© Algebra and Discrete Mathematics
Volume **41** (2026). Number 1, pp. 120–155
DOI:10.12958/adm2457

RESEARCH ARTICLE

Derivations and biderivations in dialgebras

Gabriel Gustavo Restrepo-Sánchez, José Gregorio Rodríguez-Nieto, Olga Patricia Salazar-Díaz, Andrés Sarrazola-Alzate, and Raúl Velásquez

Communicated by A. Zhuchok

Abstract. The concepts of derivations and antiderivations for Leibniz algebras naturally arise from the inner operators determined by their algebraic structure. In this paper, we introduce the corresponding analogues in the setting of dialgebras, which we call *diderivations*, and examine their structural properties in relation to classical derivations and multiplicative operators. Our approach is based on the study of left and right multiplication operators and on the construction of the Leibniz algebra generated by biderivations, thereby providing a systematic operator-theoretic framework that unifies several derivation-like structures. In addition to the general theory, we present a complete classification of the spaces of diderivations for dialgebras of dimensions two and three, obtained through explicit computations. These low-dimensional results not only illustrate the general constructions but also reveal structural patterns that inform possible extensions to higher dimensions and more intricate algebraic contexts.

The authors Andrés Sarrazola-Alzate and José Gregorio Rodríguez-Nieto would like to thank the Institute of Mathematics and Statistics, Department of Mathematics, of the Federal University of Bahia (Salvador, Bahia, Brazil) for their warm hospitality during their research visit in 2025. The last author was supported by CODI-UdeA grant 2024-77008, project Categorías y módulos sobre diálgebras.

2020 Mathematics Subject Classification: *17A32, 17A36.*

Key words and phrases: *dialgebras, derivations, diderivations, Leibniz algebras, algebraic classification.*

Introduction

The study of derivations has long played a central role in algebra, not only because they describe intrinsic symmetries of a structure, but also because they govern its deformations and extensions. In the framework of Leibniz algebras, derivations and antiderivations arise naturally from the adjoint operators determined by the bracket. Their interaction leads to the construction of the Leibniz algebra of biderivations, which contains the subalgebra of inner biderivations as a fundamental component. A similar picture appears in the category of K-B quasi-Jordan algebras, where derivation-type operators induced by multiplicative structures have also been investigated [13]. In both settings, inner operators play a structural role, serving as the basic building blocks for the corresponding operator algebras.

Dialgebras, introduced by J.-L. Loday in the early 1990s, offer a natural environment to broaden these ideas. Defined by two bilinear operations tied together by compatibility relations, dialgebras stand at a crossroads between associative and Leibniz structures: the commutator of their operations recovers precisely a Leibniz algebra. This fact situates dialgebras as a universal enveloping context in which Leibniz algebras emerge, much in the same way that associative algebras underpin Lie theory. As a result, dialgebras provide fertile ground for investigating cohomological invariants and homological phenomena, not only in the study of digroups and g-digroups, viewed as subclasses of the variety of dimonoids (see, for example, [18–21]), but also in derivations. The role of derivations within dialgebras is particularly crucial, since they capture symmetries and encode the mechanisms by which such algebras deform and interact with one another. It is important to emphasize that the concept of diassociativity, which serves as the foundation for structures such as digroups, g-digroups, dimonoids, and dialgebras, has been systematically developed in the works of A. V. Zhuchok since 2008 (see, e.g., [16, 17]).

Yet, in contrast with Leibniz algebras, the operator theory of dialgebras remains largely undeveloped. To date, only the classical notion of derivations compatible with both dialgebra products has been systematically studied, as originally formulated by Loday. The work of Lin and Zhang [6] constitutes a first step in this direction, focusing on derivations of the polynomial dialgebra $K[x, y]$. However, no general framework has been developed to incorporate operator structures analogous to antiderivations or to those arising from right multiplication in the Leibniz setting. This gap in the literature naturally raises the question: can dialgebras sustain operator structures as rich and structurally coherent as those known for Leibniz algebras?

The present article aims to take a step in this direction. On the one hand, we introduce and study a special type of derivations in dialgebras, namely those defined by multiplicative operators – the inner derivations – and extend the framework to incorporate a new operator class, which we call *diderivations*. On the other hand, we go beyond their mere definition to examine how derivations and diderivations act on some of the most intrinsic invariants of dialgebra theory: the center, the annihilator, and the bar-unit set.

In this sense, our study highlights the place of dialgebras as a meeting point where familiar algebraic ideas – derivations, inner structures, operator algebras – are reinterpreted and extended. At the same time, the introduction of diderivations opens up new paths of exploration, bridging known constructions in Leibniz algebras with novel phenomena in the dialgebraic world. We hope that these contributions will not only enrich the theory of dialgebras, but also foster further dialogue between algebraic frameworks where symmetries and derivations are at the core.

Organization of the paper

This paper is organized as follows. Section 1 is devoted to recalling the basic background on Leibniz algebras and their close relationship with dialgebras. Special emphasis is placed on the role of derivations, antiderivations, and biderivations, as these notions will be fundamental in the constructions developed later. We also revisit the seminal work of Loday [7], which established the connection between Leibniz algebras and dialgebras, thereby providing the conceptual framework in which our results are situated.

In Section 2, for a dialgebra $\mathscr{D}$ (Definition 6), we introduce the notion of *diderivations*, understood as compatible operator structures obtained by combining antiderivations with multiplicative operators arising from the dialgebra products. Within this section, we establish two structural results of central importance: first, that the space of derivations $\mathcal{D}er(\mathscr{D})$ naturally forms a Lie subalgebra of $\mathfrak{gl}(\mathscr{D})$ (Lemma 5); and second, that the space of inner derivations $\mathscr{I}nn(\mathscr{D})$ constitutes an ideal inside $\mathcal{D}er(\mathscr{D})$ (Theorem 1). These results provide a solid algebraic foundation for the subsequent classification of diderivations.

Section 3 is concerned with the *explicit classification of the vector* *spaces of diderivations in low-dimensional cases.* More precisely, we present a complete classification for dialgebras of dimensions two and three. This extends and refines the computational approaches proposed in [1, 14], where algorithms for the determination of derivations were tested in the same range of dimensions. Our results therefore contribute to completing the picture of structural operators in small dialgebras, providing the first systematic description of diderivations in these cases.

In Section 4, we turn to the study of *biderivations.* We construct the algebra $\mathcal{B}ider(\mathcal{D})$ and prove that it inherits a nontrivial Leibniz structure. More precisely, we show that for any dialgebra $\mathcal{D}$, the pair

$$\big(\mathcal{B}ider(\mathcal{D}), \langle \bullet, \bullet \rangle\big)$$

is canonically endowed with the structure of a Leibniz algebra (Theorem 5). This result highlights the robustness of the interaction between dialgebraic operators and Leibniz structures, and situates biderivations as natural algebraic objects.

The present work may be viewed as a dialgebraic counterpart of the program initiated by Mancini in [10], where biderivations of low-dimensional Leibniz algebras were classified and their Leibniz algebra structure was investigated. While the case of Leibniz algebras involves a single non-antisymmetric product, dialgebras possess two interacting products, leading to additional compatibility conditions and to the notion of diderivations introduced here.

In contrast with the algorithmic approaches developed in [4], our contribution is primarily structural: we establish that the space of derivations forms a Lie subalgebra, that inner derivations form an ideal, and that the enlarged space of biderivations naturally acquires a Leibniz algebra structure. Thus, the present paper extends the operator-theoretic framework of Leibniz algebras to the richer setting of dialgebras.

In Section 5, we present a concrete example of a dialgebra and provide the classification of both its derivations and diderivations (Theorem 9 and Lemma 11). More precisely, we analyze the dialgebra introduced by L. Lin and Y. Zhang in [6], which is generated by polynomials in two commuting variables $K[x, y]$.

In Section 6, we summarize the main classification results, underline the implications of the Leibniz structure on biderivations, and indicate several directions for further research. Among these, we stress the potential extension of our classification methods to higher-dimensional dialgebras, as well as the exploration of homological and cohomological aspects of the algebras $\mathcal{D}er(\mathcal{D})$, $\mathcal{I}nn(\mathcal{D})$, and $\mathcal{B}ider(\mathcal{D})$.

The paper is complemented by two appendices that provide additional technical support for the low-dimensional classification results. Appendix Appendix 1 describes the symbolic computational framework used to solve the linear systems defining diderivations, while Appendix Appendix 2 contains a detailed case-by-case analysis of the compact dialgebra $Dias_3^{16}$. A central tool in this analysis is the systematic use of the Schur complement, which allows us to factor determinants, track rank drops, and explain the resulting jumps in the dimension of the space of diderivations in a transparent and structured way.

1. Preliminaries

In this section we recall the basic notions that will be used throughout the paper. We first introduce derivations in arbitrary algebras and then define Leibniz algebras from a structural perspective, emphasizing the role of adjoint operators. For general background on Leibniz algebras we refer to [7]. Throughout the paper, all vector spaces are defined over a field K of characteristic zero.

Derivations in arbitrary algebras

Let $\mathscr{A}$ be a K-vector space endowed with a bilinear product

$$[\cdot,\cdot] : \mathscr{A} \times \mathscr{A} \longrightarrow \mathscr{A}.$$

No symmetry condition is imposed on this product.

Definition 1. A **derivation** of A is a linear map $d : \mathscr{A} \to \mathscr{A}$ satisfying

$$d([x,y]) = [d(x),y] + [x,d(y)]$$

for all $x, y \in \mathscr{A}$.

Thus, a derivation is a linear operator satisfying the Leibniz rule with respect to the given bilinear product.

Adjoint operators

For each $x \in \mathscr{A}$, define the linear maps

$$L_x : \mathscr{A} \to \mathscr{A}, \qquad L_x(y) := [x,y],$$

$$R_x : \mathscr{A} \to \mathscr{A}, \qquad R_x(y) := [y,x].$$

These are called the **left** and **right adjoint operators** associated with x.

The derivation properties of these adjoint maps determine important algebraic structures.

Leibniz algebras

Definition 2. A **left Leibniz algebra** is a K-vector space $\mathscr{L}$ equipped with a bilinear product $[\cdot,\cdot]$ such that, for every $x \in \mathscr{L}$, the left adjoint operator L_x is a derivation of $\mathscr{L}$.

Expanding the derivation condition for L_x, we see that this requirement is equivalent to the identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

for all $x, y, z \in \mathscr{L}$. This identity is known as the *left Leibniz identity.*

Similarly, one defines a **right Leibniz algebra** by requiring that each right adjoint operator R_x be a derivation.

In this paper, the term *Leibniz algebra* will always mean *left Leibniz algebra.*

Remark 1. If the bracket is skew-symmetric, then the Leibniz identity reduces to the Jacobi identity. Hence every Lie algebra is a Leibniz algebra. In this sense, Leibniz algebras generalize Lie algebras by retaining the derivation property of adjoint operators without assuming skew-symmetry.

Derivations, antiderivations and biderivations

Definition 3. Let $\mathscr{L}$ be a Leibniz algebra.

A **derivation** of $\mathscr{L}$ is a linear map $d : \mathscr{L} \to \mathscr{L}$ satisfying

$$d([x,y]) = [d(x),y] + [x,d(y)]$$

for all $x, y \in \mathscr{L}$.

An **antiderivation** of $\mathscr{L}$ is a linear map $D : \mathscr{L} \to \mathscr{L}$ such that

$$D([x,y]) = [D(x),y] - [D(y),x]$$

for all $x, y \in \mathscr{L}$.

Remark 2. If $\mathscr{L}$ is a Lie algebra, then derivations and antiderivations coincide.

Definition 4. A **biderivation** of a Leibniz algebra $\mathscr{L}$ is a pair (d, D) consisting of a derivation d and an antiderivation D such that

$$[x, d(y)] = -[x, D(y)]$$

for all $x, y \in \mathscr{L}$.

Straightforward calculations prove the following results.

Proposition 1. *Let $\mathscr{L}$ be a Leibniz algebra. For each $x \in \mathscr{L}$, the operator L_x is a derivation, the operator R_x is an antiderivation, and the pair*

$$(R_x, L_x)$$

is a biderivation of $\mathscr{L}$.

Definition 5. A biderivation of the form (R_x, L_x) is called an **inner biderivation** associated with x. We denote by $\mathscr{I}nn(\mathscr{L})$ the set of all inner biderivations.

The Leibniz algebra of biderivations

Let $\mathscr{B}ider(\mathscr{L})$ denote the set of all biderivations of $\mathscr{L}$. For (d, D) and (d', D') in $\mathscr{B}ider(\mathscr{L})$, define

$$[(d, D), (d', D')] := (dd' - d'd,\ Dd' - d'D).$$

Proposition 2. *With the above bracket, $\mathscr{B}ider(\mathscr{L})$ is a Leibniz algebra.*

Moreover, the map

$$\begin{array}{ccc} \mathscr{L} & \longrightarrow & \mathscr{B}ider(\mathscr{L}) \\ x & \longmapsto & (R_x, L_x) \end{array}$$

is a homomorphism of Leibniz algebras.

In particular, $\mathscr{I}nn(\mathscr{L})$ is a Leibniz subalgebra of $\mathscr{B}ider(\mathscr{L})$.

Definition 6. A **diassociative algebra**, or a **dialgebra** for short, is a K-vector space $\mathscr{D}$ endowed with two associative bilinear operations

$$\vdash : \mathscr{D} \otimes \mathscr{D} \to \mathscr{D} \quad \text{and} \quad \dashv : \mathscr{D} \otimes \mathscr{D} \to \mathscr{D}$$

satisfying the following properties:

$$x \dashv (y \dashv z) = x \dashv (y \vdash z), \tag{1}$$

$$(x \dashv y) \vdash z = (x \vdash y) \vdash z, \tag{2}$$

$$x \vdash (y \dashv z) = (x \vdash y) \dashv z \tag{3}$$

for all $x, y, z \in \mathscr{D}$.

Remark 3. In general, the identity $x \dashv (y \vdash z) = (x \dashv y) \vdash z$ does not hold, in contrast to the equality given in equation (3) of the previous definition.

Definition 7. We say that $e \in \mathscr{D}$ is a bar unit if $e \vdash x = x \dashv e = x$ for all $x \in \mathscr{D}$.

A bar unit is not necessarily unique. The set of all bar units is called the **halo** and is denoted by $\mathscr{H}(\mathscr{D})$. A **unital dialgebra** is a dialgebra with a fixed bar unit e. Since from equation (3) if $e \dashv x = x$ or $x \vdash e = x$, for $x \in \mathscr{D}$, then $\vdash = \dashv$, and therefore $\mathscr{D}$ is an associative algebra, we will always consider units on the bar side.

Example 1 (Dialgebras from differentiable algebras)**.** Let $(\mathcal{A}, d)$ be a differential associative algebra, that is, an associative algebra $\mathcal{A}$ together with a linear map $d : \mathcal{A} \to \mathcal{A}$ such that $d(ab) = d(a)b + ad(b)$, for all $a, b \in \mathcal{A}$, and $d^2 = 0$. Defining over $\mathcal{A}$ the products $x \dashv y = xd(y)$ and $x \vdash y = d(x)y$, then $(\mathcal{A}, \vdash, \dashv)$ is a dialgebra. If the associative algebra $\mathcal{A}$ is unital, with unit 1, then $d(1) = 0$, which means that 1 is not a bar unit of $(\mathcal{A}, \vdash, \dashv)$. A bar unit in such dialgebra is an element $x \in \mathcal{A}$ such that $d(x) = 1$, but this element does not necessarily exist in $\mathcal{A}$.

Example 2 (φ-dialgebras)**.** Let V be a K-vector space and $\varphi \in V^* \setminus \{0\}$ a functional in the algebraic dual of V. There exists a dialgebra structure on V induced by φ ([11, Lemma 2.1]). This is defined by:

$$\nu \vdash \omega := \varphi(\nu)\omega \qquad \text{and} \qquad \nu \dashv \omega := \nu\varphi(\omega).$$

Such a dialgebra is called a φ-dialgebra and it is denoted by V_φ.

Example 3 (Dialgebras from group actions [13])**.** Let $f(x) \in K[x]$. Suppose that $f(x)$ is not irreducible with a root $\alpha \in K$, and let $g(x)$ be an irreducible factor of $f(x)$. Let us consider K_f and K_g the splitting fields of f and g, respectively, in such a way that we have a tower of extensions $K \subseteq K_g \subseteq K_f$. We assume $\mathscr{R}(g)$ is the set of roots of g and $\mathscr{R}_{K_f}(g)$ the K_f-vector space generated by $\mathscr{R}(g)$. The Galois group $\mathscr{G}al(K_g/K)$ acts transitively on $\mathscr{R}(g)$ and by the isomorphism extension theorem, the group $\mathscr{G}al(K_f/K)$ does it as well.

Let $V = Vect_{K_f}\langle \mathscr{R}(g), \alpha \rangle$. Then α is a fixed point and, by the previous paragraph, $\mathscr{G}al(K_f/K)$ acts transitively on $V \setminus \{\alpha\} = \mathscr{R}_{K_f}(g)$. By [5, Example 4.2] $\mathscr{D} := V \times \mathscr{G}al(K_f/K)$ carries the natural dialgebra structure:

$$(\nu, g) \vdash (\omega, h) := (g\omega, gh) \qquad \text{and} \qquad (\nu, g) \dashv (\omega, h) := (\nu, gh)$$

for all $\nu, \omega \in V$ and $h, g \in \mathcal{G}al(K_f/K)$.

We now introduce some structural aspects of dialgebras, as well as the notions of ideals and subdialgebras, which are analogous to those in classical algebraic structures.

Definition 8. A **two-sided ideal** $\mathcal{I}$ of a dialgebra $\mathcal{D}$ is a subspace of $\mathcal{D}$ such that $x \vdash y$ and $x \dashv y$ are in $\mathcal{I}$, wherever $x \in \mathcal{I}$ or $y \in \mathcal{I}$. It is clear that the quotient $\mathcal{D}/\mathcal{I}$ has a canonical dialgebra structure, and the kernel of any morphism of dialgebras is an ideal of the source dialgebra.

We say that $\mathcal{B}$ is a **subdialgebra** if $x \vdash y \in \mathcal{B}$ and $x \dashv y \in \mathcal{B}$, for all $x, y \in \mathcal{B}$.

Let us now review the definition of the annihilator of a dialgebra, which will play a fundamental role in this work.

Definition 9. Let $\mathcal{D}$ be a dialgebra. Let us consider the vector subspace generated by all the elements of the form $a \dashv b - a \vdash b$, for all $a, b \in \mathcal{D}$. We denote this subspace by $\mathcal{D}^{ann}$, i.e.,

$$\mathcal{D}^{ann} := \langle a \dashv b - a \vdash b |\, a, b \in \mathcal{D} \rangle$$

and we call it the **annihilator** of $\mathcal{D}$.

In [8], J.-L. Loday shows that $\mathcal{D}^{ann}$ is an ideal of $\mathcal{D}$. Moreover, the quotient dialgebra $\mathcal{D}_{as} := \mathcal{D}/\mathcal{D}^{ann}$ is actually an associative algebra, and $\mathcal{D}^{ann}$ is the smallest ideal that satisfies this property. On the other hand, if $\mathcal{A}$ is an associative algebra, then $\mathcal{A}^{ann} = (0)$ and therefore $(\mathcal{A}_{Di})_{as} = \mathcal{A}$, where $\mathcal{A}_{Di}$ is the canonical dialgebra induced by $\mathcal{A}$. Thus, we may conclude that the functor $Di : \mathbf{As} \to \mathbf{Dias}$, where $Di(\mathcal{A}) = \mathcal{A}_{Di}$, admits a left adjoint functor $as : \mathbf{Dias} \to \mathbf{As}$. In fact, let $\mathcal{D}$ be a dialgebra and let $\mathcal{A}$ be an associative algebra. The dialgebra $\mathcal{A}_{Di}$ associated to $\mathcal{A}$, is defined by the following operators:

$$x \dashv y = xy = x \vdash y \quad \text{for all } x, y \in \mathcal{A}.$$

Let $\varphi : \mathcal{D} \to \mathcal{A}_{Di}$ be a homomorphism of dialgebras. Since $\mathcal{A}_{Di}$ satisfies $x \dashv y = x \vdash y$, the image of φ must satisfy:

$$\varphi(x \dashv y - x \vdash y) = \varphi(x) \dashv \varphi(y) - \varphi(x) \vdash \varphi(y) = 0.$$

Thus, $\mathcal{D}^{ann} \subseteq \ker \varphi$. Then there exists a unique map $\hat{\varphi} : \mathcal{D}_{as} \to \mathcal{A}$ such that $\varphi = \iota \circ \hat{\varphi}$, where $\iota : \mathcal{D} \to \mathcal{D}_{as}$ is the canonical projection. In $\mathcal{D}_{as}$, the multiplication is given by $[x][y] := [x \dashv y] = [x \vdash y]$, for all $x, y \in \mathscr{D}$. Since φ is a dialgebra morphism, we have that

$$\begin{aligned} \hat{\varphi}([x][y]) &= \hat{\varphi}([x \dashv y]) = \varphi(x \dashv y) = \varphi(x) \dashv \varphi(y) \\ &= \varphi(x) \cdot \varphi(y) = \hat{\varphi}([x]) \cdot \hat{\varphi}([y]), \end{aligned}$$

so $\hat{\varphi}$ is an algebra homomorphism. This construction yields the map

$$\Phi : \mathrm{Hom}_{\mathbf{Dias}}(\mathscr{D}, \mathcal{A}_{Di}) \longrightarrow \mathrm{Hom}_{\mathbf{As}}(\mathscr{D}_{as}, \mathcal{A}), \quad \varphi \mapsto \hat{\varphi}.$$

Reciprocally, if $\psi : \mathscr{D}_{as} \to \mathcal{A}$ is an algebra homomorphism, then $\tilde{\psi} : \mathscr{D} \to \mathcal{A}_{Di}$ defined by

$$\tilde{\psi}(x) := \psi([x])$$

is a dialgebra homomorphism, this is because the operations in $\mathcal{A}_{Di}$ are defined identically as multiplications in $\mathcal{A}$. This produces the inverse map of Φ, so we have a bijection

$$\mathrm{Hom}_{\mathbf{Dias}}(\mathscr{D}, \mathcal{A}_{Di}) \cong \mathrm{Hom}_{\mathbf{As}}(\mathscr{D}_{as}, \mathcal{A}).$$

This natural bijection witnesses that the functor

$$as : \mathbf{Dias} \to \mathbf{As}, \quad \mathscr{D} \mapsto \mathscr{D}/\mathscr{D}^{ann}$$

is left adjoint to the functor $Di : \mathbf{As} \to \mathbf{Dias}$. See [8] for more details.

We now present an example of how dialgebras can be constructed using the concept of bimodules. Indeed, it can be shown that every dialgebra arises in this way. Thereby, dialgebras can be classified in terms of an associative algebra equipped with a bimodule structure. See [3] for further details.

Example 4. Let $\mathcal{A}$ be an associative algebra and $\mathcal{M}$ be a bimodule. If $f : \mathcal{M} \to \mathcal{A}$ is an $\mathcal{A}$-bimodular application, then we can define a dialgebra structure on $\mathcal{M}$ by declaring

$$m \dashv n := mf(n) \quad \text{and} \quad m \vdash n := f(m)n$$

for all $m, n \in \mathcal{M}$.

In general, we have the following proposition.

Proposition 3. *Let $\mathscr{D}$ be a dialgebra and $\mathscr{D}_{as}$ be its canonical associative algebra. There exist a $\mathscr{D}_{as}$-bimodule structure over $\mathscr{D}$ and a morphism of $\mathscr{D}_{as}$-bimodules $\varphi : \mathscr{D} \to \mathscr{D}_{as}$ such that the dialgebra structure of $\mathscr{D}$ can be recovered by defining $a \dashv b := a \cdot \varphi(b)$ and $a \vdash b = \varphi(a) \cdot b$.*

Finally, let $\mathcal{D}$ be a dialgebra. We will denote by $Z_B(\mathcal{D})$ the subspace:

$$Z_B(\mathcal{D}) \coloneqq \{z \in \mathcal{D} \mid z \vdash x = 0 = x \dashv z, \text{ for all } x \in \mathcal{D}\}.$$

We have the following results.

Lemma 1 ([15, P. 193]). *Let $\mathcal{D}$ be a dialgebra. Then*

(a) $\mathcal{D}^{ann}$ and $Z_B(\mathcal{D})$ are ideals of $\mathcal{D}$;

(b) $\mathcal{D} * Z_B(\mathcal{D}) \subseteq \mathcal{D}^{ann}$ and $Z_B(\mathcal{D}) * \mathcal{D} \subseteq \mathcal{D}^{ann}$.

Here $*$ denotes one of the products $\vdash$ or $\dashv$.

Lemma 2 ([15, Theorem 9 & Lemma 10]). *Let e be a bar unit of $\mathcal{D}$. Then*

$$\mathcal{D}^{ann} = Z_B(\mathcal{D}) = \{z \in \mathcal{D} \mid e \dashv z = 0 = z \vdash e\}$$

and

$$\mathcal{H}(\mathcal{D}) = \{e + x \mid x \in \mathcal{D}^{ann}\}.$$

The functorial relation between the category of associative algebras and the category of Lie algebras can be extended to the categories of dialgebras and Leibniz algebras. Actually, we have the following analogue of the canonical Lie bracket defined from an associative algebra in the context of dialgebras and Leibniz algebras.

Lemma 3 ([3]). *Let $\mathcal{D}$ be a dialgebra. The bilinear map*

$$[a, b] := a \dashv b - b \vdash a$$

defines over $\mathcal{D}$ a Leibniz algebra structure. This canonical Leibniz algebra associated to $\mathcal{D}$ will be denoted by $\mathcal{D}_{\mathcal{L}}$.

Remark 4. The Leibniz bracket defined in the previous lemma is skew-symmetric if $\mathcal{D}$ is an associative algebra.

From the previous lemma we have a functor $(\cdot)_{\mathcal{L}}$: **Dias** → **Leib** from the category **Dias** of dialgebras to the category **Leib** of Leibniz algebras. Considering the classical functor $(\cdot)_{Lie}$: **As** → **Lie**, from the category **As** of associative algebras to the category **Lie** of Lie algebras, we get the following commutative diagram of functors.

Proposition 4 ([9]). *The following categorical diagram is commutative*

$$\begin{tikzcd}
\mathbf{Dias} \arrow[r, "(\cdot)_{\mathcal{L}}"] & \mathbf{Leib} \\
\mathbf{As} \arrow[u] \arrow[r, "(\cdot)_{Lie}"] & \mathbf{Lie}. \arrow[u]
\end{tikzcd}$$

2. Derivations and diderivations over dialgebras

In this section, we construct a compatible operator structure on a dialgebra by combining antiderivations with multiplicative operators induced by the dialgebra products. These objects are referred to as **diderivations**. We provide a characterization of both derivations and diderivations, while the Leibniz algebra of biderivations will be introduced in the next section.

2.1. Derivations over dialgebras

In this subsection, we investigate the concept of derivations on a dialgebra $\mathscr{D}$, extending the classical notion from associative algebras.

Definition 10. A **derivation** over $\mathscr{D}$ is a linear map $d : \mathscr{D} \to \mathscr{D}$ such that

$$d(a \vdash b) = d(a) \vdash b + a \vdash d(b) \quad \text{and} \quad d(a \dashv b) = d(a) \dashv b + a \dashv d(b)$$

for all $a, b \in \mathscr{D}$.

We will denote by $\mathcal{D}er(\mathscr{D})$ the vector space generated by all the derivations $d : \mathscr{D} \to \mathscr{D}$ defined on $\mathscr{D}$. In order to show the existence of derivations over any dialgebra, we consider the linear map $ad_a : \mathscr{D} \to \mathscr{D}$ defined by $ad_a := R_a^{\dashv} - L_a^{\vdash}$, where $R_a^{\dashv}$ and $L_a^{\vdash}$ are the multiplicative operators $R_a^{\dashv}(b) = b \dashv a$ and $L_a^{\vdash}(b) = a \vdash b$, respectively.

Lemma 4. *For all $a \in \mathscr{D}$, $ad_a \in \mathcal{D}er(\mathscr{D})$.*

Proof. We demonstrate the Leibniz rule for the product $\vdash$; the proof for $\dashv$ follows in a similar fashion. In fact, let $b, c \in \mathscr{D}$, then

$$\begin{aligned} ad_a(b) \vdash c + b \vdash ad_a(c) &= \Big(R_a^{\dashv}(b) - L_a^{\vdash}(b)\Big) \vdash c + b \vdash \Big(R_a^{\dashv}(c) - L_a^{\vdash}(c)\Big) \\ &= L^{\vdash}_{(R_a^{\dashv}(b) - L_a^{\vdash}(b))}(c) + L_b^{\vdash}(R_a^{\dashv}(c) - L_a^{\vdash}(c)) \\ &= L^{\vdash}_{R_a^{\dashv}(b)}(c) - L^{\vdash}_{L_a^{\vdash}(b)}(c) + L_b^{\vdash} R_a^{\dashv}(c) - L_b^{\vdash} L_a^{\vdash}(c). \end{aligned}$$

Besides, since

(i) $L^{\vdash}_{L_a^{\vdash}(b)}(c) = (a \vdash b) \vdash c = a \vdash (b \vdash c) = L_a^{\vdash} L_b^{\vdash}(c)$;

(ii) $L_b^{\vdash} R_a^{\dashv}(c) = b \vdash (c \dashv a) = (b \vdash c) \dashv a = R_a^{\dashv} L_b^{\vdash}(c)$;

(iii) $L_b^{\vdash} L_a^{\vdash}(c) = b \vdash (a \vdash c) = (b \vdash a) \vdash c = (b \dashv a) \vdash c = L^{\vdash}_{R_a^{\dashv}(b)}(c)$;

we have that

$$\begin{aligned} ad_a(b) \vdash c + b \vdash ad_a(c) &= L^{\vdash}_{R^{\dashv}_a(b)}(c) - L^{\vdash}_a L^{\vdash}_b(c) + R^{\dashv}_a L^{\vdash}_b(c) - L^{\vdash}_{R^{\dashv}_a(b)}(c) \\ &= -L^{\vdash}_a L^{\vdash}_b(c) + R^{\dashv}_a L^{\vdash}_b(c) \\ &= R^{\dashv}_a(b \vdash c) - L^{\vdash}_a(b \vdash c) \\ &= ad_a(b \vdash c). \end{aligned}$$

□

The derivations ad_a, $a \in \mathscr{D}$, are called **inner derivations** of $\mathscr{D}$ and we will denote by $\mathscr{I}nn(\mathscr{D}) = \langle ad_a \mid a \in \mathscr{D} \rangle$ the vector space of inner derivations. As we will show, this is an ideal of $\mathcal{D}er(\mathscr{D})$ and a Lie subalgebra of $\mathfrak{gl}(\mathscr{D}) = End(\mathscr{D})$ (Lemma 5).

Proposition 5. *Let $T : \mathscr{D} \to \mathscr{D}$ be a linear map. Then T is a derivation if and only if $L^*_{T(a)} = [T, L^*_a]$, for all $a \in \mathscr{D}$, where $*$ represents each of the products $\vdash$ or $\dashv$.*

Proof. Let T be a derivation over $\mathscr{D}$. Since for all $a, b \in \mathscr{D}$ it holds

$$T(L^*_a(b)) = T(a * b) = T(a) * b + a * T(b) = L^*_{T(a)}(b) + L^*_a(T(b)),$$

then

$$L^*_{T(a)}(b) = T(L^*_a(b)) - L^*_a(T(b)) = [T, L^*_a](b)$$

and therefore $L^*_{T(a)} = [T, L^*_a]$ for all $a \in \mathscr{D}$.

On the other hand, if $L^*_{T(a)} = [T, L^*_a]$ for all $a \in \mathscr{D}$, then

$$T(a) * b = L^*_{T(a)}(b) = T(L^*_a(b)) - L^*_a(T(b)) = T(a * b) - a * T(b),$$

for all $b \in \mathscr{D}$, in other words $T(a*b) = T(a)*b + a*T(b)$, for all $a, b \in \mathscr{D}$. This implies that T is a derivation. □

In the same way, we have the following result.

Proposition 6. *Let $T : \mathscr{D} \to \mathscr{D}$ be a linear map. Then T is a derivation if and only if $R^*_{T(a)} = [T, R^*_a]$, for all $a \in \mathscr{D}$, where $*$ represents each of the products $\vdash$ or $\dashv$.*

Lemma 5. *$\mathcal{D}er(\mathscr{D})$ is a Lie subalgebra of $\mathfrak{gl}(\mathscr{D})$.*

Proof. Let $d_1, d_2 \in \mathcal{D}er(\mathscr{D})$. Then for all $a \in \mathscr{D}$,

$$\begin{aligned} L^*_{[d_1,d_2](a)} &= L^*_{d_1(d_2(a))} - L^*_{d_2(d_1(a))} = \left[d_1, L^*_{d_2(a)}\right] - \left[d_2, L^*_{d_1(a)}\right] \\ &= [d_1, [d_2, L^*_a]] - [d_2, [d_1, L^*_a]] = [[d_1, d_2], L^*_a]. \end{aligned}$$

Thus, from Theorem 5, we have that $[d_1, d_2] \in \mathcal{D}er(\mathscr{D})$. ☐

Theorem 1. *$\mathscr{I}nn(\mathscr{D})$ is an ideal of $\mathcal{D}er(\mathscr{D})$.*

Proof. Let $ad_a \in \mathscr{I}nn(\mathscr{D})$ and $d \in \mathcal{D}er(\mathscr{D})$, then

$$[d, ad_a] = [d, (R^{\dashv}_a - L^{\vdash}_a)] = [d, R^{\dashv}_a] - [d, L^{\vdash}_a] = R^{\dashv}_{d(a)} - L^{\vdash}_{d(a)} = ad_{d(a)}.$$

Therefore, $[d, ad_a] \in \mathscr{I}nn(\mathscr{D})$. ☐

Let $\mathscr{D}$ be a dialgebra, and let $\mathscr{D}_{\mathcal{L}}$ be the corresponding Leibniz algebra generated by $\mathscr{D}$ via the bracket operation $[b, a] = b \dashv a - a \vdash b$. Then $\mathrm{ad}_a(b) = [b, a]$ is an inner derivation over $\mathscr{D}_{\mathcal{L}}$. Hence, we may conclude that inner derivations over $\mathscr{D}$ give rise to inner derivations over $\mathscr{D}_{\mathcal{L}}$.

Let us now study the action of $\mathcal{D}er(\mathscr{D})$ over bar units, and the ideals $\mathscr{D}^{ann}$ and $Z_B(\mathscr{D})$.

Lemma 6. *Let d be a derivation of a dialgebra $\mathscr{D}$. Then*

(a) $d(\mathscr{D}^{ann}) \subseteq \mathscr{D}^{ann}$;

(b) $d(Z_B(\mathscr{D})) \subseteq Z_B(\mathscr{D})$;

(c) if $\mathscr{D}$ is unital, then $d(\mathscr{H}(\mathscr{D})) \subseteq \mathscr{D}^{ann}$.

Proof. (a) Let us start proving that $d(\mathscr{D}^{ann}) \subseteq \mathscr{D}^{ann}$. In fact,

$$\begin{aligned} d(x \dashv y - x \vdash y) &= d(x \dashv y) - d(x \vdash y) \\ &= dx \dashv y + x \dashv dy - dx \vdash y - x \vdash dy \\ &= (dx \dashv y - dx \vdash y) + (x \dashv dy - x \vdash dy) \ \in \mathscr{D}^{ann}. \end{aligned}$$

(b) On the other hand, given $d \in \mathcal{D}er(\mathscr{D})$ and $z \in Z_B(\mathscr{D})$, then for all $x \in \mathscr{D}$ it holds

$$0 = d(0) = d(x \dashv z) = d(x) \dashv z + x \dashv d(z) = x \dashv d(z).$$

In a completely similar way, we prove that $d(z) \vdash x = 0$, and therefore $d(z) \in Z_B(\mathscr{D})$.

(c) Let e be a bar unit in $\mathscr{D}$. Then

$$d(e) = d(e \vdash e) = d(e) \vdash e + e \vdash d(e) = d(e) \vdash e + d(e).$$

Thus, $d(e) \vdash e = 0$. Similarly, we prove $e \dashv d(e) = 0$. Thereby, from Lemma 2 it follows that $d(e) \in \mathscr{D}^{ann}$.

□

2.2. Diderivations over dialgebras

Let us now introduce the concept of diderivation of a dialgebra and prove some of its basic properties and its relation with derivations. To begin with, we recall that in every Leibniz algebra $\mathscr{L}$, there exists the concept of an antiderivation, which is a linear map $\delta : \mathscr{L} \to \mathscr{L}$ such that

$$\delta\left([x, y]\right) = [\delta(x), y] - [\delta(y), x]$$

for all $x, y \in \mathscr{L}$. Moreover, if $\mathscr{L}$ is a Lie algebra then the concepts of derivation and antiderivation are equivalent.

On the other hand, in every Leibniz algebra $\mathscr{L}$ the linear map $Ad_a : \mathscr{L} \to \mathscr{L}$ defined by $Ad_x(y) := [x, y]$ is an inner antiderivation of $\mathscr{L}$. Assuming that $\mathscr{L} = \mathscr{D}_{\mathscr{L}}$, then in terms of the multiplicative operators we have

$$Ad_a = L_a^{\dashv} - R_a^{\vdash}.$$

To determine the action of this kind of linear maps on a dialgebra $\mathscr{D}$, let us consider the following slight modification:

$$\begin{aligned} Ad_a : \mathscr{D} &\to \ \mathscr{D} \\ b &\to Ad_a(b) = R_a^{\vdash}(b) - L_a^{\dashv}(b). \end{aligned}$$

Let us see how this transformation acts on the products $b \dashv c$ and $b \vdash c$.

$$\begin{aligned} Ad_a(b \dashv c) &= (R_a^{\vdash} - L_a^{\dashv})(b \dashv c) = (b \dashv c) \vdash a - a \dashv (b \dashv c) \\ &= b \vdash (c \vdash a) - (a \dashv b) \dashv c - b \vdash (a \dashv c) + (b \vdash a) \dashv c \\ &= b \vdash (c \vdash a - a \dashv c) + (b \vdash a - a \dashv b) \dashv c \\ &= b \vdash (R_a^{\vdash} - L_a^{\dashv})(c) + (R_a^{\vdash} - L_a^{\dashv})(b) \dashv c \\ &= Ad_a(b) \dashv c + b \vdash Ad_a(c) \end{aligned}$$

and

$$\begin{aligned} Ad_a(b \vdash c) &= (R_a^{\vdash} - L_a^{\dashv})(b \vdash c) = (b \vdash c) \vdash a - a \dashv (b \vdash c) \\ &= b \vdash (c \vdash a) - (a \dashv b) \dashv c - b \vdash (a \dashv c) + (b \vdash a) \dashv c \\ &= b \vdash (c \vdash a - a \dashv c) + (b \vdash a - a \dashv b) \dashv c \\ &= b \vdash (R_a^{\vdash} - L_a^{\dashv})(c) + (R_a^{\vdash} - L_a^{\dashv})(b) \dashv c \\ &= Ad_a(b) \dashv c + b \vdash Ad_a(c). \end{aligned}$$

From the previous discussion, the linear map Ad_a satisfies the following definition.

Definition 11. Let $\mathscr{D}$ be a dialgebra. A **diderivation** of $\mathscr{D}$ is a linear map $\delta : \mathscr{D} \to \mathscr{D}$ such that

$$\delta(a \dashv b) = \delta(a \vdash b) = \delta(a) \dashv b + a \vdash \delta(b)$$

for all $a, b \in \mathscr{D}$.

We will denote by $\mathcal{D}ider(\mathscr{D})$ the vector space generated by all the diderivations of $\mathscr{D}$. This space is not empty because for every $a \in \mathscr{D}$ we already know that $Ad_a \in \mathcal{D}ider(\mathscr{D})$. This kind of diderivations is called **inner diderivations**, and the vector space generated by such linear maps will be denoted by $\mathcal{D}\mathscr{I}nn(\mathscr{D})$.

Remark 5. If $\mathscr{D}$ is an associative algebra then $\mathcal{D}ider(\mathscr{D}) = \mathcal{D}er(\mathscr{D})$. In particular $ad_a = Ad_a$ and therefore $\mathscr{I}nn(\mathscr{D}) = \mathcal{D}\mathscr{I}nn(\mathscr{D})$.

Let us characterize the diderivations in terms of the multiplicative operators.

Theorem 2. *A linear transformation $\delta : \mathscr{D} \to \mathscr{D}$ is a diderivation if and only if $\delta L_a^{*} - L_a^{\vdash}\delta = L_{\delta(a)}^{\dashv}$, for every $a \in \mathscr{D}$, where $*$ represents one of the products $\dashv$ or $\vdash$.*

Proof. If δ is a diderivation, then for all $a, b \in \mathscr{D}$,

$$\delta\left(L_a^{*} b\right) = \delta(a * b) = \delta(a) \dashv b + a \vdash \delta(b) = L_{\delta(a)}^{\dashv} b + L_a^{\vdash}\delta(b),$$

which implies that

$$\delta\left(L_a^{*}(b)\right) - L_a^{\vdash}(\delta(b)) = L_{\delta(a)}^{\dashv}(b).$$

Consequently

$$\delta L_a^{*} - L_a^{\vdash}\delta = L_{\delta(a)}^{\dashv}, \tag{4}$$

with $*$ representing one of the products $\dashv$ or $\vdash$.

Reciprocally, if (4) holds, then

$$\delta(a) \dashv (b) = L^{\dashv}_{\delta(a)} b = \delta(L^{*}_{a} b) - L^{\vdash}_{a} \delta(b) = \delta(a * b) - a \vdash \delta(b).$$

This implies the equality $\delta(a * b) = \delta(a) \dashv b + a \vdash \delta(b)$, which means that δ is a diderivation. □

Similarly, we obtain the following equivalences.

Theorem 3. *Let $\delta : \mathcal{D} \to \mathcal{D}$ be a linear map. The following assertions are equivalent.*

(a) δ is a diderivation;

(b) $\delta R^{*}_{a} - R^{\dashv}_{a} \delta = R^{\vdash}_{\delta(a)}$, where $*$ represents one of the products $\vdash$ or $\dashv$;

(c) $L^{\dashv}_{\delta(a)} = [\delta, L^{\vdash}_{a}]$ and $R^{\vdash}_{\delta(a)} = [\delta, R^{\dashv}_{a}]$.

Using the previous characterizations and the properties of the multiplicative operators it is easy to see that if $\delta, \delta' \in \mathcal{D}ider(\mathcal{D})$, then

$$[[\delta, \delta'], L^{\vdash}_{a}] = L^{\vdash}_{[\delta,\delta'](a)} + \left(L^{\vdash}_{\delta'(a)} + L^{\dashv}_{\delta'(a)}\right) \delta - \left(L^{\vdash}_{\delta(a)} + L^{\dashv}_{\delta(a)}\right) \delta'.$$

Therefore, $\mathcal{D}ider(\mathcal{D})$ is generally not a Lie subalgebra of $\mathfrak{gl}(\mathcal{D})$. However, the following property holds.

Theorem 4. *If $\mathcal{D}$ is a dialgebra, then $[\mathcal{D}ider(\mathcal{D}), \mathcal{D}er(\mathcal{D})] \subseteq \mathcal{D}ider(\mathcal{D})$.*

Proof. We will use the properties proved in Theorems 5 and 2. Let δ be a diderivation and d be a derivation of $\mathcal{D}$. We have

$$\begin{aligned}
[\delta, d]L^{*}_{a} - L^{\vdash}_{a}[\delta, d] &= (\delta d)L^{*}_{a} - (d\delta)L^{*}_{a} - L^{\vdash}_{a}(\delta d) + L^{\vdash}_{a}(d\delta) \\
&= \delta(L^{*}_{da} + L^{*}_{a} d) - d(L^{\dashv}_{\delta a} + L^{\vdash}_{a}\delta) + (L^{\dashv}_{\delta a} - \delta L^{*}_{a})d - (dL^{\vdash}_{a} - L^{\vdash}_{da})\delta \\
&= \delta L^{*}_{da} - dL^{\dashv}_{\delta a} + L^{\dashv}_{\delta a} d - L^{\vdash}_{da}\delta \\
&= L^{\dashv}_{\delta da} + L^{\vdash}_{da} d - L^{\dashv}_{d\delta a} - L^{\dashv}_{\delta a} d + L^{\dashv}_{\delta a} d - L^{\vdash}_{da}\delta \\
&= L^{\dashv}_{\delta da} - L^{\dashv}_{d\delta a} = L^{\dashv}_{[\delta,d](a)},
\end{aligned}$$

which shows that $[\delta, d] \in \mathcal{D}ider(\mathcal{D})$. □

From the previous theorem we have the following relations.

Lemma 7. *For every dialgebra* $\mathscr{D}$ *we have,*

$$[\mathcal{D}\mathcal{I}nn(\mathscr{D}), \mathcal{D}er(\mathscr{D})] \subseteq \mathcal{D}\mathcal{I}nn(\mathscr{D}).$$

In particular, $[\mathcal{D}\mathcal{I}nn(\mathscr{D}), \mathcal{I}nn(\mathscr{D})] \subseteq \mathcal{D}\mathcal{I}nn(\mathscr{D})$.

Let us finally see how the diderivations act on the ideals $\mathscr{D}^{ann}$ and $Z_B(\mathscr{D})$, and on the halo $\mathscr{H}(\mathscr{D})$, when $\mathscr{D}$ is a unital dialgebra.

Lemma 8. *Let* $\mathscr{D}$ *be a dialgebra and* $\delta \in \mathcal{D}ider(\mathscr{D})$*. Then*

(a) $\delta(\mathcal{D}^{ann}) = \{0\}$;

(b) $x \vdash \delta(z) = \delta(z) \dashv x = 0$ *for every* $x \in \mathscr{D}$ *and* $z \in Z_B(\mathscr{D})$;

(c) if $\mathscr{D}$ *is a unital dialgebra, then* $\delta(\mathscr{H}(\mathscr{D})) = \delta(Z_B(\mathscr{D})) = \{0\}$.

Proof. Let $\delta \in \mathcal{D}ider(\mathscr{D})$.

(a) This is a straightforward computation following the definitions of $\mathscr{D}^{ann}$ and $\mathcal{D}ider(\mathscr{D})$.

(b) Let $x \in \mathscr{D}$ and $z \in Z_B(\mathscr{D})$, then $0 = \delta(z \vdash x) = \delta(z) \dashv x$. Applying δ to $x \dashv z$ we get the other property.

(c) Let $e \in \mathscr{H}(\mathscr{D})$, then $\delta(e) = \delta(e * e) = \delta(e) + \delta(e)$ and therefore $\delta(e) = 0$.

□

3. Diderivations of low dimensional dialgebras

Throughout this section, we will always assume that a dialgebra $\mathscr{D}$ carries a complex vector space structure. In [1, 14] the authors introduced an algorithm to compute derivations of dialgebras and tested such an algorithm in low dimensional cases, this is, on dialgebras of dimension two and three. In this section, we will follow [1] to give a complete classification of the vector space of diderivations for dialgebras of dimensions two and three.

$Dias_2^1$	$Dias_2^3$
$e_1 \vdash e_1 = e_1 \dashv e_1 = e_1$ $e_2 \dashv e_1 = e_2$	$e_1 \vdash e_1 = e_2$ $e_1 \dashv e_1 = \lambda e_2,\ \lambda \in \mathbb{C}$
$Dias_2^2$	$Dias_2^4$
$e_1 \vdash e_1 = e_1 \dashv e_1 = e_1$ $e_1 \dashv e_2 = e_2$	$e_1 \vdash e_1 = e_1 \dashv e_1 = e_1$ $e_1 \vdash e_2 = e_2 \dashv e_1 = e_2$

Table 1.1: Two-dimensional complex dialgebras.

3.1. Diderivations of two-dimensional dialgebras

We recall for the reader that from [14] any complex dialgebra belongs to one of the isomorphism classes given in Table 1.1.

Let $\mathcal{D}$ be an n-dimensional dialgebra with basis $\{e_1, \cdots, e_n\}$ and $\delta \in \mathcal{D}ider(\mathcal{D})$. Representing the diderivation δ in matrix form $\delta = (d_{ij})$ and considering Theorem 3 we may construct the system of equations:

$$[\delta, L_{e_i}^{\vdash}] = \sum_{j=1}^{n} d_{ij} L_{e_j}^{\dashv} \qquad \text{and} \qquad [\delta, R_{e_i}^{\dashv}] = \sum_{j=1}^{n} d_{ij} R_{e_j}^{\vdash}, \tag{5}$$

which helps us to reconstruct the diderivation δ.

Let $\mathcal{D}$ be a two-dimensional dialgebra with basis $\{e_1, e_2\}$, then any diderivation can be written in matrix form as

$$\delta = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

In order to find the coefficients of δ using System (5), we first need to find the matrices $R_{e_i}^{\vdash}, R_{e_i}^{\dashv}, L_{e_i}^{\vdash}$, and $L_{e_i}^{\dashv}$.

Let us start considering the dialgebra $Dias_2^1$. From Table 1.1 we have

$$L_{e_1}^{\dashv} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad L_{e_1}^{\vdash} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad R_{e_1}^{\dashv} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad R_{e_1}^{\vdash} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$L_{e_2}^{\dashv} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad L_{e_2}^{\vdash} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad R_{e_2}^{\dashv} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad R_{e_2}^{\vdash} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which, together with the previous description of the multiplicative operators, shows that $d_{11} = d_{12} = d_{22} = 0$, and therefore

$$dim_{\mathbb{C}}(\mathcal{D}ider(Dias_2^1)) = 1.$$

For the other three isomorphism classes of two-dimensional dialgebras we have used the code in Appendix Appendix 1.

Diderivation Space	**Base**	**Dimension**
$\mathcal{D}ider(Dias_2^1)$ $\mathcal{D}ider(Dias_2^2)$ $\mathcal{D}ider(Dias_2^3)$	$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	1
$\mathcal{D}ider(Dias_2^4)$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	0

Table 1.2: Isomorphism classes of the space of diderivations in dimension two.

Remark 6. Actually, for $Dias_2^3$ we have

$$\delta = \begin{pmatrix} d_{11} & 0 \\ d_{21} & (1+\lambda)d_{11} \end{pmatrix}$$

with $\lambda = 0$ or $\lambda = 1$, but if $\lambda = 1$ then δ is a derivation ([12, Main Theorem]) and for $\lambda = 0$ we get $d_{11} = 0$ and $\delta \in \mathscr{D}ider(Dias_2^3)$.

3.2. Diderivations of three-dimensional dialgebras

In the following table, we present the dimensions of the space of diderivations for three-dimensional dialgebras, using the classification of complex dialgebras given in [1,12]. The case $Dias_3^{16}$ is substantially more involved; its dimension formulas, expressed in terms of structural parameters, are summarized in Table 3.3, while the detailed computations are deferred to Appendix Appendix 2.

Diderivation Space	**Base**	**Dimension**
$\mathcal{D}ider(Dias_3^1)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1
$\mathcal{D}ider(Dias_3^2)$ $\mathcal{D}ider(Dias_3^3)$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0
$\mathcal{D}ider(Dias_3^4)$ $\mathcal{D}ider(Dias_3^5)$ $\mathcal{D}ider(Dias_3^7)$	$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1
$\mathcal{D}ider(Dias_3^6)$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0
$\mathcal{D}ider(Dias_3^8)$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1
$\mathcal{D}ider(Dias_3^9)$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1
$\mathcal{D}ider(Dias_3^{10})$, $\mathcal{D}ider(Dias_3^{11})$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
$\mathcal{D}ider(Dias_3^{12})$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	1
$\mathcal{D}ider(Dias_3^{13})$, $\mathcal{D}ider(Dias_3^{14})$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	2
$\mathcal{D}ider(Dias_3^{15})$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	0
$\mathcal{D}ider(Dias_3^{17})$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	1

3.3. Compact case table $\boldsymbol{Dias_3^{16}}$

The following table summarizes the case-by-case structure of the space of diderivations for the compact dialgebra $Dias_3^{16}$, according to the vanishing of structural parameters and the resulting compatibility constraints.

Branch	**Conditions**	**Description**	**Dim**
$m \neq 0$	$\Delta_1, \Delta_2 \neq 0$	Generic; only d_{21}, d_{23} survive	2
$m \neq 0$	$\Delta_1 = 0,\ \Delta_2 \neq 0$	One compat. constraint	3
$m \neq 0$	$\Delta_1 \neq 0,\ \Delta_2 = 0$	One compat. constraint	3
$m \neq 0$	$\Delta_1 = \Delta_2 = 0$	Two compat. constraints	4
$m \neq 0$	add $p = -1, q = 0$	(24)(2) drops $\Rightarrow$ +1 dim	+1
$m = 0$	$n + k \neq 0,\ k \neq np$	Generic in branch	2
$m = 0$	$n + k \neq 0,\ k = np$	Extra freedom in (d_{11}, d_{33})	3
$m = 0$	prev. + $p = -1, q = 0$	(24)(2) drops $\Rightarrow$ +1 dim	4
$m = 0$	$n + k = 0,\ q \neq 0$	d_{31} unconstrained; generic	3
$m = 0$	$n + k = 0,\ q \neq 0,\ k = -1$	Also $n = 1 \Rightarrow (1 - n)d_{22} = 0$ void	4
$m = 0$	$n + k = 0,\ q = 0$	(24)(2) drops; generic	4
$m = n = k = q = 0$	$p \neq -1$	Free: $d_{21}, d_{23}, d_{31}, d_{32}, d_{33}$	5
$m = n = k = q = 0$	$p = -1$	Also d_{13} free	6

4. The Leibniz algebra $\mathcal{B}ider(\mathscr{D})$

Let us now introduce the Leibniz algebra associated to derivations and diderivations over dialgebras. The construction is based on the fact that $\mathcal{D}ider(\mathscr{D})$ is a $\mathcal{D}er(\mathscr{D})$-bimodule when considering the actions $\delta \cdot d = [\delta, d]$ and $d \cdot \delta = [d, \delta]$, where $[\cdot, \cdot]$ is the canonical bracket of $\mathfrak{gl}(\mathscr{D})$.

In the previous context, suppose that we have an equivariant map $\varphi : \mathcal{D}ider(\mathscr{D}) \to \mathcal{D}er(\mathscr{D})$, then it would be possible to endow $\mathcal{D}ider(\mathscr{D})$ with a Leibniz algebra structure. Given that so far there is not such an equivariant map, then we need to extend the space to a bigger bimodule where an equivariant map can be built. To do that, let us consider the space $\mathcal{D}ider(\mathscr{D}) \oplus \mathcal{D}er(\mathscr{D})$ which is a $\mathcal{D}er(\mathscr{D})$-bimodule under the actions $(\delta + d) \cdot d' := [\delta, d'] + [d, d']$ and $d' \cdot (\delta + d) := [d', \delta] + [d', d]$. The canonical projection $\pi : \mathcal{D}ider(\mathscr{D}) \oplus \mathcal{D}er(\mathscr{D}) \to \mathcal{D}er(\mathscr{D})$ is the sought equivariant map.

Theorem 5. *Let $\mathscr{D}$ be a dialgebra, then $(\mathcal{B}ider(\mathscr{D}), \langle \bullet, \bullet \rangle)$ is a Leibniz algebra, where the bracket is defined by*

$$\begin{aligned} \langle \bullet, \bullet \rangle : \mathcal{B}ider(\mathscr{D}) \times \mathcal{B}ider(\mathscr{D}) \to \ & \mathcal{B}ider(\mathscr{D}) \\ (\tilde{d} \oplus d, \tilde{d}' \oplus d') \to \ & \left\langle \tilde{d} \oplus d, \tilde{d}' \oplus d' \right\rangle := [\tilde{d}, d'] \oplus [d, d']. \end{aligned}$$

Proof. Let $\delta_1, \delta_2, \delta_3 \in \mathcal{D}ider(\mathscr{D})$ and $d_1, d_2, d_3 \in \mathcal{D}er(\mathscr{D})$, then

i)
$$\begin{aligned} \langle \langle \delta_1 \oplus d_1, \delta_2 \oplus d_2 \rangle, \delta_3 \oplus d_3 \rangle &= \langle [\delta_1, d_2] \oplus [d_1, d_2], \delta_3 \oplus d_3 \rangle \\ &= [[\delta_1, d_2], d_3] \oplus [[d_1, d_2], d_3]. \end{aligned}$$

ii)
$$\begin{aligned} \langle \langle \delta_1 \oplus d_1, \delta_3 \oplus d_3 \rangle, \delta_2 \oplus d_2 \rangle &= \langle [\delta_1, d_3] \oplus [d_1, d_3], \delta_2 \oplus d_2 \rangle \\ &= [[\delta_1, d_3], d_2] \oplus [[d_1, d_3], d_2]. \end{aligned}$$

iii)
$$\begin{aligned}\langle \delta_1 \oplus d_1, \langle \delta_2 \oplus d_2, \delta_3 \oplus d_3 \rangle\rangle &= \langle \delta_1 \oplus d_1, [\delta_2, d_3] \oplus [d_2, d_3]\rangle \\ &= [\delta_1, [d_2, d_3]] \oplus [d_1, [d_2, d_3]].\end{aligned}$$

In light of the skew-symmetry property of Jacobi's identity, it is clear that $[[\delta_1, d_2], d_3] = [[\delta_1, d_3], d_2] + [\delta_1, [d_2, d_3]]$ and $[[d_1, d_2], d_3] = [[d_1, d_3], d_2] + [d_1, [d_2, d_3]]$. Using this identity in **i)**, **ii)**, and **iii)**, we have

$$\begin{aligned}\left\langle \left\langle \tilde{d}_1 \oplus d_1, \tilde{d}_2 \oplus d_2 \right\rangle, \tilde{d}_3 \oplus d_3 \right\rangle = {}& \left\langle \left\langle \tilde{d}_1 \oplus d_1, \tilde{d}_3 \oplus d_3 \right\rangle, \tilde{d}_2 \oplus d_2 \right\rangle + \\ & \left\langle \tilde{d}_1 \oplus d_1, \left\langle \tilde{d}_2 \oplus d_2, \tilde{d}_3 \oplus d_3 \right\rangle \right\rangle\end{aligned}$$

and therefore $(\mathcal{B}ider(\mathcal{D}), \langle \bullet, \bullet \rangle)$ is a Leibniz algebra. ☐

From Theorems 5 and 6 we have the following result.

Lemma 9. *The subspaces* $\mathcal{DI}nn(\mathcal{D}) \oplus \mathcal{D}er(\mathcal{D})$ *and* $\mathcal{DI}nn(\mathcal{D}) \oplus \mathcal{I}nn(\mathcal{D})$ *are ideals of* $\mathcal{B}ider(\mathcal{D})$*. Moreover* $\mathcal{B}ider(\mathcal{D})^{ann} \subseteq \mathcal{D}ider(\mathcal{D}) \oplus \{0\}$*.*

Remark 7. The Leibniz algebra $\mathcal{B}ider(\mathcal{D})$ encodes the full first-order operator structure of the dialgebra $\mathcal{D}$. While $\mathcal{D}er(\mathcal{D})$ describes infinitesimal symmetries of the dialgebra products, $\mathcal{D}ider(\mathcal{D})$ captures compatible operator corrections arising from the interaction between the two multiplicative structures. The algebra $\mathcal{B}ider(\mathcal{D})$ organizes these operators into a unified framework, analogous to a semidirect extension, where derivations act naturally on diderivations via the adjoint representation.

In particular, studying biderivations provides a systematic way to understand how multiplicative operators, inner structures, and symmetry-type transformations interact. This makes $\mathcal{B}ider(\mathcal{D})$ a natural operator-theoretic envelope of $\mathcal{D}$, reflecting both its internal symmetries and the compatibility constraints imposed by its double product structure.

5. Derivations and diderivations of the dialgebra $K[x, y]$

Finally, we present an example of a dialgebra, and we classify both its derivations and diderivations. We consider the dialgebra introduced by L. Lin and Y. Zhang in [6] generated by polynomials in two commuting variables $K[x, y]$.

Let us start by defining the products $\vdash$ and $\dashv$:

$$f(x, y) \dashv g(x, y) = f(x, y)g(y, y), \quad f(x, y) \vdash g(x, y) = f(x, x)g(x, y).$$

An easy calculation shows that $(K[x,y],\vdash,\dashv)$ is a dialgebra, [6, Theorem 1.2.1]. Moreover, it is clear that $\{x^m y^n \mid m,n\in\mathbb{N}\}$ is a basis for the dialgebra $K[x,y]$.

Now, since the constant polynomial $e(x,y)=1$ is a bar unit of $K[x,y]$:

$$\begin{aligned} f(x,y)\dashv e(x,y) &= f(x,y)e(y,y) = f(x,y)1 = f(x,y),\\ e(x,y)\vdash f(x,y) &= e(x,x)f(x,y) = 1f(x,y) = f(x,y),\end{aligned}$$

we have that $(K[x,y],\vdash,\dashv)$ is a unital dialgebra, and by Lemma 2 we may conclude that

$$\begin{aligned}(K[x,y])^{ann} &= Z_B(K[x,y])\\ &= \{f(x,y)\mid 1\dashv f(x,y)=0 \text{ and } f(x,y)\vdash 1=0\}\\ &= \{f(x,y)\mid f(x,x)=f(y,y)=0\}\,,\end{aligned}$$

in other words, the ideal $(K[x,y])^{ann}$ is generated by the polynomial $(x-y)$. According to Lemma 2 we have

$$\mathscr{H}(\mathscr{D}) = \{1+(y-x)h(x,y)\mid h(x,y)\in K[x,y]\}\,.$$

This property was also established by L. Lin and Y. Zhang in [6, Lemma 1.2.5].

Let us recall the results about derivations in $K[x,y]$ proved by L. Lin and Y. Zhang. On the one hand, by definition, we know that $x\dashv x = x\dashv y$, $y\dashv y = y\dashv x$, $x\dashv x = x\dashv y$, $x\vdash 1 = y\vdash 1 = x$ and $1\dashv x = 1\dashv y = y$. Therefore

$$x^m y^n = x^m\,1\,y^n = x^m\vdash 1\dashv x^n = \underbrace{x\vdash x\vdash\cdots\vdash x}_{m}\vdash 1\dashv\underbrace{x\dashv x\cdots\dashv x}_{n}.$$

On the other hand, if $d\in\mathcal{D}er(K[x,y])$ then

$$d(x) = d(x\vdash 1) = d(x)\vdash 1 + x\vdash d(1) = d(x)\vdash 1 + x\,d(1).$$

Since $d(1)\in(K[x,y])^{ann}$ (see the argument before [6, Theorem 2.13]), there exists $g(x,y)\in K[x,y]$ such that $d(1)=(x-y)g(x,y)$. Considering $d(x)=f(x,y)$, then $f(x,y)=f(x,x)+x[(x-y)g(x,y)]$ and we have

$$\begin{aligned} d(x^m y^n) =& d(x\vdash x\vdash\cdots\vdash x\vdash 1\dashv x\dashv x\cdots\dashv x)\\ =& (d(x)\vdash x\vdash\cdots\vdash x\vdash 1\dashv x\dashv x\cdots\dashv x)\\ &+\cdots+(x\vdash x\vdash\cdots\vdash x\vdash d(1)\dashv x\dashv x\cdots\dashv x)\\ &+(x\vdash x\vdash\cdots\vdash x\vdash 1\dashv d(x)\dashv x\cdots\dashv x)\\ &+\cdots+(x\vdash x\vdash\cdots\vdash x\vdash 1\dashv x\dashv x\cdots\dashv d(x))\\ &= mx^{m-1}y^n f(x,x) + x^m y^n d(1) + nx^m y^{n-1} f(y,y).\end{aligned}$$

From this identity L. Lin and Y. Zhang proved the following theorems.

Theorem 6 ([6, Theorem 2.3.2]). *Let* $d \in \mathcal{D}er(F[x,y])$. *There exist* $f(x) \in K[x]$ *and* $g(x,y) \in K[x,y]$, *such that*

$$d(x^m y^n) = mx^{m-1}y^n f(x) + x^m y^n (x-y)g(x,y) + nx^m y^{n-1} f(y). \tag{6}$$

Conversely, for any $f(x) \in K[x]$ *and* $g(x,y) \in K[x,y]$, *the linear transformation defined by Equation (6) is a derivation of* $K[x,y]$.

Theorem 7 ([6, Theorem 2.3.3]). *Let* $d \in \mathcal{I}nn(K[x,y])$. *There exist* $h(x) \in K[x]$ *and* $g(x,y) \in K[x,y]$ *satisfying*

$$(x-y)g(x,y) = h(y) - h(x) \tag{7}$$

such that (6) holds with $f(x) = 0$.

Conversely, let $h(x) \in K[x]$, $g(x,y) \in K[x,y]$ *and* $f(x) = 0$, *such that (6) holds, then the linear transformation* d *generated by (6) is an inner derivation of* $K[x,y]$.

Remark 8. It is important to point out that

1. $d(y) = d(1) \dashv x + 1 \dashv d(x)$, because of $d(y) = d(1 \dashv x)$.
2. In the first implication of Theorem 6 $(x-y)g(x,y) = d(1)$ and $f(x) = d(x) \vdash 1$. This means that every derivation is determined by its action over $d(x)$ and $d(1)$.

Let us now characterize diderivations over $K[x,y]$. To simplify the reasoning, let us introduce the following notation:

$$x_{\vdash}^{m} = \underbrace{x \vdash x \vdash x \cdots \vdash x}_{m\text{-times}} \quad \text{and} \quad y_{\dashv}^{n} = \underbrace{y \dashv y \dashv y \cdots \dashv y}_{n\text{-times}}.$$

By definition, we know that $x_{\vdash}^{m} = y^{m-1} \vdash x$ and $y_{\dashv}^{n} = y \dashv x^{n-1}$.

Theorem 8. *Let* $\delta \in \mathcal{D}ider(K[x,y])$, *then*

1. $\delta(x^m) = \sum_{k=0}^{m-1} (x_{\vdash}^{k}) \vdash \delta(x) \dashv (y_{\dashv}^{m-k-1}) = \delta(x) \sum_{k=0}^{m-1} x^k y^{m-k-1}$.
2. $\delta(y^n) = \sum_{k=0}^{n-1} (x_{\vdash}^{k}) \vdash \delta(y) \dashv (y_{\dashv}^{n-k-1}) = \delta(y) \sum_{k=0}^{n-1} x^k y^{n-k-1}$.

3. $\delta(x^m y^n) = \delta(x) y^n \sum_{k=0}^{m-1} x^k y^{m-k-1} + \delta(y) x^m \sum_{k=0}^{n-1} x^k y^{n-k-1}$.

Proof. 1. To show $\delta(x^m) = \delta(x) \sum_{k=0}^{m-1} x^k y^{m-k-1}$ we proceed by induction. If $m = 2$,

$$\begin{aligned} \delta(x^2) &= \delta(x \vdash x) = \delta(x) \dashv x + x \vdash \delta(x) \\ &= \delta(x)y + x\delta(x) = \delta(x)(y+x) = \delta(x) \sum_{k=0}^{1} x^k y^{1-k}. \end{aligned}$$

Let us assume the result for $m \in \mathbb{N}_{>2}$ and let us show that it also holds for $m+1$. We have,

$$\begin{aligned} \delta(x^{m+1}) &= \delta(x^m \vdash x) = \delta(x^m) \dashv x + x^m \vdash \delta(x) \\ &= \left(\delta(x) \sum_{k=0}^{m-1} x^k y^{m-k-1} \right) y + x^m \delta(x) \\ &= \delta(x) \sum_{k=0}^{m-1} x^k y^{m-k} + x^m \delta(x) \\ &= \delta(x) \left(\sum_{k=0}^{m-1} x^k y^{m-k} + x^m \right) \\ &= \delta(x) \sum_{k=0}^{m} x^k y^{m-k}. \end{aligned}$$

2. The proof of this item follows the same lines of reasoning as in 1.

3. All in all, we have

$$\begin{aligned} \delta(x^m y^n) &= \delta(x^m) \dashv y^n + x^m \vdash \delta(y^n) \\ &= \left(\delta(x) \sum_{k=0}^{m-1} x^k y^{m-k-1} \right) \dashv y^n + x^m \vdash \left(\delta(y) \sum_{k=0}^{n-1} x^k y^{n-k-1} \right) \\ &= \delta(x) y^n \sum_{k=0}^{m-1} x^k y^{m-k-1} + \delta(y) x^m \sum_{k=0}^{n-1} x^k y^{n-k-1}. \end{aligned}$$

□

Let us finally consider the identity

$$(x-y)\left(\sum_{k=0}^{n-1} x^k y^{n-k-1}\right) = x^n - y^n.$$

From the third part in the previous theorem

$$\delta(x^m y^n) = \delta(x)\, y^n \left(\frac{x^m - y^m}{x-y}\right) + \delta(y)\, x^m \left(\frac{x^n - y^n}{x-y}\right),$$

which shows the following characterization.

Lemma 10. *If $\delta \in \mathcal{D}ider(K[x,y])$, then*

$$\delta(h(x,y)) = \frac{1}{x-y}\left[\delta(x)\Big(h(x,y) - h(y,y)\Big) - \delta(y)\Big(h(x,y) - h(x,x)\Big)\right]$$

for every $h(x,y) \in K[x,y]$.

Theorem 9. *Let $f(x,y),\ g(x,y) \in K[x,y]$. The linear transformation δ of $K[x,y]$ induced by the identity*

$$\delta(x^m y^n) = f(x,y)\, y^n \left(\frac{x^m - y^m}{x-y}\right) + g(x,y)\, x^m \left(\frac{x^n - y^n}{x-y}\right),$$

is a diderivation of $K[x,y]$.

Furthermore, $\delta(x) = f(x,y)$ and $\delta(y) = g(x,y)$.

Finally, like for diderivations, we have:

Lemma 11. *For every $p(x,y) \in K[x,y]$,*

$$Ad_{p(x,y)}(h(x,y)) = p(x,y)\Big(h(x,x) - h(y,y)\Big) \tag{8}$$

for all $h(x,y) \in K[x,y]$. Therefore $Ad_{p(x,y)}\big(K[x,y]\big) \subseteq (K[x,y])^{ann}$.

Conversely, if δ is a linear transformation that satisfies (8), then $\delta = Ad_{p(x,y)}$.

6. Conclusion

In this article, we have investigated derivations and biderivations in the framework of dialgebras, introducing the new concept of diderivations and analyzing their interaction with derivations through multiplicative operators. A further central result is the construction of the algebra of biderivations and the proof that it naturally carries a Leibniz algebra structure, thereby extending the classical correspondence between associative, Lie, and Leibniz settings. In addition, we have presented a complete classification of the vector spaces of diderivations for dialgebras of dimensions two and three. These low dimensional computations not only corroborate the general framework but also reveal structural patterns that inform the extension of the theory. The article also completes the classification of operators on the polynomial dialgebra $F[x, y]$, concluding with an explicit description of its diderivations.

Appendices

Appendix 1. Python code

To compute explicitly the space of diderivations of $\mathscr{D} = Dias_2^1$, we solve the linear system (5) by symbolic computation using `SymPy`.

We represent a linear map $\delta\colon \mathscr{D} \to \mathscr{D}$ by its matrix

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

with respect to the fixed basis $\{e_1, e_2\}$. The left and right multiplication operators $L^{\vdash}_{e_i}, L^{\dashv}_{e_i}, R^{\vdash}_{e_i}, R^{\dashv}_{e_i}$ are encoded as 2×2 matrices according to the multiplication table of $Dias_2^1$ given in Section 3.

The defining relations of a diderivation are implemented in matrix form as

$$[D, L^{\vdash}_{e_i}] = \sum_{j=1}^{2} d_{ji} L^{\dashv}_{e_j}, \qquad [D, R^{\dashv}_{e_i}] = \sum_{j=1}^{2} d_{ji} R^{\vdash}_{e_j}, \qquad i = 1, 2,$$

which yields a homogeneous linear system in the variables

$$d_{11}, d_{12}, d_{21}, d_{22}.$$

```
import sympy as sp

# unknown matrix D
d11, d12, d21, d22 = sp.symbols('d11 d12 d21 d22')
D = sp.Matrix([[d11, d12], [d21, d22]])

# multiplication operators for Dias_2^1
L_vdash = [sp.Matrix([[1,0],[0,0]]), sp.zeros(2)]
L_dashv = [sp.Matrix([[1,0],[0,0]]), sp.Matrix
    ([[0,0],[1,0]])]
```

```
R_vdash = [sp.Matrix([[1,0],[0,0]]), sp.zeros(2)]
R_dashv = [sp.Matrix([[1,0],[0,1]]), sp.zeros(2)]

# build the linear system
eqs = []
for i in range(2):
    eqs += list(D*L_vdash[i] - L_vdash[i]*D
                - (D[0,i]*L_dashv[0] + D[1,i]*L_dashv[1])
                  )
    eqs += list(D*R_dashv[i] - R_dashv[i]*D
                - (D[0,i]*R_vdash[0] + D[1,i]*R_vdash[1])
                  )

# solve the homogeneous system
sol = sp.linsolve(eqs, [d11, d12, d21, d22])
print(sol)
```

Appendix 2. Diderivations over $\boldsymbol{Dias_3^{16}}$

Let

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

and let us fix scalar parameters: $k, m, n, p, q \in \mathbb{C}$, which come from the previous table.

The left operators:

$$L_{1\mathrm{pb}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_{2\mathrm{pb}} = 0, \quad L_{3\mathrm{pb}} = \begin{pmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L_{1\mathrm{bp}} = \begin{pmatrix} 0 & 0 & 0 \\ m & 0 & n \\ 0 & 0 & 0 \end{pmatrix}, \quad L_{2\mathrm{bp}} = 0, \quad L_{3\mathrm{bp}} = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & q \\ 0 & 0 & 0 \end{pmatrix},$$

and the right operators:

$$R_{1\mathrm{pb}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & k \\ 0 & 0 & 0 \end{pmatrix}, \quad R_{2\mathrm{pb}} = 0, \quad R_{3\mathrm{pb}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$R_{1\mathrm{bp}} = \begin{pmatrix} 0 & 0 & 0 \\ m & 0 & p \\ 0 & 0 & 0 \end{pmatrix}, \quad R_{2\mathrm{bp}} = 0, \quad R_{3\mathrm{bp}} = \begin{pmatrix} 0 & 0 & 0 \\ n & 0 & q \\ 0 & 0 & 0 \end{pmatrix},$$

together with equation (5) give rise to the six equalities:

$$DL_{1\text{bp}} - L_{1\text{bp}}D = d_{11}L_{1\text{pb}} + d_{21}L_{2\text{pb}} + d_{31}L_{3\text{pb}}, \tag{9}$$

$$DL_{2\text{bp}} - L_{2\text{bp}}D = d_{12}L_{1\text{pb}} + d_{22}L_{2\text{pb}} + d_{32}L_{3\text{pb}}, \tag{10}$$

$$DL_{3\text{bp}} - L_{3\text{bp}}D = d_{13}L_{1\text{pb}} + d_{23}L_{2\text{pb}} + d_{33}L_{3\text{pb}}, \tag{11}$$

$$DR_{1\text{pb}} - R_{1\text{pb}}D = d_{11}R_{1\text{bp}} + d_{21}R_{2\text{bp}} + d_{31}R_{3\text{bp}}, \tag{12}$$

$$DR_{2\text{pb}} - R_{2\text{pb}}D = d_{12}R_{1\text{bp}} + d_{22}R_{2\text{bp}} + d_{32}R_{3\text{bp}}, \tag{13}$$

$$DR_{3\text{pb}} - R_{3\text{pb}}D = d_{13}R_{1\text{bp}} + d_{23}R_{2\text{bp}} + d_{33}R_{3\text{bp}}. \tag{14}$$

Remark 9. Since $L_{2\text{pb}} = R_{2\text{pb}} = 0$, the coefficients d_{21}, d_{23} never occur. In other words, they are always free.

An independent set of scalar equations extracted from (9)–(14) is:

(i) From (9): $m\,d_{12} = 0, \quad n\,d_{12} = 0, \quad m\,d_{32} = 0, \quad n\,d_{32} = 0,$ (15)

$(-d_{11} + d_{22})\,m - d_{31}\,n = d_{31}\,k,$ (16)

$-\,d_{13}\,m + (d_{22} - d_{33})\,n = d_{11}.$ (17)

(ii) From (10): $k\,d_{32} = 0, \qquad d_{12} = 0.$ (18)

(iii) From (11): $(-d_{11} + d_{22})\,p - d_{31}\,q = d_{33}\,k,$ (19)

$-\,d_{13}\,p + (d_{22} - d_{33})\,q = d_{13}.$ (20)

(iv) From (12): $-\,d_{31}\,k = d_{11}\,m + d_{31}\,n,$ (21)

$k\,(d_{22} - d_{33}) = d_{11}\,p + d_{31}\,q.$ (22)

(v) From (13): $d_{12}\,m + d_{32}\,n = 0, \qquad d_{12}\,p + d_{32}\,q = 0.$ (23)

(vi) From (14): $-d_{11} + d_{22} = d_{13}m + d_{33}n, \quad (p+1)d_{13} + qd_{33} = 0.$ (24)

Initial reduction

From Equations (18) and (24): $d_{12} = 0$ and, except in the extreme $m = n = k = q = 0$, also $d_{32} = 0$. If $m \neq 0$, then combining (16) and (21) yields $d_{22} = 0$.

We work with the 5×5 linear subsystem in the unknowns

$$\mathbf{x} = (d_{11}, d_{13}, d_{22}, d_{31}, d_{33})^{\top},$$

extracted from (R1a), (R1b), (L3a), (R3a.2), (R3a.1):

$$M = \begin{pmatrix} m & 0 & 0 & (n+k) & 0 \\ -p & 0 & -k & -q & k \\ -p & 0 & p & -q & -k \\ 0 & (p+1) & 0 & 0 & q \\ -1 & -m & 1 & 0 & -n \end{pmatrix}. \tag{25}$$

Let us define the two key factors

$$\Delta_1 := -k - mq + np, \qquad \Delta_2 := (k+n)(p+1) - mq. \tag{26}$$

Row 4 is particularly simple:

$$(p+1)\, d_{13} + q\, d_{33} = 0. \tag{27}$$

Permuting rows/columns so that the d_{13} equation (row 4) and the d_{13} variable (column 2) come first. Then M has the following block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad A = (p+1) \in \mathbb{C}. \tag{28}$$

Here A is 1×1, B is 1×4, C is 4×1, and D is 4×4 (explicit expressions are immediate from (25)).

Schur complement

Taking into account the decomposition (28), it is well-known that if A is invertible, then

$$\det M = \det A \cdot \det(D - CA^{-1}B). \tag{29}$$

Here $A = (p+1)$, so provided $p \neq -1$ we may form the Schur complement

$$\widetilde{A} = D - \tfrac{1}{p+1}\, CB.$$

Computing CB explicitly gives

$$CB = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -mq \end{pmatrix},$$

so that

$$\widetilde{A} = \begin{pmatrix} m & 0 & (n+k) & 0 \\ -p & -k & -q & k \\ -p & p & -q & -k \\ -1 & 1 & 0 & -n + \frac{mq}{p+1} \end{pmatrix}.$$

Multiplying the last row by $(p+1)$ yields the cleaner form

$$\widetilde{A} = \begin{pmatrix} m & 0 & (n+k) & 0 \\ -p & -k & -q & k \\ -p & p & -q & -k \\ -(p+1) & (p+1) & 0 & mq - n(p+1) \end{pmatrix}. \tag{30}$$

Thus

$$\det M \doteq \det \widetilde{A}, \tag{31}$$

where $\doteq$ means equality up to a nonzero unit.

Apply the row operations

$$R_2 \leftarrow mR_2 + pR_1, \qquad R_3 \leftarrow mR_3 + pR_1, \qquad R_4 \leftarrow mR_4 + (p+1)R_1,$$

we obtain

$$\widetilde{A}' = \begin{pmatrix} m & 0 & (n+k) & 0 \\ 0 & -km & -mq + (k+n)p & km \\ 0 & mp & -mq + (k+n)p & -km \\ 0 & m(p+1) & -mq + (k+n)(p+1) & m^2 q \end{pmatrix}.$$

Hence the first column has pivot m and zeros below, so $\det \widetilde{A} = m \cdot \det C$, where

$$C = \begin{pmatrix} -km & -mq + (k+n)p & km \\ mp & -mq + (k+n)p & -km \\ m(p+1) & -mq + (k+n)(p+1) & m^2 q \end{pmatrix}. \tag{32}$$

Let $S := -mq + (k+n)p$. Then the middle column of the matrix C is $(S,\ S,\ S + (k+n))^{\top}$. Using $R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 - R_1$, and expanding cofactors along the zero at C_{22} yields

$$\begin{aligned} \det C = (mp + km)\left(-\det \begin{pmatrix} S & km \\ k+n & m^2 q - km \end{pmatrix} \right) \\ - 2km \left(-\det \begin{pmatrix} -km & S \\ m(p+1) + km & k+n \end{pmatrix} \right). \end{aligned}$$

Evaluating the 2×2 minors and simplifying with $S = -mq + (k+n)p$ one finds

$$\det C \doteq \Delta_2 \cdot \Delta_1 \ . \tag{33}$$

Combining (29), (32), and (33):

$$\det M \doteq m\, \Delta_2\, \Delta_1 \ .$$

Remark 10. The vanishing locus of the determinant, and hence the rank drops giving rise to extra solution dimensions, is precisely

$$\{\, m = 0 \,\} \ \cup\ \{\, \Delta_2 = 0 \,\} \ \cup\ \{\, \Delta_1 = 0 \,\}.$$

Resolution with $m \neq 0$

Case A: trivial solution (generic)

If

$$\Delta_1 \neq 0 \quad \text{and} \quad \Delta_2 \neq 0,$$

then the only possibility is

$$d_{11} = d_{12} = d_{13} = d_{22} = d_{31} = d_{32} = d_{33} = 0,$$

with d_{21}, d_{23} always free.

Case B: non-trivial family when $\Delta_1 = 0$

Assume

$$\Delta_1 = -k - mq + np = 0, \qquad m \neq 0.$$

From (21): $d_{11} = -\dfrac{n+k}{m}\, d_{31}$. From (22) and $\Delta_1 = 0$: $d_{33} = \dfrac{p+1}{m}\, d_{31}$. Using $(p+1)d_{13} + qd_{33} = 0$: $d_{13} = -\dfrac{q}{m}\, d_{31}$. With parameter $t := d_{33}$:

$$(d_{11}, d_{12}, d_{13}, d_{22}, d_{31}, d_{32}, d_{33}) = t\left(-\frac{k+n}{p+1}, 0, \frac{k-np}{m(p+1)}, 0, \frac{m}{p+1}, 0, 1 \right).$$

Case C: non-trivial family when $\Delta_2 = 0$

Assume

$$\Delta_2 = (k+n)(p+1) - mq = 0, \qquad m \neq 0.$$

From (21): $d_{11} = -\dfrac{n+k}{m}\, d_{31}$. From (22) and $\Delta_2 = 0$: $d_{33} = -\dfrac{k+n}{mk}\, d_{31}$. Using $(p+1)d_{13} + qd_{33} = 0$ and $\Delta_2 = 0$: $d_{13} = -\dfrac{k+n}{m}\, d_{33}$. With parameter $t := d_{33}$:

$$(d_{11}, d_{12}, d_{13}, d_{22}, d_{31}, d_{32}, d_{33}) = t\left(k,\ 0,\ -\frac{k+n}{m},\ 0,\ -\frac{k\,m}{k+n},\ 0,\ 1 \right),$$

assuming $k + n \neq 0$.

Case D: $m \neq 0$ and $\Delta_1 = \Delta_2 = 0$

Necessarily

$$n = -k(p+2), \qquad q = -\frac{k(p+1)^2}{m}, \qquad k+n = -k(p+1),$$

and, from the initial reduction for $m \neq 0$,

$$d_{12} = d_{22} = d_{32} = 0.$$

The solution set is

$$(d_{11}, d_{13}, d_{31}, d_{33}) = \Big(\tfrac{k(p+1)}{m}\, d_{31},\ \tfrac{k(p+1)}{m}\, d_{33},\ d_{31},\ d_{33}\Big),$$

Degenerate branches ($m = 0$)

When $m = 0$ we do *not* force $d_{22} = 0$. The governing block is

$$(n+k)d_{31} = 0, \quad -d_{11} + d_{22} = n\, d_{33}, \quad k(d_{22} - d_{33}) = d_{11}p + d_{31}q,$$
$$p(-d_{11} + d_{22}) - d_{31}q = d_{33}k,$$

together with (24)(2) and (20). Also $d_{12} = 0$ and d_{32} is forced to 0 unless $n = k = q = 0$.

- $m = 0$**,** $n + k \neq 0$: then $d_{31} = 0$.
  - If $k \neq np$ (generic in this branch): dim $= 2$ when $q \neq 0$; if $q = 0$ still dim $= 2$ generically.
  - If $k = np$: one extra free parameter appears in (d_{11}, d_{33}); hence dim $= 3$. If moreover $p = -1$ and $q = 0$ (so (24)(2) vanishes), add $+1$: dim $= 4$.
- $m = 0$**,** $n + k = 0$: then d_{31} is not constrained by (21).
  - For $q \neq 0$ (so $d_{32} = 0$): generically dim $= 3$ (one free parameter in $(d_{11}, d_{13}, d_{22}, d_{33})$ plus d_{21}, d_{23}). If additionally $k = -1$ (i.e., $n = 1$), the relation $(1-n)d_{22} = 0$ becomes void and gives one more degree: dim $= 4$.
  - For $q = 0$: (24)(2) drops; generically this adds $+1$ vs. the previous line, so dim $= 4$ (and can rise on further coincidences).
- **Extreme** $m = n = k = q = 0$: Here $d_{12} = 0$, d_{32} is free, and from (17) one gets $d_{11} = 0$, then from (24)(1) $d_{22} = 0$.

- If $p \neq -1$: the free directions are d_{21}, d_{23}, d_{31}, d_{32}, d_{33}, so $\dim = 5$.
- If $p = -1$: $(p+1)\, d_{13} = 0$ vanishes and (20) becomes tautological, so d_{13} is also free and $\dim = 6$.

References

[1] Abubakar, M., Rakhimov, I.S., Rikhsiboev, I.M.: On derivations of low dimensional associative dialgebras. AIP Conf. Proc. **1602**, 730–735 (2014). https://doi.org/10.1063/1.4882566

[2] Birkhoff, G.: Lattice Theory (3rd ed.). Amer. Math. Soc., vol. **25**. Colloquium Publications, Providence, RI (1967)

[3] Frabetti, A.: Dialgebra (co)homology with coefficients. In: Dialgebras and Related Operads. Lecture Notes in Mathematics, vol. **1763**, pp. 67–103. Springer, Berlin (2001). https://doi.org/10.1007/3-540-45328-8_3

[4] Hongsopa, P.: On algorithms for computing derivations and antiderivations of Leibniz algebras. Sci Ess. J. **40**(2), 45–57 (2024)

[5] Kinyon, M.K.: Leibniz algebras, Lie racks, and digroups. J. Lie Theory. **17**(1), 99–114 (2007). https://doi.org/10.5802/jolt.439

[6] Lin, L., Zhang, Y.: $F[x, y]$ as a dialgebra and a Leibniz algebra. Comm. Algebra **38**(9), 3417–3447 (2010). https://doi.org/10.1080/00927870903164677

[7] Loday, J.-L.: Une version non-commutative des algèbres de Lie: les algèbres de Leibniz. Les rencontres physiciens-mathématiciens de Strasbourg – RCP25. **44**, 127–151 (1993)

[8] Loday, J.-L.: Dialgebras. In: Dialgebras and Related Operads. Lecture Notes in Mathematics, vol. **1763**, pp. 7–66. Springer, Berlin (2001). https://doi.org/10.1007/3-540-45328-8_2

[9] Loday, J.-L., Pirashvili, T.: Universal enveloping algebras of Leibniz algebras and (co)homology. Math. Ann. **296**, 139–158 (1993). https://doi.org/10.1007/BF01445099

[10] Mancini, M.: Biderivations of low-dimensional Leibniz algebras. In: Albuquerque, H., Brox, J., Martínez, C., Saraiva, P. (eds) Non-Associative Algebras and Related Topics. NAART. Springer Proc. Math. Statis., vol. **427**, pp. 127–136. Springer, Cham (2020). https://doi.org/10.1007/978-3-031-32707-0_8

[11] Ongay, F.: φ-dialgebras and a class of matrix "coquecigrues". Canadian Math. Bull. **50**(1), 126–137 (2007). https://doi.org/10.4153/CMB-2007-013-6

[12] Rakhimov, I.S., Masutova, K.K., Omirov, B.A.: On derivations of semisimple Leibniz algebras. Bull. Malays. Math. Sci. Soc. **40**, 295–306 (2017). https://doi.org/10.1007/s40840-015-0113-5

[13] Restrepo-Sánchez, G.G., Rodríguez-Nieto, J.G., Salazar-Díaz, O.P. Sarrazola-Alzate, A., Velásquez, R.: Dialgebra structure of $F[x] \otimes F[x]$, derivations and diderivations. Mediterr. J. Math. **22**, 159 (2025). https://doi.org/10.1007/s00009-025-02905-3

[14] Rikhsiboev, I.M., Rakhimov, I.S., Basri, W.: Diassociative algebras and their derivations. J. Phys.: Conf. Ser. **553**, 012006 (2014). https://doi.org/10.1088/1742-6596/553/1/012006

[15] Velásquez, R., Felipe, R.: Split dialgebras, split quasi-Jordan algebras and regular elements. J. Algebra Appl. **8**(2), 191–218 (2009). https://doi.org/10.1142/S0219498809003278

[16] Zhuchok, A.V.: Napivretraktsiyi dimonoyidiv (Semiretractions of dimonoids). Pratsi Instytutu prykladnoi matematyky i mekhaniky NAN Ukrainy. **17**, 42–50 (2008) (in Ukrainian)

[17] Zhuchok, A.V.: Dimonoids. Algebra Logic. **50**, 323–340 (2011) https://doi.org/10.1007/s10469-011-9144-7

[18] Zhuchok, A.V.: Relatively free dimonoids and bar-units. Int. J. Algebra Comput. **31**(8), 1587–1599 (2021). https://doi.org/10.1142/S0218196721500570

[19] Zhuchok, A.V., Zhuchok, Y.V.: On two classes of digroups. São Paulo J. Math. Sci. **11**(1), 240–252 (2017). https://doi.org/10.1007/s40863-016-0038-4

[20] Zhuchok, Y.V.: Free abelian dimonoids. Algebra Discrete Math. **20**(2), 330–342 (2015)

[21] Zhuchok, Y.V.: Automorphisms of the endomorphism semigroup of a free abelian diband. Algebra Discrete Math. **25**(2), 322–332 (2018)

CONTACT INFORMATION

G. G. Restrepo-Sánchez,
R. Velásquez
Instituto de Matemáticas, Universidad de Antioquia, Medellín, Colombia
E-Mail: `ggustavo.restrepo@udea.edu.co`, `raul.velasquez@udea.edu.co`

J. G. Rodríguez-Nieto,
O. P. Salazar-Díaz
Departamento de Matemáticas, Universidad Nacional de Colombia, Medellín, Colombia
E-Mail: `jgrodrig@unal.edu.co`, `opsalazard@unal.edu.co`

A. Sarrazola-Alzate
Ciencias Básicas e Ingenierías, Universidad EIA, Envigado, Colombia
E-Mail: `andres.sarrazola@eia.edu.co`

Received by the editors: 28.12.2025
and in final form 19.02.2026.