© Algebra and Discrete Mathematics
Volume **40** (2025). Number 2, pp. 215–225
DOI:10.12958/adm2449

RESEARCH ARTICLE

LocalNR package and some of its applications

Iryna Raievska, Maryna Raievska, and Yaroslav Sysak

Communicated by A. Petravchuk

Abstract. In 2024, the authors developed well-known algorithms for the construction and analysis of finite nearrings, implementing it to compute local nearrings of order greater than 31. The new package was named LocalNR package. In this paper, using our method we

- indicate seventeen 2-generated groups of exponent 8 which are the additive groups of zero-symmetric local nearrings of order 128; and formulate a conjecture on all such nearrings according to which there left only 2 open cases;
- prove that there are no local nearrings with the Miller–Moreno multiplicative group of order 64 and the additive group to be a 2-generated group of order 128.

Introduction

We study algebraic structures called nearrings, which are interesting examples of generalised rings (i.e. addition need not to be commutative, and only one distributive law is assumed) with applications in noncommutative homological algebra, algebraic topology, functional analysis

The first and second authors would like to acknowledge partial funding from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No 873071, SOMPATY.

2020 Mathematics Subject Classification: *16W30.*

Key words and phrases: *nearring, local nearring, Miller–Moreno group, LocalNR package, GAP.*

and in categories with group objects. The classification of nearrings up to certain orders is a challenging problem, and requires extensive computations, which we will implement using GAP [4].

GAP is an open source system for computational discrete algebra, with particular emphasis on Computational Group Theory. GAP provides a programming language, also called GAP, a library of mathematical algorithms implemented in this language, and various libraries of mathematical objects. The SONATA package provides methods for the construction and analysis of finite nearrings, as well as the library of all nearrings up to order 15 and all nearrings with identity up to order 31. For the researchers in nearrings, the list of all 698 local nearrings of order at most 31 up to isomorphism is provided by the GAP package SONATA [1]; however, classifying nearrings of order 32 is a significant challenge. Motivated by this, we have developed and implemented algorithms to compute all local nearrings of further orders, in a new GAP package called LocalNR [11]. The current version of this package (not yet redistributed with GAP) contains all 37441 local nearrings of order at most 361, except those of orders 32, 64, 128, 243, and 256.

A study of local nearrings was initiated by Maxson [6] who defined a number of their basic properties and proved in particular that the additive group of a finite local nearring is a p-group. It follows from Clay, Malone [3] that local nearrings with cyclic additive group are commutative local rings.

A classification of all non-abelian groups of order less than 32 that can be additive groups of a nearring with identity was established in [2]. The same paper also determined the number of non-isomorphic nearrings with identity on such groups. There are 51 non-isomorphic groups of order 32 from which 19 are the additive groups of local nearrings [9]. There exist 267 non-isomorphic groups of order $64 = 2^5$ from which 53 are 2-generated groups and only 24 of these groups are the additive groups of local nearrings [10]. The next natural step is the investigation of all 162 non-isomorphic 2-generated groups of order $128 = 2^7$ with this property. As the result we will provide the classification of local nearrings on such groups.

Local nearrings with commutative multiplicative groups that are not rings were studied in [5]. Also, it was obtained the full classification of such nearrings with cyclic multiplicative groups. We study 2-generated groups of order 128 as the additive groups of local nearrings with the Miller–Moreno multiplicative groups of order 64 using GAP.

Basic definitions and many results concerning nearrings can be for instance found in [7].

1. Preliminaries

Let $[n, i]$ be the i-th group of order n in the SmallGroups library in GAP.

Recall that each finite non-abelian group whose proper subgroups are abelian is called a *Miller–Moreno group* or in other terminology a *minimal non-abelian group*.

We recall some definitions.

Definition 1. A set R with two binary operations " $+$ " and " $\cdot$ " is a nearring if:

1) $(R, +)$ is a group with neutral element 0;

2) $(R, \cdot)$ is a semigroup;

3) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all x, y, $z \in R$.

Such a nearring is called a left nearring. If axiom 3) is replaced by an axiom $(x + y) \cdot z = x \cdot z + y \cdot z$ for all x, y, $z \in R$, then we get a right nearring.

The group $(R, +)$ of a nearring R is denoted by R^+ and called the *additive group* of R. If in addition $0 \cdot x = 0$ for all $x \in R$, then the nearring R is called *zero-symmetric*. Furthermore, R is a *nearring with an identity* i if the semigroup $(R, \cdot)$ is a monoid with identity element i.

Definition 2. A nearring R with identity is called local if the set L of all non-invertible elements of R forms a subgroup of the additive group R^+ and a nearfield, if $L = 0$.

2. LocalNR package: Group functions

The LocalNR package contains the library of local nearrings up to order 361 and some functions to analyze finite nearrings. The current version of this package is version 1.0.4, released on 2024-03-15. We give some functions for local nearrings from the LocalNR package.

2.1. IsMinimalNonAbelianGroup

IsMinimalNonAbelianGroup(G) (property)

Returns: a boolean

The argument is a group G. The output is true if G is a minimal non-abelian group, otherwise the output is false.

```
gap> H:=SmallGroup(120,4);
<pc group of size 120 with 5 generators>
gap> IsMinimalNonAbelianGroup(H);
false
gap> K:=SmallGroup(16,6);
<pc group of size 16 with 4 generators>
gap> IsMinimalNonAbelianGroup(K);
true
gap> IsMinimalNonAbelianGroup(SmallGroup(16,8));
false
```

2.2. IsMetacyclicPGroup

IsMetacyclicPGroup(G) (property)

Returns: a boolean

The argument is a group G. The output is true if G is a metacyclic p-group, otherwise the output is false.

```
gap> IsMetacyclicPGroup(K);
true
gap> IsMetacyclicPGroup(SmallGroup(81,4));
true
gap> IsMetacyclicPGroup(SmallGroup(81,15));
false
```

2.3. EndoOrbitsOfGroup

EndoOrbitsOfGroup(G) (operation)

Returns: EndoOrbitsOfGroup

The argument is a group G.

```
gap> D:=SmallGroup(81,2);
<pc group of size 81 with 4 generators>
```

```
gap> T:=EndoOrbitsOfGroup(D);;
gap> Length(T);
1
gap> Size(T[1][2]);
81
```

2.4. IsEndoCyclicGroup

IsEndoCyclicGroup(G) (property)

Returns: a boolean

The argument is a group G. The output is true if G is a endocyclic group, otherwise the output is false.

Let G be a group and $\operatorname{End} G$ be the set of all its endomorphisms, which can be considered as a semigroup with respect to the composition operation of endomorphisms. For each $g \in G$ we denote by $g^{\operatorname{End} G}$ the set $\{g^\alpha | \alpha \in \operatorname{End} G\}$ of all images of the element g with respect to endomorphisms of $\operatorname{End} G$. A group G is called endocyclic if it contains an element g with $G = g^{\operatorname{End} G}$.

```
gap> D:=SmallGroup(81,2);
<pc group of size 81 with 4 generators>
gap> IsEndoCyclicGroup(D);
true
```

3. LocalNR package: Nearring functions

3.1. UnitsOfNearRing

UnitsOfNearRing(R) (attribute)

Returns: a set

The argument is a nearring R. The output is true if R is a nearring with identity, otherwise the output is `Error, no units exist`.

```
gap> N:=LocalNearRing(32,5,16,3,8);
ExplicitMultiplicationNearRing ( <pc group of size 32 with
5 generators> , multiplication )
gap> U:=UnitsOfNearRing(N);
```

```
[ (f1), (f1*f5), (f1*f4), (f1*f4*f5), (f1*f3), (f1*f3*f5), (f1*f3*f4),
(f1*f3*f4*f5), (f1*f2), (f1*f2*f5), (f1*f2*f4), (f1*f2*f4*f5), (f1*f2*f3),
(f1*f2*f3*f5), (f1*f2*f3*f4), (f1*f2*f3*f4*f5) ]
gap> Un:=NearRingUnits(N);;
gap> U=Un;
true
```

3.2. IsLocalNearRing

IsLocalNearRing(R) (property)

Returns: a boolean

The argument is a nearring R. The output is true if R is a local nearring, otherwise the output is false.

```
gap> H:=SmallGroup(16,6);
<pc group of size 16 with 4 generators>
gap> A:= AutomorphismNearRing(H);
AutomorphismNearRing( <pc group of size 16 with 4 generators> )
gap> Size(A);
64
gap> IsLocalNearRing(A);
true
gap> K:=LibraryNearRingWithOne(SmallGroup(8,2),1);
#I using isomorphic copy of the group LibraryNearRing(8/2, 814)
gap> IsLocalNearRing(K);
false
```

3.3. IsLocalRing

IsLocalRing(R) (property)

Returns: a boolean

The argument is a local nearring R. The output is true if R is a local ring, otherwise the output is false.

```
gap> L:=AllLocalNearRings(16,14,8,4);;
gap> Size(L);
24
gap> F:=Filtered(L,x->IsLocalRing(x));;
gap> Size(F);
1
```

3.4. NearRingNonUnits

NearRingNonUnits(R) (attribute)

Returns: a set

The argument is a nearring R. The output is the set of non-invertible elements of R.

```
gap> T:=LocalNearRing(49,2,42,1,1);
ExplicitMultiplicationNearRing ( <pc group of size 49 with
2 generators> , multiplication )
gap> Nu:=NearRingNonUnits(T);
[ (<identity> of ...), (f2), (f2^2), (f2^3), (f2^4), (f2^5), (f2^6) ]
gap> Size(Nu);
7
gap> R:=LibraryNearRing(SmallGroup(8,4),3);
#I using isomorphic copy of the group LibraryNearRing(8/5, 3)
gap> N:=NearRingNonUnits(R);
[ (()), ((1,2,3,4)(5,6,7,8)), ((1,3)(2,4)(5,7)(6,8)), ((1,4,3,2)(5,8,7,6)),
((1,5,3,7)(2,8,4,6)), ((1,6,3,8)(2,5,4,7)), ((1,7,3,5)(2,6,4,8)),
((1,8,3,6)(2,7,4,5)) ]
```

3.5. NonUnitsAsAdditiveSubgroup

NonUnitsAsAdditiveSubgroup(R) (attribute)

Returns: a subgroup

The argument is a local nearring R. The output is the additive subgroup of non-units of R.

```
gap> T:=LocalNearRing(125,4,100,9,1);
ExplicitMultiplicationNearRing ( <pc group of size 125 with
3 generators> , multiplication )
gap> L:=NonUnitsAsAdditiveSubgroup(T);
Group([ <identity> of ..., f2, f3, f2^2, f2*f3, f3^2, f2^3, f2^2*f3,
f2*f3^2, f3^3, f2^4, f2^3*f3, f2^2*f3^2, f2*f3^3, f3^4, f2^4*f3, f2^3*f3^2,
f2^2*f3^3, f2*f3^4, f2^4*f3^2, f2^3*f3^3, f2^2*f3^4, f2^4*f3^3, f2^3*f3^4,
f2^4*f3^4 ])
gap> IdGroup(L);
[ 25, 2 ]
```

4. On local nearrings of order 128

4.1. 2-Generated groups of exponent 8 as the additive groups of zero-symmetric local nearrings of order 128

There exist 2328 non-isomorphic groups of order $128 = 2^7$ from which 162 are 2-generated groups (5 groups are of exponent 64, 18 groups are of exponent 32, 65 groups are of exponent 16, 72 groups are of exponent 8, and 2 groups are of exponent 4).

We denote by C_n and Q_n the cyclic and quaternion groups of order n, respectively.

The following statement partially answers Question 1 of paper [8].

Proposition 1. *The following* 2*-generated groups of exponent* 8 *are the additive groups of zero-symmetric local nearrings of order* 128*:*

IdGroup	*Structure Description*	*Number of LNR*
[128, 2]	$((C_8 \times C_2) \rtimes C_4) \rtimes C_2$	> 41184
[128, 4]	$(C_2 \times Q_8) \rtimes C_8$	> 103424
[128, 5]	$(C_8 \times C_2) \rtimes C_8$	> 1536
[128, 6]	$(C_8 \times C_4) \rtimes C_4$	> 73728
[128, 7]	$(C_8 \times C_2) \rtimes C_8$	> 4160
[128, 8]	$(C_4 \rtimes C_8) \rtimes C_4$	> 10240
[128, 12]	$((C_8 \times C_2) \rtimes C_2) \rtimes C_4$	> 1336
[128, 13]	$(C_8 \times C_2) \rtimes C_8$	> 33928
[128, 27]	$(C_8 \rtimes C_4) \rtimes C_4$	> 106240
[128, 38]	$((C_8 \times C_2) \rtimes C_2) \rtimes C_4$	> 80384
[128, 48]	$(((C_8 \times C_2) \rtimes C_2) \rtimes C_2) \rtimes C_2$	> 194080
[128, 49]	$(C_4 \times C_2 \times C_2) \rtimes C_8$	> 191520
[128, 50]	$((C_4 \times C_2) \rtimes C_8) \rtimes C_2$	> 16992
[128, 51]	$(C_2 \times Q_8) \rtimes C_8$	> 16992
[128, 56]	$(C_4 \times C_4) \rtimes C_8$	> 254208
[128, 57]	$(C_4 \times C_4) \rtimes C_8$	> 127488
[128, 126]	$(C_2.((C_4 \times C_2) \rtimes C_2) =$ $= (C_2 \times C_2).(C_4 \times C_2)) \rtimes C_4$	> 153488

Conjecture. The following 2-generated groups of exponent 8 and only they are the additive groups of zero-symmetric local nearrings of order 128:

- the groups from Proposition 1;
- $(C_8 \times C_2) \rtimes C_8$ [128, 9];
- $(C_4 \rtimes C_8) \rtimes C_4$ [128, 28].

The algorithm for construction of finite local nearrings on the additive group [128, 160] can be extracted from [14] via GAP using the LocalNR package and the dataset of local nearrings on 2-generated groups of order 128 [12].

4.2. On Miller–Moreno groups as multiplicative groups of local nearrings of order 128

It is an easy exercise for example in GAP to get the following assertion.

Remark 1. There are 6 non-isomorphic Miller–Moreno groups of order 64, i.e.:

- $C_8 \rtimes C_8$ [64, 3];
- $(C_8 \times C_2) \rtimes C_4$ [64, 17];
- $C_{16} \rtimes C_4$ [64, 27];
- $(C_{16} \times C_2) \rtimes C_2$ [64, 29];
- $C_4 \rtimes C_{16}$ [64, 44];
- $C_{32} \rtimes C_2$ [64, 51].

We provide the GAP-code for these calculations.

```
gap> G:=AllSmallGroups(64);;
gap> Size(G);
267
gap> F:=Filtered(G,x->IsMinimalNonAbelianGroup(x)=true);;
gap> Size(F);
6
gap> List(F,IdGroup);
[[64, 3], [64, 17], [64, 27], [64, 29], [64, 44], [64, 51]]
gap> List(F,StructureDescription);
["C8 ⋊ C8", "(C8 × C2) ⋊ C4", "C16 ⋊ C4", "(C16 × C2) ⋊ C2", "C4 ⋊ C16", "C32 ⋊ C2"]
```

Let R be a local nearring on the 2-generated groups of order 128, whose multiplicative group R^* is a Miller–Moreno group of order 64. As follows from Lemma 9 [13], then R^* can be only a non-metacyclic group. Therefore, R^* is one of two groups G with $IdGroup(G) = [64, 17]$ and $IdGroup(G) = [64, 29]$.

The following theorem is the result of calculations performed using GAP and the LocalNR package.

After analyzing the 2-generated groups of order 128 as the additive groups of local nearrings with the non-metacyclic Miller–Moreno multiplicative groups of order 64 using GAP, it was found that there are no such local nearrings. This allows us to make the following assumption.

Theorem 1. *There is no local nearring on* 2*-generated groups of order* 128*, whose multiplicative group is a Miller–Moreno group of order* 64*.*

References

[1] Aichinger, E., Binder, F., Ecker, Ju., Mayr, P., Noebauer, C.: SONATA — system of near-rings and their applications. GAP package, Version 2.9.1. https://gap-packages.github.io/sonata/ (2018)

[2] Boykett, T.H.H., Noebauer, C.: A class of groups which cannot be the additive groups of near-rings with identity. In: Contributions to general algebra **10** (Klagenfurt, 1997). Verlag Johannes Heyn, Klagenfurt, pp. 89–99 (1998)

[3] Clay, J.R., Malone, Jr.: The near-rings with identities on certain finite groups. Math. Scand. **19**, 146–150 (1966). https://doi.org/10.7146/math.scand.a-10803

[4] The GAP Group, GAP — Groups, Algorithms, and Programming. Version 4.14.0. https://www.gap-system.org (2024)

[5] Gorodnik, A.: Local near-rings with commutative groups of units. Houston J. Math. **25**, 223–234 (1999)

[6] Maxon, C.J.: On local near-rings. Math. Z. 106, 197–205 (1968). https://doi.org/10.1007/BF01110133

[7] Pilz, G.: Near-rings: the theory and its applications. North Holland, Amsterdam (1977)

[8] Raievska. I.: Local nearrings with additive groups of order 128. Naukovyi visnyk Uzhhorodskoho universytetu. Seriia Matematyka i informatyka **45**(2), 110–114 (2024). https://doi.org/10.24144/2616-7700.2024.45(2).110-114

[9] Raievska, I., Raievska, M.: Skinchenni lokalni mayzhe-kiltsya (Finite local nearrings). Mohylyanskyy Mat. Zh. 1, 38–48 (2018) (in Ukrainian). https://doi.org/10.18523/2617-7080i2018p38-48

[10] Raievska, I., Raievska, M.: Local nearrings, their structure, and the GAP system. Ukr. Math. J. 76(11), 1831–1848 (2025); translation from Ukr. Mat. Zh. **76**(11), 1629–1644 (2024). https://doi.org/10.1007/s11253-025-02426-y

[11] Raievska, I., Raievska, M., Sysak. Y.: LocalNR, Package of local nearrings. GAP package, Version1.0.4. https://gap-packages.github.io/LocalNR (2024)

[12] Raievska, I., Raievska, M., Sysak, Ya.: DatabaseEndom128: (v0.2) [Data set]. Zenodo. https://doi.org/10.5281/zenodo.7224766 (2024)

[13] Raievska, M.Yu., Sysak, Ya.P.: On local nearrings with Miller–Moreno multiplicative group. Ukr. Math. J. **64**(6), 930–937 (2012). https://doi.org/10.1007/s11253-012-0688-z

[14] Sysak, Ya., Raievska, I., Raievska, M.: Construction-of-local-nearrings-on-additive-group-[128,160]. https://github.com/raemarina/128_160 (2025)

Contact information

I. Raievska, M. Raievska, Ya. Sysak

Institute of Mathematics of NAS of Ukraine, Tereshchenkivska str., 3, 01024, Kyiv, Ukraine
E-Mail: raeirina@imath.kiev.ua, raemarina@imath.kiev.ua, sysak@imath.kiev.ua

Received by the editors: 01.12.2025
and in final form 12.12.2025.