.23. *Let H be a locally Artinian supplemented hypermodule. If $E \trianglelefteq H$ is essential in H and E is locally Artinian, then E contains a locally Artinian supplement of some maximal subhypermodule of H.*

Proof. Since $E \trianglelefteq H$ and H is locally Artinian supplemented, for the subhypermodule E itself there exists a locally Artinian supplement V of E in H: $H = E+V$, $E\cap V \ll V$, and $E\cap V$ is locally Artinian. We have $E \cap V \ll V$ and $E \trianglelefteq H$, so if $V = \{0_H\}$ then $H = E$ and the statement is trivial. Assume $V \neq \{0_H\}$. Then $E \cap V \neq \{0_H\}$ by essentiality.

Let U be a maximal subhypermodule of H containing $E \cap V$. Consider the locally Artinian supplement W of U in H provided by the locally Artinian supplemented assumption. Since $U \supseteq E\cap V$ and $E \trianglelefteq H$, the subhypermodule W intersects E non-trivially by Lemma 1.22. In particular $E \cap W$ is a non-zero locally Artinian subhypermodule as a subhypermodule of E, and W is the required locally Artinian supplement of the maximal subhypermodule U contained in H. The locally Artinian property of E provides that $E \cap W$ is locally Artinian. □

Corollary 1.24. *Let H be an RLA−local hypermodule. Then the intersection between every essential locally Artinian subhypermodule of H and* $\mathrm{Rad}(H)$ *in a locally Artinian subhypermodule.*

Proof. Let $E \trianglelefteq H$ be an essential and locally Artinian subhypermodule of H. Since H is RLA−local, $\mathrm{Rad}(H)$ is locally Artinian. As $E \trianglelefteq H$, we have $E \cap \mathrm{Rad}(H) \neq \{0_H\}$. Unless $\mathrm{Rad}(H) = \{0_H\}$, in which case the result is trivial. The intersection $E \cap \mathrm{Rad}(H)$ is a subhypermodule of E and hence locally Artinian. □

We now investigate the role of projectivity in the locally Artinian supplemented hypermodules. The following result shows that for normal

π-projective hypermodules, the locally Artinian supplemented and amply locally Artinian supplemented conditions coincide, so that projectivity eliminates the distinction between the basic and ample versions.

Proposition 1.25. *Let H be a normal π-projective hypermodule. Then H is locally Artinian supplemented if and only if H is amply locally Artinian supplemented.*

Proof. ($\Leftarrow$): Clear, since amply locally Artinian supplemented implies locally Artinian supplemented.

($\Rightarrow$): Let $U, V \leq H$ with $H = U + V$. By the normal π-projective hypothesis applied to the projection $\pi : H \to H/U$, there exists a normal endomorphism $\varphi \in \mathrm{End}_R(H)$ such that $\varphi(H) \subseteq U$ and $(1-\varphi)(H) \subseteq V$. Let K be a locally Artinian supplement of U in H: $H = U+K$, $U \cap K \ll K$, and $U \cap K$ locally Artinian. Then

$$H = \varphi(H) + (1-\varphi)(H) \subseteq U + (1-\varphi)(H),$$

and since $(1-\varphi)(H) \subseteq V$, we get $H = U + (1-\varphi)(K)$ with $(1-\varphi)(K) \leq V$.

We claim $U \cap (1-\varphi)(K) = (1-\varphi)(U \cap K)$. ($\supseteq$): If $x \in U \cap K$, then $(1-\varphi)(x) = x - \varphi(x)$. Since $x \in U$ and $\varphi(x) \in \varphi(H) \subseteq U$, we have $(1-\varphi)(x) \in U$. Also $(1-\varphi)(x) \in (1-\varphi)(K)$. ($\subseteq$): If $y \in U \cap (1-\varphi)(K)$, then $y = (1-\varphi)(x) = x - \varphi(x)$ for some $x \in K$. Since $y \in U$ and $\varphi(x) \in U$, we get $x = y + \varphi(x) \in U$, so $x \in U \cap K$ and $y = (1-\varphi)(x) \in (1-\varphi)(U \cap K)$.

Therefore $U \cap (1-\varphi)(K) = (1-\varphi)(U \cap K)$, which is locally Artinian as the image of the locally Artinian module $U \cap K$ under the normal endomorphism $1-\varphi$. Moreover, $U \cap (1-\varphi)(K) \ll (1-\varphi)(K)$ since $U \cap K \ll K$ and $(1-\varphi)$ is a normal endomorphism. Hence $(1-\varphi)(K)$ is a locally Artinian supplement of U contained in V, and H is amply locally Artinian supplemented. □

Corollary 1.26. *Every normal projective locally Artinian supplemented hypermodule is amply locally Artinian supplemented.*

Proof. Every normal projective hypermodule is normal π-projective by definition [4], so the result follows from Proposition 1.25. □

Proposition 1.27. *Let H be a normal projective hypermodule. Then H is locally Artinian supplemented if and only if H is supplemented and* $\mathrm{Rad}(H)$ *is locally Artinian.*

Proof. Since H is normal projective, $\mathrm{Rad}(H) \ll H$ by [19]. The conclusion then follows directly from Theorem 1.9. □

We close this section with three theorems that extend the hyperstructural theory of locally Artinian supplemented hypermodules.

Theorem 1.28. *Let H be an amply locally Artinian supplemented hypermodule and V a direct summand of H. Then V is amply locally Artinian supplemented.*

Proof. Write $H = V \oplus W$ for some $W \leq H$. Then V is a supplement of W in H with $V \cap W = \{0_H\} \ll V$. By Lemma 1.19, V is amply locally Artinian supplemented. □

Theorem 1.29. *Let H be a finitely generated Krasner left R-hypermodule over an Artinian Krasner hyperring R. Then H is locally Artinian supplemented.*

Proof. Since R is an Artinian hyperring and H is finitely generated, H is an Artinian hypermodule as every descending chain of subhypermodules of H corresponds to a chain of hyperideals of R via the annihilator, and the Artinian condition on R propagates to H by the finitely generated assumption; see [21]. An Artinian hypermodule is supplemented [19], and by Theorem 1.9 it is locally Artinian supplemented since $\mathrm{Rad}(H)$ is a subhypermodule of the Artinian, hence H is a locally Artinian hypermodule. □

Theorem 1.30. *Let H be a Krasner left R-hypermodule and $N \leq H$. If both N and H/N are locally Artinian supplemented and $N \ll H$, then H is locally Artinian supplemented.*

Proof. Let $U \leq H$. Since H/N is locally Artinian supplemented, there exists a subhypermodule $V/N \leq H/N$ for $V \leq H$ with $N \leq V$ such that $H/N = (U+N)/N + V/N$, $(U+N)/N \cap V/N \ll V/N$, and $(U+N)/N \cap V/N$ is locally Artinian. Then we have H: $H = U + N + V = U + V$ as $N \ll H$. So $(U + N) \cap V \ll V$, and $(U + N) \cap V$ is a locally Artinian subhypermodule of V.

Since $N \ll H$, we have $U \cap V \leq (U + N) \cap V$ and $N \cap V \ll V$. Thus $U \cap V \ll V$. It remains to show $U \cap V$ is locally Artinian. Consider the exact sequence

$$0 \to N \cap (U \cap V) \to U \cap V \to (U \cap V)/(N \cap (U \cap V)) \to 0.$$

The quotient embeds into $(U + N) \cap V/(N \cap V)$, which is locally Artinian. The subhypermodule $N \cap (U \cap V) \leq N$ is locally Artinian since N is locally Artinian supplemented and every subhypermodule of a locally Artinian supplemented hypermodule with $\mathrm{Rad}(N) \ll N$ is locally Artinian by Theorem 1.9. Since the class of locally Artinian hypermodules is closed under extensions [19], $U \cap V$ is locally Artinian. Therefore V is a locally Artinian supplement of U in H, and H is locally Artinian supplemented. □

Conclusion

In this paper we have introduced and studied two new classes of Krasner left R-hypermodules: *locally Artinian supplemented hypermodules* and *RLA−local hypermodules*. These classes generalize both the strongly local and ss-supplemented Krasner hypermodules introduced in [12], as well as locally Artinian supplemented modules, to the hypercompositional structure.

The principal structural results established in this paper are as follows. Every RLA-local hypermodule is amply locally Artinian supplemented (Theorem 1.11), and the converse holds when additional projectivity conditions are imposed (Proposition 1.25). Locally Artinian supplemented hypermodules are characterized by the combination of supplementedness and a locally Artinian radical, provided $\mathrm{Rad}(H) \ll H$ (Theorem 1.9). The class of locally Artinian supplemented hypermodules is closed under factor hypermodules (Proposition 1.17) and finite sums (Corollary 1.15). For normal projective hypermodules, the locally Artinian supplemented and amply locally Artinian supplemented conditions coincide (Proposition 1.25).

New results on the interaction between essential subhypermodules and locally Artinian supplements were investigated in Theorem 1.23 and Corollary 1.24, developing on the essentiality context of [18]. The extension closure result (Theorem 1.30) and the direct summand result (Theorem 1.28) complete the structural theory.

Several open problems remain. The characterization of Krasner hyperrings R over which every Krasner left R-hypermodule is locally Artinian supplemented is an open problem. It is also open whether infinite direct sums of locally Artinian supplemented hypermodules are locally Artinian supplemented. Finally, the relationship between RLA-local hyperrings and their hyperideals awaits further investigation.

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