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RESEARCH ARTICLE

Regularity of the partial Baer-Levi semigroups with restricted range

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Abstract. Let Y be a fixed nonempty subset of an infinite set X and let q be an infinite cardinal such that $q \leq |X|$. Let $PS(X,Y,q)$ denote the semigroup of all partial injective transformations from X into Y for which the complement of its range has cardinality q. Then $PS(X,Y,q)$ is a generalization of the partial Baer-Levi semigroup. In this paper, we study several types of regularity on $PS(X,Y,q)$. We characterize all regular, left regular, right regular, completely regular, intra-regular and coregular elements and determine the largest regular subsemigroup of this semigroup. Furthermore, when Y is finite, we present formulas for counting the total number of elements of each type mentioned above.

1. Introduction

Let X be a nonempty set and let $I(X)$ denote the symmetric inverse semigroup on X: that is, the semigroup of all injective mappings α whose domain, dom α, and range, $X\alpha$ are subsets of X. When X is infinite and q is a fixed cardinal such that $\aleph_0 \leq q \leq |X|$, we write

$$BL(X,q) = \{\alpha \in I(X) : \operatorname{dom}\alpha = X \ \text{ and } \ d(\alpha) = q\},$$

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where $d(\alpha) = |X \backslash X\alpha|$ is called the *defect* of α. Then $BL(X, q)$ is a semigroup which is called the Baer-Levi semigroup of type $(|X|, q)$. It is known that $BL(X, q)$ is a right cancellative, right simple semigroup without idempotents. The semigroup $BL(X, q)$ plays a crucial role in semigroup theory, since every semigroup S satisfying these three properties can be embedded in a Baer-Levi semigroup of type (p, p), where $p = |S|$ ([2], Section 8.1).

In 1975, Sullivan [11] introduced the following semigroup:

$$PS(X, q) = \{\alpha \in I(X) : d(\alpha) = q\},$$

and call this the partial Baer-Levi semigroup on X. It is clear that $BL(X, q)$ is a subsemigroup of $PS(X, q)$. Later, Pinto and Sullivan [5] described Green's equivalences, regular elements and ideals of $PS(X, q)$. Moreover, they presented an interesting result that $PS(X, q)$ is neither right simple nor right cancellative and always contains idempotents, this result is completely opposite to what occurs in $BL(X, q)$. In addition, partial orders and maximal subsemigroups on $PS(X, q)$ have been studied in [8–10].

Recently, in 2024, Singha [7] introduced and studied the subsemigroup $PS(X, Y, q)$ of $PS(X, q)$, which consists of all elements of $PS(X, q)$ whose range is contained in a fixed nonempty subset Y of X. Symbolically, this subsemigroup of $PS(X, q)$ is defined as :

$$PS(X, Y, q) = \{\alpha \in I(X) : d(\alpha) = q \text{ and } X\alpha \subseteq Y\}.$$

If $X = Y$, then $PS(X, Y, q) = PS(X, q)$, hence $PS(X, Y, q)$ is a generalization of $PS(X, q)$. The author in [7] characterized the Green's relations and described the natural partial order on $PS(X, Y, q)$. He also showed that $PS(X, Y, q)$ is not a regular semigroup. These findings, along with those from other research on $PS(X, q)$ are the motivation for this study.

The main objective of this paper is to study many types of regularity on $PS(X, Y, q)$. We provide necessary and sufficient conditions for each element of this semigroup to be regular, left regular, right regular, completely regular, intra-regular and coregular. In addition, we determine the largest regular subsemigroup of $PS(X, Y, q)$ and we show that the set of all completely regular elements of $PS(X, Y, q)$ is a disjoint union of permutation groups on some appropriate sets. In the final part of this research, we present some combinatorial proofs for the number of these elements in $PS(X, Y, q)$ when Y is finite.

2. Preliminary

Throughout this paper, unless otherwise specified, we suppose that X is an infinite set, q is an infinite cardinal such that $q \leq |X|$ and Y is a nonempty subset of X. For each mapping $\alpha \in PS(X, Y, q)$, we write

$$\alpha = \begin{pmatrix} a_i \\ y_i \end{pmatrix},$$

where the subscript i belongs to an index set I, the abbreviation $\{y_i\}$ denotes $\{y_i : i \in I\}$, $X\alpha = \{y_i\} \subseteq Y$, $\operatorname{dom} \alpha = \{a_i\}$ and $a_i\alpha = y_i$. We also write

$$g(\alpha) = |X \backslash \operatorname{dom} \alpha| \text{ and } r(\alpha) = |X\alpha|,$$

and refer to these cardinals as the *gap* and the *rank* of α, respectively. Observe that, $|X \setminus Y| \leq |X \setminus X\alpha| = q$ for any $\alpha \in PS(X, Y, q)$, therefore $PS(X, Y, q) \neq \emptyset$ only when $|X \setminus Y| \leq q$.

For a subset A of X, we denote by $\alpha|_A$ the restriction of α to A and we write $A\alpha$ instead of $(A \cap \operatorname{dom} \alpha)\alpha$ for convenience and brevity. Also, we let id_A denote the identity mapping on A, and we write $A = B \mathbin{\dot\cup} C$ to denote A is a disjoint union of B and C. As usual, $\emptyset$ denotes the emptyset, but in some contexts, $\emptyset$ is used to denote the empty (one-to-one) transformation, which is the zero element of $I(X)$. We refer to Lemma 1 of [6] that, if $\alpha \in I(X)$ and $Y, Z \subseteq X$, then $(Y \backslash Z)\alpha = Y\alpha \backslash Z\alpha$. This fact will be used throughout this paper without further discussion.

An element a in a semigroup S is said to be regular if $a = axa$ for some $x \in S$. We also recall that a is called left regular if $a = xa^2$ for some $x \in S$, right regular if $a = a^2x$ for some $x \in S$ and completely regular if $a = axa$ and $ax = xa$ for some $x \in S$. An element a is said to be intra-regular if $a = ya^2z$ for some $y, z \in S$. Moreover, a is said to be a coregular element in S if $a = axa = xax$ for some $x \in S$.

By the above definitions, it can be verified that every coregular element is a regular, left regular, right regular, completely regular and intra-regular element. In addition, if $a^3 = a$, then it is clear that a is a coregular element in S. On the other hand, if a is a coregular, i.e., $a = axa = xax$ for some $x \in S$, then $a = xax = x(axa)x = (xax)ax = (xax)(axa)x = (xax)a(xax) = a^3$. This implies that a is coregular if and only if $a^3 = a$. Next, if a is completely regular, then it is clear that a is regular. Moreover, $a = (ax)a = (xa)a = xa^2$ and $a = a(xa) = a(ax) = a^2x$. Thus, every completely regular element is also left and right regular. Additionally, Petrich and Reilly [4] proved that an element a of a semigroup S is completely regular if and only if a is both a left and a right regular element of S. In addition, if an element a of S is both left regular and right regular such that $a = ba^2$ and $a = a^2c$ for some $b, c \in S$, then $a = baa = ba(a^2c) = (ba)a^2(c)$. Therefore, a is intra-regular. It follows that every completely regular element is an intra-regular element.

3. Regularity

We recall from [7, Proposition 1] that, when $|X| = q$, $PS(X, Y, q)$ contains a zero element which is the empty transformation $\emptyset$ with an empty domain. In this case, it can be directly verified by definition that $\emptyset$ is a regular, left regular, right regular, completely regular, intra-regular and coregular element of $PS(X, Y, q)$. Therefore, for convenience and brevity in proving the results in this section, we will omit the case where $\alpha = \emptyset$. However, after the proof in this section is completed, we will find that $\alpha = \emptyset$ still meets the criteria for being a regular, left regular, right regular, completely regular, intra-regular and coregular element, which will be presented in Theorem 1, Theorem 3, Theorem 4, Corollary 1, Theorem 5, and Theorem 7, respectively.

From [7, Proposition 3], the author proved that $PS(X, Y, q)$ is not a regular semigroup. So, we begin by characterizing all regular elements of $PS(X, Y, q)$.

Theorem 1. *Let $\alpha \in PS(X, Y, q)$. Then α is regular if and only if* $\operatorname{dom} \alpha \subseteq Y$ *and $g(\alpha) = q$. Moreover, in this case, α^{-1} is also a regular element of $PS(X, Y, q)$.*

Proof. Suppose that α is regular, then there exists $\beta \in PS(X, Y, q)$ such that $\alpha = \alpha\beta\alpha$. Then $x\alpha = x\alpha\beta\alpha$ for all $x \in \operatorname{dom} \alpha$. As α is injective, we have $x = x\alpha\beta \in X\beta \subseteq Y$, whence $\operatorname{dom} \alpha \subseteq X\beta \subseteq Y$. Next, we claim that $|X\beta \setminus \operatorname{dom} \alpha| \leq q$. Suppose that $X\beta \setminus \operatorname{dom} \alpha = \{c_i\}$. Then there exists a set $\{b_i\} \subseteq \operatorname{dom} \beta$ and $b_i\beta = c_i$. As $\alpha = \alpha\beta\alpha$, we have that $\beta\alpha|_{X\alpha} = \mathrm{id}_{X\alpha}$. Thus, if $b_i \in X\alpha$, then $b_i = b_i\beta\alpha = c_i\alpha$, this is a contradiction since $c_i \notin \operatorname{dom} \alpha$. Therefore, $\{b_i\} \subseteq X \setminus X\alpha$, whence $|X\beta \setminus \operatorname{dom} \alpha| = |\{b_i\}| \leq |X \setminus X\alpha| = d(\alpha) = q$ as required. Now, as $\operatorname{dom} \alpha \subseteq X\beta$, we have

$$X \setminus \operatorname{dom} \alpha = (X \setminus X\beta) \,\dot{\cup}\, (X\beta \setminus \operatorname{dom} \alpha),$$

where $|X \setminus X\beta| = q$ and $|X\beta \setminus \operatorname{dom} \alpha| \leq q$, whence $g(\alpha) = |X \setminus \operatorname{dom} \alpha| = q$.

For the converse, we suppose that $\operatorname{dom}\alpha \subseteq Y$ and $g(\alpha) = q$. Then $X\alpha^{-1} = \operatorname{dom}\alpha \subseteq Y$ and $d(\alpha^{-1}) = g(\alpha) = q$, whence $\alpha^{-1} \in PS(X, Y, q)$. It is clear that $\alpha = \alpha\alpha^{-1}\alpha$ and $\alpha^{-1} = \alpha^{-1}\alpha\alpha^{-1}$. Hence, α and α^{-1} are regular elements of $PS(X, Y, q)$. □

We recall that, an inverse semigroup is a semigroup S such that for every element $a \in S$, there exists a unique *inverse* $b \in S$ such that $a = aba$ and $b = bab$. Equivalently, S is an inverse semigroup if S is a regular semigroup whose idempotents commute. By [7, Proposition 2], the set of all idempotents in $PS(X, Y, q)$ is defined as

$$E(PS(X, Y, q)) = \{\mathrm{id}_A : A \subseteq Y \text{ and } |X \setminus A| = q\}.$$

Let $R(X, Y, q)$ be the set of all regular elements of $PS(X, Y, q)$, that is,

$$R(X, Y, q) = \{\alpha \in PS(X, Y, q) : \operatorname{dom}\alpha \subseteq Y \text{ and } g(\alpha) = q\}.$$

The next result shows that $R(X, Y, q)$ is the largest regular subsemigroup of $PS(X, Y, q)$.

Theorem 2. *The set $R(X, Y, q)$ is the largest regular subsemigroup of $PS(X, Y, q)$. Moreover, it is also an inverse subsemigroup of $PS(X, Y, q)$.*

Proof. Since every element of $R(X, Y, q)$ is regular, it remains to show $R(X, Y, q)$ is closed. Let $\alpha, \beta \in R(X, Y, q)$, then $\operatorname{dom}\alpha \subseteq Y$, $\operatorname{dom}\beta \subseteq Y$ and $g(\alpha) = q = g(\beta)$. Since $\operatorname{dom}\alpha\beta \subseteq \operatorname{dom}\alpha$, we have

$$\begin{aligned} X \setminus \operatorname{dom}\alpha\beta &= (X \setminus \operatorname{dom}\alpha) \,\dot\cup\, (\operatorname{dom}\alpha \setminus \operatorname{dom}\alpha\beta) \\ &= (X \setminus \operatorname{dom}\alpha) \,\dot\cup\, ((X\alpha)\alpha^{-1} \setminus (X\alpha \cap \operatorname{dom}\beta)\alpha^{-1}) \\ &= (X \setminus \operatorname{dom}\alpha) \,\dot\cup\, (X\alpha \setminus (X\alpha \cap \operatorname{dom}\beta))\alpha^{-1} \\ &= (X \setminus \operatorname{dom}\alpha) \,\dot\cup\, (X\alpha \setminus \operatorname{dom}\beta)\alpha^{-1}. \end{aligned}$$

Since $|(X\alpha \setminus \operatorname{dom}\beta)\alpha^{-1}| \le |(X \setminus \operatorname{dom}\beta)\alpha^{-1}| \le |X \setminus \operatorname{dom}\beta| = q$ and $|X \setminus \operatorname{dom}\alpha| = q$, we get $g(\alpha\beta) = |X \setminus \operatorname{dom}\alpha\beta| = q$, whence $\alpha\beta \in R(X, Y, q)$. Therefore, $R(X, Y, q)$ is the largest regular subsemigroup of $PS(X, Y, q)$. Next, since all idempotents of $PS(X, Y, q)$ have the form id_A for some $A \subseteq Y$ such that $|X \setminus A| = q$ and all of these mappings commute. Therefore, $R(X, Y, q)$ is an inverse subsemigroup of $PS(X, Y, q)$. □

Next, we characterize left regular elements of $PS(X, Y, q)$.

Theorem 3. *Let $\alpha \in PS(X, Y, q)$. Then α is left regular if and only if* $\operatorname{dom}\alpha \subseteq X\alpha$.

Proof. Suppose that α is left regular. Then $\alpha = \lambda\alpha^2$ for some $\lambda \in PS(X,Y,q)$. Thus, $X\alpha \subseteq X\alpha^2$. In general, as $X\alpha^2 \subseteq X\alpha$, we have $X\alpha = X\alpha^2 = (X\alpha \cap \text{dom } \alpha)\alpha$. Consequently, as α is injective, we have $X\alpha \cap \text{dom } \alpha = \text{dom } \alpha$, whence $\text{dom } \alpha \subseteq X\alpha$.

Conversely, suppose that $\text{dom } \alpha \subseteq X\alpha$. Then

$$q = |X \setminus X\alpha| \leq |X \setminus \text{dom } \alpha| = g(\alpha).$$

As $\text{dom } \alpha^2 \subseteq \text{dom } \alpha$, we get $q \leq g(\alpha) \leq g(\alpha^2)$ and we may write

$$X \setminus \text{dom } \alpha^2 = (X \setminus \text{dom } \alpha) \,\dot\cup\, (\text{dom } \alpha \setminus \text{dom } \alpha^2),$$

which implies

$$g(\alpha^2) = g(\alpha) + |\text{dom } \alpha \setminus \text{dom } \alpha^2|. \tag{1}$$

Suppose, for contradiction, that $g(\alpha^2) = t > g(\alpha)$. Then substitute $g(\alpha^2) = t$ in (1), we get $|\text{dom } \alpha \setminus \text{dom } \alpha^2| = t$. As $\text{dom } \alpha \subseteq X\alpha$, we may write

$$X \setminus \text{dom } \alpha = (X \setminus X\alpha) \,\dot\cup\, (X\alpha \setminus \text{dom } \alpha),$$

which implies

$$\begin{aligned} g(\alpha) &= d(\alpha) + |X\alpha \setminus \text{dom } \alpha| \\ &= d(\alpha) + |(X\alpha \setminus \text{dom } \alpha)\alpha^{-1}| \\ &= d(\alpha) + |(X\alpha)\alpha^{-1} \setminus (X\alpha \cap \text{dom } \alpha)\alpha^{-1}| \\ &= d(\alpha) + |\text{dom } \alpha \setminus \text{dom } \alpha^2| \\ &= q + t \\ &= t. \end{aligned}$$

which is a contradiction, whence $q \leq g(\alpha) = g(\alpha^2)$. Next, as $\text{dom } \alpha \subseteq X\alpha$, we get $X\alpha^2 = (X\alpha \cap \text{dom } \alpha)\alpha = (\text{dom } \alpha)\alpha = X\alpha$. Thus, we can write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix} \text{ and } \alpha^2 = \begin{pmatrix} b_i \\ x_i \end{pmatrix},$$

where $\{b_i\} = \text{dom } \alpha^2 \subseteq \text{dom } \alpha = \{a_i\} \subseteq X\alpha = \{x_i\} = X\alpha^2 \subseteq Y$. We consider two cases.

Case 1. If $g(\alpha^2) = q$, then $d((\alpha^2)^{-1}) = g(\alpha^2) = q$ and $X(\alpha^2)^{-1} = \{b_i\} \subseteq Y$, whence $(\alpha^2)^{-1} \in PS(X,Y,q)$. Let $\lambda = \alpha(\alpha^2)^{-1} \in PS(X,Y,q)$, we have $\alpha = \lambda\alpha^2$.

Case 2. If $g(\alpha^2) > q$, then we have $q < g(\alpha) = g(\alpha^2)$. We see that

$$X \setminus \text{dom } \alpha^2 = (Y \setminus \text{dom } \alpha^2) \,\dot\cup\, ((X \setminus \text{dom } \alpha^2) \cap (X \setminus Y)), \tag{2}$$

where $|X \setminus \operatorname{dom} \alpha^2| = g(\alpha^2) > q$ and $|(X \setminus \operatorname{dom} \alpha^2) \cap (X \setminus Y)| \leq |X \setminus Y| \leq q$. Then, from (2), we have $|Y \setminus \operatorname{dom} \alpha^2| = |X \setminus \operatorname{dom} \alpha^2| > q$. Now, write $Y \setminus \operatorname{dom} \alpha^2 = \{v_k\} \dot{\cup} U$, where $|\{v_k\}| = g(\alpha^2)$ and $|U| = q$. As $g(\alpha) = g(\alpha^2)$, we can suppose $X \setminus \operatorname{dom} \alpha = \{u_k\}$ and define

$$\lambda = \begin{pmatrix} a_i & u_k \\ b_i & v_k \end{pmatrix}.$$

It is clear that λ is injective, $X\lambda \subseteq Y$ and $\alpha = \lambda\alpha^2$. Moreover, we have $d(\lambda) = |X \setminus (\{b_i\} \cup \{v_k\})| = |U| + |X \setminus Y| = q$, whence $\lambda \in PS(X, Y, q)$.

Hence, in both cases, α is a left regular element of $PS(X, Y, q)$ as required. □

Theorem 4. *Let $\alpha \in PS(X, Y, q)$. Then α is right regular if and only if $X\alpha \subseteq \operatorname{dom} \alpha$.*

Proof. Suppose that α is right regular. Then $\alpha = \alpha^2\lambda$ for some $\lambda \in PS(X, Y, q)$. Thus, $\operatorname{dom} \alpha \subseteq \operatorname{dom} \alpha^2$. As $\operatorname{dom} \alpha^2 \subseteq \operatorname{dom} \alpha$, we get that $\operatorname{dom} \alpha = \operatorname{dom} \alpha^2 = (X\alpha \cap \operatorname{dom} \alpha)\alpha^{-1}$. Hence $X\alpha \cap \operatorname{dom} \alpha = X\alpha$. Therefore $X\alpha \subseteq \operatorname{dom} \alpha$.

Conversely, suppose that $X\alpha \subseteq \operatorname{dom} \alpha$. Then $\operatorname{dom} \alpha^2 = (X\alpha \cap \operatorname{dom} \alpha)\alpha^{-1} = (X\alpha)\alpha^{-1} = \operatorname{dom} \alpha$. We may write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \alpha^2 = \begin{pmatrix} a_i \\ c_i \end{pmatrix}.$$

As $\{b_i\} = X\alpha \subseteq Y$ and $d(\alpha) = q$, we can define $\lambda = \begin{pmatrix} c_i \\ b_i \end{pmatrix} \in PS(X, Y, q)$. Then $\alpha = \alpha^2\lambda$. Therefore α is right regular. □

As we discussed earlier, an element a in a semigroup S is completely regular if and only if it is both left regular and right regular. Hence, the result below is readily deduced from Theorem 3 and Theorem 4.

Corollary 1. *Let $\alpha \in PS(X, Y, q)$. Then α is completely regular if and only if $X\alpha = \operatorname{dom} \alpha$.*

In view of Corollary 1, a completely regular α in $PS(X, Y, q)$ is a permutation on a set A, i.e., a bijection on a set A, where $A = X\alpha = \operatorname{dom} \alpha$. In addition, as $X\alpha \subseteq Y$ and $d(\alpha) = q$, we have that $A \subseteq Y$ and $|X \setminus A| = q$. Let S_A denote the group of permutations on a set A for which $A \subseteq Y$ and $|X \setminus A| = q$. It is clear that all mappings in S_A satisfy the condition in Corollary 1. By using these results, we can describe the set of all completely regular elements of $PS(X, Y, q)$ in terms of a disjoint union of permutation groups as follows.

Corollary 2. *Let $T = \{A \subseteq Y : |X \setminus A| = q\}$. The set of all completely regular elements in $PS(X, Y, q)$ is precisely the set*

$$\bigcup_{A \in T} S_A.$$

Unlike regular, left regular, right regular and completely regular elements, a classification of intra-regular elements of $PS(X, Y, q)$ must be divided into two cases according to the cardinality of X.

Theorem 5. *Suppose that $|X| = q$. Let $\alpha \in PS(X, Y, q)$. Then α is intra-regular if and only if $|\text{dom } \alpha| = |\text{dom } \alpha^2 \cap Y|$.*

Proof. Suppose that α is intra-regular. Then $\alpha = \lambda\alpha^2\mu$ for some $\lambda, \mu \in PS(X, Y, q)$. Since $X\lambda \subseteq Y$, we have

$$\begin{aligned}|\text{dom } \alpha| &= |\text{dom } \lambda\alpha^2\mu| \\ &= |(X\lambda\alpha^2 \cap \text{dom } \mu)(\lambda\alpha^2)^{-1}| \\ &= |X\lambda\alpha^2 \cap \text{dom } \mu| \\ &\leq |X\lambda\alpha^2| \\ &= |(X\lambda \cap \text{dom } \alpha^2)\alpha^2| \\ &= |X\lambda \cap \text{dom } \alpha^2| \\ &\leq |Y \cap \text{dom } \alpha^2|.\end{aligned}$$

As $\text{dom } \alpha^2 \subseteq \text{dom } \alpha$, we have $|\text{dom } \alpha^2 \cap Y| \leq |\text{dom } \alpha^2| \leq |\text{dom } \alpha|$. Then the equality follows, whence $|\text{dom } \alpha| = |\text{dom } \alpha^2 \cap Y|$.

Conversely, suppose that $|\text{dom } \alpha| = |\text{dom } \alpha^2 \cap Y|$. Write $\text{dom } \alpha^2 \cap Y = \{a_i\}$ and $\alpha = \begin{pmatrix} c_i \\ d_i \end{pmatrix}$, $i \in I$. If $|I| < q$, then we define

$$\lambda = \begin{pmatrix} c_i \\ a_i \end{pmatrix} \text{ and } \mu = \begin{pmatrix} a_i\alpha^2 \\ d_i \end{pmatrix}.$$

Then $\alpha = \lambda\alpha^2\mu$. We also see that $d(\lambda) = |X \setminus \{a_i\}| = q$. Moreover, $X\lambda \subseteq Y$, $X\mu = X\alpha \subseteq Y$ and $d(\mu) = d(\alpha) = q$, whence $\lambda, \mu \in PS(X, Y, q)$. On the other hand, if $|I| = q$, then we write $\{a_i\} = \{u_i\} \dot\cup \{v_i\}$ and define

$$\lambda' = \begin{pmatrix} c_i \\ u_i \end{pmatrix} \text{ and } \mu' = \begin{pmatrix} u_i\alpha^2 \\ d_i \end{pmatrix}.$$

Then $\alpha = \lambda'\alpha^2\mu'$. Moreover, $X\mu' = X\alpha \subseteq Y$ and $d(\mu') = d(\alpha) = q$. Hence $\mu' \in PS(X, Y, q)$. In addition, $X\lambda' \subseteq Y$ and $\{v_i\} \subseteq X \setminus X\lambda'$ where $|\{v_i\}| = q$, so $d(\lambda') = q$. Therefore $\lambda' \in PS(X, Y, q)$, whence α is intra-regular. □

Theorem 6. *Suppose that $|X| > q$. Let $\alpha \in PS(X, Y, q)$. Then α is intra-regular if and only if $g(\alpha^2) \leq q$ or $q < g(\alpha^2) = g(\alpha)$.*

Proof. Suppose that α is intra-regular. Then $\alpha = \lambda\alpha^2\mu$ for some $\lambda, \mu \in PS(X, Y, q)$. We also assume that $g(\alpha^2) \nleq q$, then $g(\alpha^2) = t$ for some infinite cardinal t greater than q. Since

$$|X \setminus \text{dom } \alpha^2| = |(X \setminus \text{dom } \alpha^2) \cap X\lambda| + |(X \setminus \text{dom } \alpha^2) \cap (X \setminus X\lambda)|,$$

where $|(X \setminus \text{dom } \alpha^2) \cap (X \setminus X\lambda)| \leq |X \setminus X\lambda| = q < t$, the above equation implies $|(X \setminus \text{dom } \alpha^2) \cap X\lambda| = t$. Then, for each $v_i \in (X \setminus \text{dom } \alpha^2) \cap X\lambda$, $v_i = u_i\lambda$ for some $u_i \in \text{dom } \lambda$, where $u_i\lambda \notin \text{dom } \alpha^2$. So $u_i\alpha = (u_i\lambda)\alpha^2\mu$ is not defined, whence $u_i \notin \text{dom } \alpha$ for all i. Therefore, $\{u_i\} \subseteq X \setminus \text{dom } \alpha$, where $|\{u_i\}| = t$. It follows that $q < g(\alpha^2) = t \leq g(\alpha)$. Clearly, as $\text{dom } \alpha^2 \subseteq \text{dom } \alpha$, we must have $g(\alpha) \leq g(\alpha^2)$, whence $q < g(\alpha^2) = g(\alpha)$ as required.

Conversely, suppose that $g(\alpha^2) \leq q$ or $q < g(\alpha^2) = g(\alpha)$. Let $|X| = p > q$. As the defect of each mapping in $PS(X, Y, q)$ is q, we have that its rank is p. We write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \alpha^2 = \begin{pmatrix} c_i \\ d_i \end{pmatrix},$$

where $i \in I$, $|I| = p$. Moreover, since

$$\text{dom } \alpha^2 = (\text{dom } \alpha^2 \cap Y) \,\dot{\cup}\, (\text{dom } \alpha^2 \setminus Y),$$

where $|\text{dom } \alpha^2 \setminus Y| \leq |X \setminus Y| \leq q$, we have $|\text{dom } \alpha^2 \cap Y| = p$. Then we can write $\text{dom } \alpha^2 \cap Y = \{x_i\} \,\dot{\cup}\, Q$, where $|Q| = q$ and define

$$\mu = \begin{pmatrix} x_i\alpha^2 \\ b_i \end{pmatrix}.$$

Clearly, $X\mu = X\alpha \subseteq Y$ and $d(\mu) = d(\alpha) = q$, whence $\mu \in PS(X, Y, q)$. Now, if $g(\alpha^2) \leq q$, then we define

$$\lambda = \begin{pmatrix} a_i \\ x_i \end{pmatrix}.$$

Clearly, $\alpha = \lambda\alpha^2\mu$ and $X\lambda \subseteq Y$. Moreover,

$$d(\lambda) = |X \setminus \{x_i\}| = |X \setminus \text{dom } \alpha^2| + |\text{dom } \alpha^2 \setminus Y| + |Q| = q,$$

whence, $\lambda \in PS(X, Y, q)$. Finally, suppose that $q < g(\alpha^2) = g(\alpha)$. Since

$$X \setminus \text{dom } \alpha^2 = ((X \setminus \text{dom } \alpha^2) \cap Y) \mathbin{\dot{\cup}} ((X \setminus \text{dom } \alpha^2) \setminus Y),$$

where $|(X \setminus \text{dom } \alpha^2) \setminus Y| \leq |X \setminus Y| \leq q$, we have $|(X \setminus \text{dom } \alpha^2) \cap Y| = g(\alpha^2) > q$. We write $(X \setminus \text{dom } \alpha^2) \cap Y = \{u_k\} \mathbin{\dot{\cup}} R$, where $|\{u_k\}| = g(\alpha^2)$ and $|R| = q$. Since $g(\alpha^2) = g(\alpha)$, we can assume that $X \setminus \text{dom } \alpha = \{v_k\}$ and define

$$\lambda' = \begin{pmatrix} a_i & v_k \\ x_i & u_k \end{pmatrix}.$$

We see that $\alpha = \lambda'\alpha^2\mu$ and $X\lambda' = \{x_i\} \cup \{u_k\} \subseteq Y$. Moreover, $d(\lambda') = |X \setminus (\{x_i\} \cup \{u_k\})| = |X \setminus Y| + |Q| + |R| = q$, whence $\lambda' \in PS(X, Y, q)$. Hence, α is intra-regular. □

In the following theorem, we characterize the coregular elements of $PS(X, Y, q)$.

Theorem 7. *Let $\alpha \in PS(X, Y, q)$. Then α is coregular if and only if α satisfies the following condition: if $x\alpha = y$, then $y\alpha = x$ for all $x, y \in \text{dom } \alpha$. Equivalently, $\alpha^2 = \text{id}_{\text{dom } \alpha}$.*

Proof. Suppose that α is coregular. Then $\alpha^3 = \alpha$ as discussed in Section 2. For each $x_i \in \text{dom } \alpha$, we suppose that $x_i\alpha = y_i$. Then $x_i\alpha = x_i\alpha^3 = y_i\alpha^2$. Consequently, since α is injective, we get that $y_i\alpha = x_i$, which satisfies the desired condition. In addition, we get that $x_i\alpha^2 = y_i\alpha = x_i$, which implies $\alpha^2 = \text{id}_{\text{dom } \alpha}$ as required. Conversely, suppose that $\alpha^2 = \text{id}_{\text{dom } \alpha}$. Then $\alpha^3 = \alpha^2\alpha = \text{id}_{\text{dom } \alpha}\alpha = \alpha$. Hence, α is a coregular element of $PS(X, Y, q)$. □

By taking $Y = X$, the results obtained in this research can be extended to the semigroup $PS(X, q)$ as follows.

Corollary 3. *Let $\alpha \in PS(X, q)$. Then the following hold:*

(1) *α is regular if and only if $g(\alpha) = q$. Moreover, whenever this occurs, α^{-1} is also a regular element in $PS(X, q)$.*

(2) *α is left regular if and only if $\text{dom } \alpha \subseteq X\alpha$.*

(3) *α is right regular if and only if $X\alpha \subseteq \text{dom } \alpha$.*

(4) *α is completely regular if and only if $X\alpha = \text{dom } \alpha$.*

(5) *Suppose that* $|X| = q$. *Then* α *is intra-regular if and only if* $|\text{dom }\alpha| = |\text{dom }\alpha^2|$.

(6) *Suppose that* $|X| > q$. *Then* α *is intra-regular if and only if* $g(\alpha^2) \le q$ *or* $q < g(\alpha^2) = g(\alpha)$.

(7) α *is coregular if and only if* α *satisfies the following condition: if* $x\alpha = y$, *then* $y\alpha = x$ *for all* $x, y \in \text{dom }\alpha$. *Equivalently,* $\alpha^2 = \text{id}_{\text{dom }\alpha}$.

4. Combinatorial results

In this section, Y is assumed to be a nonempty finite subset of X. Then, as X is infinite, we have $|X| = |X \setminus Y| \le q$. Therefore, $|X| = q$ and so $PS(X, Y, q)$ contains a zero element which is the empty transformation $\emptyset$.

First, let us reconsider the results obtained in Section 3 under the assumption that Y is finite. We observe that, since Y is finite, if $\alpha \in PS(X, Y, q)$ such that $\text{dom }\alpha \subseteq Y$, then $\text{dom }\alpha$ is also finite. It follows that $g(\alpha) = |X \setminus \text{dom }\alpha| = |X| = q$. Hence, the following result is a direct consequence of Theorem 1.

Corollary 4. *Let* Y *be a nonempty finite subset of* X *and let* $\alpha \in PS(X, Y, q)$. *Let* $\alpha \in PS(X, Y, q)$. *Then* α *is regular if and only if* $\text{dom }\alpha \subseteq Y$.

For the next result, by using the condition that Y is a finite set, the results proved in Section 3 appear simpler.

Corollary 5. *Let* Y *be a nonempty finite subset of* X *and let* $\alpha \in PS(X, Y, q)$. *Then the following statements are equivalent:*

(1) α *is left regular.*
(2) α *is right regular.*
(3) α *is completely regular.*
(4) α *is intra-regular.*
(5) $X\alpha = \text{dom }\alpha$.

Proof. We first show that (4) implies (2). To do this, we let α be an intra-regular element. Since $|X| = q$, Theorem 5 implies that $|\text{dom }\alpha| = |\text{dom }\alpha^2 \cap Y|$. Consequently, since Y is finite, we have $\text{dom }\alpha^2 \cap Y$ is finite, whence $\text{dom }\alpha$ is also a finite set. Since $\text{dom }\alpha^2 \cap Y \subseteq \text{dom }\alpha^2 \subseteq \text{dom }\alpha$, it follows that $\text{dom }\alpha = \text{dom }\alpha^2 \cap Y \subseteq Y$. Now we have $\text{dom }\alpha^2 \subseteq \text{dom }\alpha \subseteq Y$, which implies $\text{dom }\alpha = \text{dom }\alpha^2 \cap Y = \text{dom }\alpha^2$. It follows that $\text{dom }\alpha = (X\alpha \cap \text{dom }\alpha)\alpha^{-1}$, which means $X\alpha \cap \text{dom }\alpha = X\alpha$, whence $X\alpha \subseteq \text{dom }\alpha$. Therefore, α is right regular by Theorem 4. Next, we show that (2) implies (3). Suppose that α is right regular, we have $X\alpha \subseteq \operatorname{dom}\alpha$ by Theorem 4. Since $X\alpha \subseteq Y$, we have $X\alpha$ is also a finite set. Consequently, as $|\operatorname{dom}\alpha| = |X\alpha|$, the condition $X\alpha \subseteq \operatorname{dom}\alpha$ implies that $X\alpha = \operatorname{dom}\alpha$. Therefore, α is completely regular by Corollary 1. As discussed in Section 2 that (3) implies (4), it can be deduced that (2), (3) and (4) are equivalent. Next, we show that (1) implies (3). Suppose that α is left regular, then $\operatorname{dom}\alpha \subseteq X\alpha$ by Theorem 3. Since Y is finite and $\operatorname{dom}\alpha \subseteq X\alpha \subseteq Y$. Then $\operatorname{dom}\alpha$ is a finite subset of a finite set $X\alpha$, where $|\operatorname{dom}\alpha| = |X\alpha|$. This implies $X\alpha = \operatorname{dom}\alpha$, whence α is completely regular by Corollary 1. We now recall what was discussed in Section 2, namely that (3) implies (1), so (1) and (3) are equivalent. Finally, since (3) and (5) are equivalent by Corollary 1, it follows that all five conditions are equivalent when Y is a finite set. □

We know from Corollary 5 that the numbers of left regular, right regular, completely regular, and intra-regular elements of $PS(X,Y,q)$ are equal, while the numbers of regular and coregular elements may differ.

Recall that if A and B are finite sets with equal cardinality k, then there are exactly $k!$ bijections from A onto B.

Theorem 8. *Let Y be a nonempty finite subset of X such that $|Y| = n$. Then*

(1) *The total number of regular elements of $PS(X,Y,q)$ is*

$$\sum_{k=0}^{n} k!\binom{n}{k}^2.$$

(2) *The total number of left regular (right regular, completely regular, intra-regular) elements of $PS(X,Y,q)$ is*

$$\sum_{k=0}^{n} k!\binom{n}{k}.$$

Proof. To prove (1), let α be a regular element of $PS(X,Y,q)$. By Corollary 4, we have $\operatorname{dom}\alpha \subseteq Y$. Now, $\operatorname{dom}\alpha$ and $X\alpha$ are two finite subsets of Y with the same size, say $|\operatorname{dom}\alpha| = |X\alpha| = k$ for some integer k such that $1 \leq k \leq n$. Then, there are $\binom{n}{k}\cdot\binom{n}{k} = \binom{n}{k}^2$ ways to choose the domain set and the image set from the set Y. Next, since α is a bijection from $\operatorname{dom}\alpha$ onto $X\alpha$, there are $k!\binom{n}{k}^2$ possible ways to define a regular element α. In addition, since $\emptyset$ is also regular and $0! = 1 = \binom{n}{0}$, we will also add the number of this mapping to the formula. Hence, the total number of regular elements of $PS(X,Y,q)$ is $\sum_{k=0}^{n} k!\binom{n}{k}^2$.

For the proof of (2), we have dom $\alpha = X\alpha$ by Corollary 5. So, there are $\binom{n}{k}$ ways to choose the domain set, which coincides with the image set. The remaining proof can be continued following the same steps as in the proof of (1). □

In what follows, let $\alpha \in PS(X,Y,q)$ and $x, y \in \operatorname{dom} \alpha$ such that $x \neq y$. We call an unordered pair $\{x,y\}$ a symmetric pair of α if x and y satisfy the condition : if $x\alpha = y$, then $y\alpha = x$. We also call x a fixed point of α if $x\alpha = x$. It follows by Theorem 7 that, α is coregular if every element of dom α is contained in some symmetric pairs, or is a fixed point. For example, let $X = \mathbb{N}$ denote the set of all positive integers, $Y = \{1,2,3,...,10\}$ and let $q = \aleph_0$. Define

$$\theta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 2 & 4 & 3 & 5 & 6 & 1 \end{pmatrix}, \ \delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 4 & 5 & 3 & 6 & 8 & 7 \end{pmatrix},$$

then $\theta, \delta \in PS(X,Y,q)$. We can verify, according to the condition of Theorem 7, that θ is coregular while δ is not. We also see that $\{1,7\}$ and $\{3,4\}$ are all symmetric pairs of θ and $2,5,6$ are all fixed points of θ, which can be seen that every element of dom θ is either in some symmetric pairs or is a fixed point. For δ, we see that $\{1,2\}$ and $\{7,8\}$ are the only two symmetric pairs of δ, but 3, 4, 5 are neither members of these pairs nor fixed points of δ. We notice that, if $\alpha = \mathrm{id}_A$ for some subset A of Y, then α is coregular and all elements in dom α are fixed points of α. Consequently, the identity mapping on A has no symmetric pairs.

To count the number of coregular elements of $PS(X,Y,q)$, we need to refer to the floor function of a real number x that gives the greatest integer less than or equal to x, and it is denoted by $\lfloor x \rfloor$. We observe that there are at most $\lfloor \frac{p}{2} \rfloor$ symmetric pairs for a coregular element α with $|\operatorname{dom} \alpha| = p$. We also refer to the double factorial of an integer n, where $n \geq -1$, which is denoted by $n!!$. The double factorial is defined as follows [1, p. 544–547]:

$$n!! = \begin{cases} 2 \times 4 \times 6 \times \ldots \times n & \text{if } n > 0 \text{ and } n \text{ is even;} \\ 1 \times 3 \times 5 \times \ldots \times n & \text{if } n > 0 \text{ and } n \text{ is odd;} \end{cases}$$

and $0!! = (-1)!! = 1$. To clarify, $n!!$ is not equal to $(n!)!$. For example,

$3!! = 1 \times 3 = 3$, whereas $(3!)! = 6! = 720$. For the proof of Theorem 9, the following result which has been discussed in [3] is required.

Lemma 1. *Let n be a positive number. The number of ways to partition $2n$ distinct objects into n unordered pairs, where the order of the pairs does not matter is $(2n-1)!!$.*

To prove the following theorem, we also recall that, if α is coregular in $PS(X, Y, q)$, then α is left (right, completely, intra) regular. So, by Corollary 5, we have $\operatorname{dom} \alpha = X\alpha \subseteq Y$.

Theorem 9. *Let Y be a nonempty finite subset of X. Then the following hold:*

(1) *If $|Y| = 1$, then there are exactly two coregular elements of $PS(X, Y, q)$.*

(2) *If $|Y| = n \geq 2$, then the total number of coregular elements of $PS(X, Y, q)$ is*

$$(n+1) + \sum_{p=2}^{n} \left(\binom{n}{p} \cdot \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{2k} (2k-1)!! \right).$$

Proof. To prove (1), suppose that $Y = \{a\}$. Then, by Theorem 7, we can see that $\emptyset$ and the identity mapping on a singleton set $\{a\}$ are the only two coregular elements of $PS(X, Y, q)$.

To prove (2), suppose that $|Y| = n \geq 2$. We will count the total number of possible ways to construct a coregular element $\alpha \in PS(X, Y, q)$ by considering the cases according to the size of $\operatorname{dom} \alpha$.

Case 1 : $|\operatorname{dom} \alpha| = 0$. In this case, the only coregular element with an empty domain is $\alpha = \emptyset$.

Case 2 : $|\operatorname{dom} \alpha| = 1$. By Theorem 7, it is clear that a coregular element with a single domain is precisely the identity mapping on a singleton set $\{a\}$ for some $a \in Y$. Since $|Y| = n$, we get n possible ways to construct such coregular elements.

Case 3 : $|\operatorname{dom} \alpha| = p$, where $2 \leq p \leq n$. The process for constructing coregular elements containing p members in a domain is as follows:

Step 1. Form the domain of α. This step can be done in $\binom{n}{p}$ ways since $\operatorname{dom} \alpha \subseteq Y$, $|Y| = n$ and $|\operatorname{dom} \alpha| = p$.

Step 2. Form disjoint symmetric pairs in the domain of α, where the number of these pairs can range from 0 (when α is the identity mapping) to $\lfloor \frac{p}{2} \rfloor$. To find the number of ways to form exactly k disjoint symmetric pairs, where $0 \le k \le \lfloor \frac{p}{2} \rfloor$, we perform the following steps:

Step 2.1 Select $2k$ elements out of p elements to be in k symmetric pairs. This can be done in $\binom{p}{2k}$ ways.

Step 2.2 Arrange these $2k$ elements into k unordered pairs, and each pair will create one symmetric pair. By Lemma 1, the number of ways to do this is equal to $(2k-1)!!$.

Step 2.3 The remaining $p-2k$ elements (if any) will be designated as fixed points, i.e., $x\alpha = x$. This step can only be done in one way.

In Step 2.2, Lemma 1 cannot be applied when $k = 0$. However, in this case, the coregular element α has no symmetric pairs, i.e., it is the identity mapping on its own domain and $\binom{p}{2k}(2k-1)!! = \binom{p}{0}(-1)!! = 1$, which already represents the number of α. Therefore, we can include the case $k = 0$ in all the steps above. Then, multiplying these steps gives the number of coregular elements with exactly k pairs, which is $\binom{n}{p} \cdot \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{2k}(2k-1)!!$. Next, by summing over all possible values of p from 2 to n gives the total number of coregular elements in Case 3, which is $\sum_{p=2}^{n} \left(\binom{n}{p} \cdot \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{2k}(2k-1)!! \right)$.

Finally, the total number of coregular elements is obtained by combining the numbers from all three cases, which is equal to

$$(n+1) + \sum_{p=2}^{n} \left(\binom{n}{p} \cdot \sum_{k=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{2k}(2k-1)!! \right).$$

□

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