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RESEARCH ARTICLE

A note on determinability of free objects in four classes of strong doppelsemigroups

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Abstract. The problem of determinability of free algebras in a given variety up to an isomorphism by their endomorphism semigroups, originating in B. Plotkin's universal algebraic geometry, is a classical question in algebra. We prove that the free strong doppelsemigroup, the free commutative strong doppelsemigroup, the free n-dinilpotent strong doppelsemigroup, and the free n-nilpotent strong doppelsemigroup are determined up to an isomorphism by their endomorphism semigroups.

1. Introduction and preliminaries

The study of initial mathematical objects via related derived objects represents a central approach in mathematics. Since the 1930s, M. Stone, I. M. Gel'fand, A. N. Kolmogorov, E. Hewitt, L. Gillman, M. Henriksen, K. Magill, and others have studied how topological spaces can be determined by various algebraic systems of functions defined on them [12]. In a related algebraic context, a classical problem, first posed by É. Galois and later formulated by S. Ulam [11], deals with the "determination of a mathematical structure from a given set of endomorphisms". If $\mathrm{End}(A)$ and $\mathrm{Aut}(A)$ denote, respectively, the endomorphism semigroup and the automorphism group of an algebra A, the main question becomes: given $\mathrm{End}(A) \cong \mathrm{End}(B)$, what can be said about the relationship between algebras A and B? This determinability problem has been studied extensively. B. I. Plotkin [10] posed it for free algebras in a given variety

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within the framework of universal algebraic geometry. It was solved for free groups by E. Formanek [2], for free semigroups and free monoids by G. Mashevitsky and B. M. Schein [8], and for free (strict) n-tuple semigroups by A.V. Zhuchok [14].

In this paper, we study the determinability problem for the free strong doppelsemigroup, the free commutative strong doppelsemigroup, the free n-dinilpotent strong doppelsemigroup, and the free n-nilpotent strong doppelsemigroup by their endomorphism semigroups.

An algebra A of some class Σ *is determined up to an isomorphism* by its endomorphism semigroup in the class Σ if for any algebra $B \in \Sigma$ the condition $\mathrm{End}(A) \cong \mathrm{End}(B)$ implies $A \cong B$. The converse implication is obvious. The main result of this paper is the following theorem.

Theorem 1.1. *The free strong doppelsemigroup $\widetilde{F}[X]$, the free commutative strong doppelsemigroup $\widetilde{F}^*[X]$, the free n-dinilpotent strong doppelsemigroup $FDSD_n(X)$, and the free n-nilpotent strong doppelsemigroup $FNSD_n(X)$ are determined up to an isomorphism by their endomorphism semigroups.*

Recall that a *doppelsemigroup* [13] is a nonempty set D equipped with two binary operations $\dashv$ and $\vdash$ satisfying the following axioms:

$$(x \dashv y) \vdash z = x \dashv (y \vdash z),$$

$$(x \vdash y) \dashv z = x \vdash (y \dashv z),$$

$$(x \dashv y) \dashv z = x \dashv (y \dashv z),$$

$$(x \vdash y) \vdash z = x \vdash (y \vdash z).$$

Free doppelsemigroups of rank 1 and an arbitrary rank were constructed in [9] and [13], respectively. Relatively free doppelsemigroups were studied in [15, 18] and in the references cited therein. The structure of doppelsemigroups of k-linked upfamilies is discussed in [3]. In [1], doppelsemigroups appeared as algebras over some operad. They are also closely related to the notion of interassociativity in semigroup theory (see, e.g., [4]) and to the theory of cubical dialgebras [7].

Recall necessary defnitions from [13, 15, 16].

A doppelsemigroup $(D, \dashv, \vdash)$ is called commutative if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are commutative. As usual, $\mathbb{N}$ denotes the set of all positive integers. A semigroup S is called nilpotent if $S^{n+1} = 0$ for some $n \in \mathbb{N}$. The least such n is called the nilpotency index of S. For $k \in \mathbb{N}$ a nilpotent semigroup of nilpotency index $\leq k$ is called

k-nilpotent. An element 0 of a doppelsemigroup $(D, \dashv, \vdash)$ is called zero if $x * 0 = 0 = 0 * x$ for all $x \in D$ and $* \in \{\dashv, \vdash\}$. A doppelsemigroup $(D, \dashv, \vdash)$ with zero is called dinilpotent if $(D, \dashv)$ and $(D, \vdash)$ are nilpotent semigroups. A dinilpotent doppelsemigroup $(D, \dashv, \vdash)$ is called n-dinilpotent if $(D, \dashv)$ and $(D, \vdash)$ are n-nilpotent semigroups.

A doppelsemigroup $(D, \dashv, \vdash)$ with zero 0 is called nilpotent if for some $n \in \mathbb{N}$ and any $x_i \in D$ with $1 \leq i \leq n+1$, and $*_j \in \{\dashv, \vdash\}$ with $1 \leq j \leq n$,

$$x_1 *_1 x_2 *_2 \ldots *_n x_{n+1} = 0.$$

The least such n is called the nilpotency index of $(D, \dashv, \vdash)$. For $k \in \mathbb{N}$ a nilpotent doppelsemigroup of nilpotency index $\leq k$ is called k-nilpotent.

A doppelsemigroup $(D, \dashv, \vdash)$ is called *strong* [16] if it satisfies the axiom

$$x \dashv (y \vdash z) = x \vdash (y \dashv z).$$

Every semigroup $(S, \cdot)$ can be viewed as a strong doppelsemigroup T if assume that $T = (S, \cdot, \cdot)$. Consequently, the class of strong doppelsemigroups naturally generalizes the class of semigroups. Nevertheless, there exist strong doppelsemigroups that cannot be obtained from semigroups in this manner. For instance, commutative dimonoids in the sense of Loday [6] are examples of strong doppelsemigroups and two strongly interassociative semigroups [5] give rise to a strong doppelsemigroup. For further details on strong doppelsemigroups, see [16].

We now construct the free strong doppelsemigroup, the free commutative strong doppelsemigroup, the free n-dinilpotent strong doppelsemigroup, and the free n-nilpotent strong doppelsemigroup.

We will denote $\mathbb{N}$ with zero by $\mathbb{N}^0$. Let X be an arbitrary nonempty set, $F[X]$ the free semigroup in the alphabet X and ω an arbitrary word over X. The length of ω is denoted by l_ω. Define operations $\dashv$ and $\vdash$ on $C = \{(w, m) \in F[X] \times \mathbb{N}^0 \,|\, l_w > m\}$ by

$$(w_1, m_1) \dashv (w_2, m_2) = (w_1 w_2, m_1 + m_2 + 1), \quad (1)$$

$$(w_1, m_1) \vdash (w_2, m_2) = (w_1 w_2, m_1 + m_2) \quad (2)$$

for all (w_1, m_1), $(w_2, m_2) \in C$. The algebra $(C, \dashv, \vdash)$ is denoted by $\widetilde{F}[X]$. By Theorem 4.1 and Lemma 4.3 of [16], $\widetilde{F}[X]$ is the free strong doppelsemigroup.

In the construction of $\widetilde{F}[X]$, instead of the free semigroup on X, take the free commutative semigroup $F^*[X]$ on X. In this case, denote by $\widetilde{F}^*[X]$ the algebra $(C, \dashv, \vdash)$ with operations defined by (1) and (2), respectively. By Corollary 6.2 and Lemma 6.3 of [16], $\widetilde{F}^*[X]$ is the free commutative strong doppelsemigroup.

Let $n \in \mathbb{N}$ and $\overline{n}^0 = \{0, 1, \ldots, n-1\}$. Then we consider

$$D_n = \{(w, m) \in F[X] \times \overline{n}^0 \mid l_w > m,\, l_w - m \leq n\} \cup \{0\}.$$

Define operations $\dashv$ and $\vdash$ on D_n by

$$(w_1, m_1) \dashv (w_2, m_2)$$
$$= \begin{cases} (w_1w_2,\, m_1 + m_2 + 1) & \text{if } \begin{array}{l} m_1 + m_2 + 2 \leq n, \\ l_{w_1w_2} - m_1 - m_2 - 1 \leq n, \end{array} \\ 0 & \text{otherwise,} \end{cases}$$
$$(w_1, m_1) \vdash (w_2, m_2)$$
$$= \begin{cases} (w_1w_2,\, m_1 + m_2) & \text{if } \begin{array}{l} m_1 + m_2 + 1 \leq n, \\ l_{w_1w_2} - m_1 - m_2 \leq n, \end{array} \\ 0 & \text{otherwise,} \end{cases}$$
$$(w_1, m_1) * 0 = 0 * (w_1, m_1) = 0 * 0 = 0$$

for all $(w_1, m_1), (w_2, m_2) \in D_n \backslash \{0\}$ and $* \in \{\dashv, \vdash\}$. The algebra obtained in this way is denoted by $FDSD_n(X)$.

Theorem 1.2 ([16, Theorem 3])**.** *$FDSD_n(X)$ is the free n-dinilpotent strong doppelsemigroup.*

Let us turn to constructing the free n-nilpotent strong doppelsemigroup.

Consider $C_n = \{(w, m) \in \widetilde{F}[X] \mid l_w \leq n\} \cup \{0\}$. Define operations $\dashv$ and $\vdash$ on C_n by

$$(w_1, m_1) \dashv (w_2, m_2) = \begin{cases} (w_1w_2,\, m_1 + m_2 + 1) & \text{if } \; l_{w_1w_2} \leq n, \\ 0 & \text{if } \; l_{w_1w_2} > n, \end{cases}$$

$$(w_1, m_1) \vdash (w_2, m_2) = \begin{cases} (w_1w_2,\, m_1 + m_2) & \text{if } \; l_{w_1w_2} \leq n, \\ 0 & \text{if } \; l_{w_1w_2} > n, \end{cases}$$

$$(w_1, m_1) * 0 = 0 * (w_1, m_1) = 0 * 0 = 0$$

for all $(w_1, m_1), (w_2, m_2) \in C_n \backslash \{0\}$ and $* \in \{\dashv, \vdash\}$. The algebra $(C_n, \dashv, \vdash)$ is denoted by $FNSD_n(X)$.

Theorem 1.3 ([16, Theorem 3])**.** *$FNSD_n(X)$ is the free n-nilpotent strong doppelsemigroup.*

2. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1, then discuss the idempotent case of strong doppelsemigroups, and formulate an open problem.

Proof. Let $F \in \{\widetilde{F}[X], \widetilde{F}^*[X], FDSD_n(X), FNSD_n(X)\}$. From [16] we know that $X \times \{0\}$ is the least generating set of F, and therefore, the automorphism group $\mathrm{Aut}(F)$ of the algebra F is isomorphic to the symmetric group $S(X)$ on X. Hence, for every $x \in X$, we have

$$\{(x,0)\phi \,|\, \phi \in \mathrm{Aut}(F)\} = X \times \{0\}. \tag{3}$$

An endomorphism f of F is called constant if there exists an element $a \in F$ such that $(x,0)f = a$ for all $x \in X$. This endomorphism is denoted by f_a. Let $g \in \mathrm{End}(F)$. For all $x \in X$, we get $(x,0)(f_a g) = ((x,0)f_a)g = ag = (x,0)f_{ag}$, hence

$$f_a g = f_{ag}, \tag{4}$$

because $X \times \{0\}$ is the least generating set of F.

Suppose that $g \in \mathrm{End}(F)$ is constant, and let $\phi \in \mathrm{Aut}(F)$. In this case, $g = f_a$ for some $a \in F$, and for all $x \in X$, we have $(x,0)(\phi f_a) = ((x,0)\phi)f_a = a = (x,0)f_a$. Hence

$$\phi f_a = f_a. \tag{5}$$

Conversely, suppose $g \in \mathrm{End}(F)$ satisfies $\phi g = g$ for all $\phi \in \mathrm{Aut}(F)$. Fix $x \in X$. Then $(x,0)g = (x,0)(\phi g) = ((x,0)\phi)g$. Applying (3) to this equality, we obtain $(x,0)g = (c,0)g$ for all $c \in X$. It follows that $g = f_a$, where $a = (x,0)g$; that is, g is a constant endomorphism.

Let further $g \in \mathrm{End}(F)$ be constant and idempotent. Then

$$g = f_a \tag{6}$$

for some $a \in F$, and by (4) we have $f_a = f_a f_a = f_{af_a}$. This implies $a = af_a$.

We now provide the proof by a case-by-case analysis for $\widetilde{F}[X]$, $\widetilde{F}^*[X]$, $FDSD_n(X)$, and $FNSD_n(X)$.

Case $F = \widetilde{F}[X]$. Here we suppose that $a \notin X \times \{0\}$. Then $a = (x_1 x_2 \dots x_k, m)$ $(k > 1)$, where $x_1, x_2, \dots, x_k \in X$. In this case, a can be represented as

$$(x_1, 0) *_1 (x_2, 0) *_2 \dots *_{k-1} (x_k, 0)$$

for some $*_1, *_2, \dots, *_{k-1} \in \{\dashv, \vdash\}$. We establish

$$af_a = (x_1x_2\dots x_k, m)f_a$$

$$= ((x_1, 0) *_1 (x_2, 0) *_2 \dots *_{k-1} (x_k, 0))f_a$$

$$= (x_1, 0)f_a *_1 (x_2, 0)f_a *_2 \dots *_{k-1} (x_k, 0)f_a$$

$$= (x_1x_2\dots x_k, m) *_1 (x_1x_2\dots x_k, m) *_2 \dots *_{k-1} (x_1x_2\dots x_k, m)$$

$$= ((x_1x_2\dots x_k)^k, m') \neq (x_1x_2\dots x_k, m) = a$$

for some $m' \in \mathbb{N}^0$ as $k > 1$. Consequently, we arrive at a contradiction. So, $a \in X \times \{0\}$. Conversely, considering an endomorphism $f_{(x,0)}$ of $\widetilde{F}[X]$ with $x \in X$, for all $(x_1x_2\dots x_k, m) \in \widetilde{F}[X]$, we have

$$(x_1x_2\dots x_k, m)f^2_{(x,0)}$$

$$= (((x_1, 0) *_1 (x_2, 0) *_2 \dots *_{k-1} (x_k, 0))f_{(x,0)})f_{(x,0)}$$

$$= ((x_1, 0)f_{(x,0)} *_1 (x_2, 0)f_{(x,0)} *_2 \dots *_{k-1} (x_k, 0)f_{(x,0)})f_{(x,0)}$$

$$= ((x, 0) *_1 (x, 0) *_2 \dots *_{k-1} (x, 0))f_{(x,0)}$$

$$= (x, 0) *_1 (x, 0) *_2 \dots *_{k-1} (x, 0)$$

$$= (x_1x_2\dots x_k, m)f_{(x,0)}.$$

Therefore, $f_{(x,0)}$ is constant and idempotent.

Case $F = \widetilde{F}^*[X]$. Applying all arguments from Case $F = \widetilde{F}[X]$ equally to this case, we obtain that $g \in \mathrm{End}(\widetilde{F}^*[X])$ is constant and idempotent if and only if $g = f_a$ for some $a \in X \times \{0\}$.

Case $F = FDSD_n(X)$. If $n = 1$, then clearly the operations of $FDSD_1(X)$ coincide, and hence $FDSD_1(X)$ is a zero semigroup. Consequently, as constant endomorphisms, f_0 and each $f_{(x,0)}$ with $x \in X$ are idempotent. Suppose that $n > 1$. Then, as in the case $n = 1$, f_0 is again an idempotent constant endomorphism of $FDSD_n(X)$. Moreover, from the reasoning in Case $F = \widetilde{F}[X]$, we conclude that for f_a with $a \in D_n \setminus \{0\}$, the element a belongs only to $X \times \{0\}$. Thus, for $n \geq 1$, an endomorphism g of $FDSD_n(X)$ is constant and idempotent if and only if $g = f_a$ for some $a \in (X \times \{0\}) \cup \{0\}$.

Case $F = FNSD_n(X)$. The fact that $g \in \mathrm{End}(FNSD_n(X))$ is constant and idempotent if and only if $g = f_a$ for some $a \in (X \times \{0\}) \cup \{0\}$ can be proved in the same way as Case $F = FDSD_n(X)$.

Now, let Y be an arbitrary nonempty set and let

$$\Omega_i : \mathrm{End}(F_i) \to \mathrm{End}(F_i')$$

be an arbitrary isomorphism for all $i \in \{1, 2, 3, 4\}$, where $F_1 = \widetilde{F}[X]$ and $F_1' = \widetilde{F}[Y]$, $F_2 = \widetilde{F}^*[X]$ and $F_2' = \widetilde{F}^*[Y]$, $F_3 = FDSD_n(X)$ and $F_3' = FDSD_n(Y)$, $F_4 = FNSD_n(X)$ and $F_4' = FNSD_n(Y)$. Since $\mathrm{Aut}(F_i)\Omega_i = \mathrm{Aut}(F_i')$, $i \in \{1, 2, 3, 4\}$, it follows that for every automorphism $\phi \in \mathrm{Aut}(F_i)$, the image $\phi\Omega_i \in \mathrm{Aut}(F_i')$. Consider a constant and idempotent endomorphism f_a of F_i ($i \in \{1, 2, 3, 4\}$) for some $a \in X \times \{0\}$. Using (5), we have

$$f_a\Omega_i = (\phi f_a)\Omega_i = (\phi\Omega_i)(f_a\Omega_i) \tag{7}$$

for all $\phi \in \mathrm{Aut}(F_i)$. Taking into account the above established facts that $g \in \mathrm{End}(F)$ is constant if and only if $\phi g = g$ for all $\phi \in \mathrm{Aut}(F)$, that $\mathrm{Aut}(F_i') = \mathrm{Aut}(F_i)\Omega_i$, and that (7) holds, we conclude that $f_a\Omega_i$ is a constant endomorphism of F_i' ($i \in \{1, 2, 3, 4\}$). Moreover, the zero endomorphism $f_0 \in \mathrm{End}(F_i)$ is mapped under Ω_i to the zero endomorphism $f_0' \in \mathrm{End}(F_i')$ for $i \in \{3, 4\}$, since Ω_i is an isomorphism of endomorphism semigroups and therefore must send the zero element f_0 of $\mathrm{End}(F_i)$ to the zero element f_0' of $\mathrm{End}(F_i')$. Therefore, for each $i \in \{1, 2, 3, 4\}$, there exists $d \in F_i'$ such that $f_a\Omega_i = f_d$. Furthermore, f_d is an idempotent endomorphism since f_a is idempotent. Thus, from the above facts and (6), we have $f_d = f_{(y,0)}$ for some $y \in Y$.

Finally, define a map $\theta_i : X \to Y$, where $x\theta_i = y$ if and only if $f_{(x,0)}\Omega_i = f_{(y,0)}$, $i \in \{1, 2, 3, 4\}$. It is clear that θ_i is a bijection, since Ω_i is an isomorphism between the endomorphism semigroups, and the constant idempotents $f_{(x,0)}$ and $f_{(y,0)}$ correspond uniquely to elements $x \in X$ and $y \in Y$. Consequently, $F_i \cong F_i'$ for every $i \in \{1, 2, 3, 4\}$. This proves that each of the free algebras $\widetilde{F}[X]$, $\widetilde{F}^*[X]$, $FDSD_n(X)$, and $FNSD_n(X)$ is determined, up to an isomorphism, by its endomorphism semigroup. □

Following the idea presented above, we now discuss idempotent strong doppelsemigroups, which have not received attention so far. A doppelsemigroup is called *idempotent* [15, 17] if both its operations are idempotent. It is known that in a doppelsemigroup $(D, \dashv, \vdash)$, the operations $\dashv$ and $\vdash$ coincide whenever $(D, \dashv)$ or $(D, \vdash)$ is a rectangular band, a left or right zero semigroup, or when the doppelsemigroup is commutative and idempotent (see [15]). At the same time, there exist idempotent doppelsemigroups whose binary operations $\dashv$ and $\vdash$ are distinct [17]. Nevertheless, this phenomenon does not occur in the class of strong doppelsemigroups. In fact, if $(D, \dashv, \vdash)$ is an idempotent strong doppelsemigroup then

$$x \dashv y = x \dashv (y \vdash y) = x \vdash (y \dashv y) = x \vdash y$$

for all $x, y \in D$. This shows:

Proposition 2.1. *There do not exist idempotent strong doppelsemigroups with distinct operations.*

The problem of constructing a free idempotent doppelsemigroup remains open [15].

Following the classical question of B. I. Plotkin [2] concerning the structure of $\mathrm{Aut}(\mathrm{End}(G))$ for a free group G (or, more generally, a free algebra), one may naturally pose an analogous question for free strong doppelsemigroups. In [19], the group $\mathrm{Aut}(\mathrm{End}(\mathfrak{D}))$ was described for a free dimonoid $\mathfrak{D}$, which was first constructed in [6]. This motivates the following problem:

Problem 2.1. Determine the structure of automorphism groups of the semigroups $\mathrm{End}(\widetilde{F}[X])$, $\mathrm{End}(\widetilde{F}^*[X])$, $\mathrm{End}(FDSD_n(X))$, $\mathrm{End}(FNSD_n(X))$.

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