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Divisor function of the Gaussian integers weighted by the Kloosterman sum

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Abstract. We study the mean values of the divisor function $\tau(\omega)$ over the ring of Gaussian integers G when weighted by Kloosterman sums. For $\alpha, \beta, \gamma \in G$ with $\gamma \neq 0$, let

$$K(\alpha, \beta; \gamma) = \sum_{x \in G_\gamma^*} \exp\left(2\pi i \Re\left(\frac{\alpha x + \beta x^{-1}}{\gamma}\right)\right).$$

We obtain an asymptotic formula for

$$\sum_{N(\omega) \leq X} \tau(\omega) \cdot K(1, \alpha\omega; \gamma),$$

uniformly in α co-prime to γ and with explicit dependence on $N(\gamma)$. Our approach combines a Selberg–Kuznetsov–type identity over G with bounds for $K(\alpha, \beta; \gamma)$ in prime-power modulus, together with Dirichlet–series methods for twisted sums

$$Z_m(s; \delta_1, \delta_2) = \sum_{\omega \in G} \frac{e^{4mi \arg(\omega + \delta_1)} \cdot e^{2\pi i \cdot \Re(\delta_2 \omega)}}{N(\omega + \delta_1)^s}.$$

We prove a truncated functional equation for Z_m, establish mean-square bounds on the critical line, and deduce the required cancellation in the Kloosterman–weighted average of $\tau(\omega)$. As by-products we record a generalized Selberg–Kuznetsov identity in G and Weil–type bounds for $K(\alpha, \beta; \mathfrak{p}^m)$. These results extend classical techniques for $\mathbb{Z}$ to the Gaussian setting and may be of independent interest for additive problems in G involving divisor-type functions and exponential sums.

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1. Introduction

Let G denote the ring of Gaussian integers. For $\gamma \in G$, let G_γ denote the residue class ring modulo γ. For $\gamma \in G$, write G_γ^* for the multiplicative group of units modulo γ.

For $\alpha, \beta, \gamma \in G$ the Kloosterman sum $K(\alpha, \beta; \gamma)$ is defined by the equality

$$K(\alpha, \beta; \gamma) = \sum_{x \in G_\gamma^*} exp\left(2\pi i \Re\left(\frac{\alpha x + \beta x^{-1}}{\gamma}\right)\right),$$

where x^{-1} is the multiplicative inverse modulo γ for x.

We derive an asymptotic formula for the mean value of the divisor function $\tau(\omega)$ over G weighted by the Kloosterman sums.

We use the following notations:

- let $\nu(\alpha)$ denote the number of non-associated prime divisors $\alpha \in G$;
- $\mathfrak{p}$ will always denote the Gaussian prime;
- $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ be the greatest common divisor of $\alpha_1, \alpha_2, \ldots, \alpha_k$, $\alpha_j \in G$;
- $\widetilde{\varphi}(q)$ denotes the Euler totient on G; write $\widetilde{\varphi}(q) = N(q) \prod\limits_{\mathfrak{p}|q} \left(\frac{N(\mathfrak{p})-1}{N(\mathfrak{p})}\right)$;
- by $\tau(\alpha)$ we denote the number of non-associated divisors of α, $\alpha \in G$, $\tau^{(m)}(\alpha) = \sum\limits_{\substack{\delta \in G \\ \delta | \alpha}} e^{4mi \arg \delta}$;
- unless stated otherwise, sums over G are taken over non-associated Gaussian integers;
- $N(\alpha) = |\alpha|^2$, $\alpha \in G$;
- let H be a sufficiently large positive number; letters k, m, n denote integers, while α, β, γ, δ denote Gaussian integers;
- by ε we denote an arbitrary small positive number, not necessarily the same in different occurrences. Unless explicitly stated otherwise, the constants in the Vinogradov and Landau symbols are absolute or depend on ε;
- we denote by $\mu(\alpha)$ the Möbius function over G;

- as usual $[t]$ denotes the integer part of t;
- further $\exp(2\pi i t) = e^{2\pi i t}$;
- for any α and γ such that $(\alpha, \gamma) = 1$ we denote by α^{-1} the inverse of α modulo γ. If the value of the modulus is clear from the context then we write for simplicity α^{-1};
- and finally, by $\square$ we mark an end of a proof or its absence.

2. Auxiliary

First, we recall the necessary background about the Kloosterman sums over G.

Lemma 1. *Let α, β, γ be the Gaussian integers and let $\nu(\gamma)$ be the number of different prime divisors of γ. Then for $\gamma \neq 0$ the following estimate*

$$|K(\alpha, \beta; \gamma)| \leq 2^{\nu(\gamma)+1} N((\alpha, \beta, \gamma))^{\frac{1}{2}} \cdot N(\gamma)^{\frac{1}{2}}$$

holds.

Moreover, if γ_1 be square-free and γ_2 be square-free part of γ_1 then for $\gamma = \gamma_1 \cdot \gamma_2$, $(\gamma_1, \gamma_2) = 1$ and for any $\beta \in G$, $(\beta, \gamma) = \delta$, $\delta_1 | \gamma_1$, $\delta_2 | \gamma_2$, $N(\delta) > 1$, we have

$$K(1, \beta; \gamma) = \begin{cases} 0 & if \quad N(\delta_2) > 1, \\ \mu(\delta_1) K\left(1, \beta\delta_1^{-2}, \frac{\gamma}{\delta_1}\right) & if \quad N(\delta_2) = 1. \end{cases}$$

Here, δ_1^{-1} is a multiplicative inverse for $\delta \bmod \frac{\gamma}{\delta_1}$, $\delta^{-2} = (\delta^{-1})^2$.

Proof. The claim follows by reduction to prime powers. Indeed, using the multiplicativity of $K(\alpha, \beta; \gamma)$ with respect to γ, it suffices to prove the equality for $\gamma = \mathfrak{p}$. For $m = 1$ we have $\delta_2 = 1$ and $(\beta, \gamma) = (\beta, \mathfrak{p}) = \mathfrak{p}$, and hence

$$K(1, \beta; \mathfrak{p}^m) = K(1, 0; \mathfrak{p}) = -1 = \mu(\delta_1) K(1, 0; 1).$$

For $m \geq 2$, $(\beta, \mathfrak{p}^m) = \mathfrak{p}^k$, $k \geq 1$. Thus $\delta_1 = 1$, $\delta_2 = \mathfrak{p}^k$. Set $m_1 = \left[\frac{m}{2}\right] \geq 1$.

Let $x = x_0(1 + \mathfrak{p}^{m-m_1} y)$, $x_0 \in G^*_{\mathfrak{p}^{m-m_1}}$, $y \in G_{\mathfrak{p}^m}$. Then

$$x^{-1} = x_0^{-1}(1 - \mathfrak{p}^{m-m_1} y + \mathfrak{p}^{2(m-m_1)} y^2), \quad (x_0^{-1} \pmod{p^m})).$$

Denoting $\beta = \mathfrak{p}^k\beta_0$, $(\beta_0, \mathfrak{p}) = 1$, we can write

$$K(1, \beta_0; \mathfrak{p}^k; \mathfrak{p}^m) = \sum_{x_0 \in G^*_{\mathfrak{p}^{m-m_1}}} \sum_{y \in G_{\mathfrak{p}^{m_1}}} \exp\left(2\pi i \Re \frac{f(x_0, y)}{\mathfrak{p}^m}\right),$$

where

$$f(x_0, y) = x_0 + \beta_0\mathfrak{p}^k x_0^{-1} + \mathfrak{p}^{m-m_1}(y - b_0\mathfrak{p}^k x_0^{-1}\mathfrak{p}^{m-m_1} + \beta_0\mathfrak{p}^k x_0^{-1}\mathfrak{p}^{m-m_1}y^2).$$

Therefore,

$$\begin{aligned} K(1, \mathfrak{p}^k\beta_0; \mathfrak{p}^m) = &\sum_{x_0 \in G^*_{\mathfrak{p}^{m-m_1}}} \exp\left(2\pi i \Re \frac{x_0 + \beta_0\mathfrak{p}^k x_0^{-1}}{\mathfrak{p}^m}\right) \times \\ &\times \sum_{y \in G_{\mathfrak{p}^{m_1}}} \exp\left(2\pi i \Re \frac{y}{\mathfrak{p}^{m_1}}\right) \\ = 0. \end{aligned}$$

At last, for $\delta_2 = 1$ we have $\delta_1 = \mathfrak{p}$, $\mathfrak{p}^m = \mathfrak{p}$. But this case we already considered. □

Corollary 1. *Let $f(\omega)$ be a multiplicative function over G for which the series $\sum\limits_{\omega \in G} f(\omega)N(\omega)^{-s}$ converges absolutely. Then in semiplane $\Re s > 1$ the equality*

$$\sum_{\omega \in G} \frac{f(\omega)K(1, \omega; \gamma)}{N(\omega)^s} = \sum_{\delta | \gamma_1} \mu(\delta) \sum_{\substack{\omega \in G \\ (\omega, \gamma) = \delta}} \frac{f(\omega)}{N(\omega)^s} K\left(1, \omega\delta^{-1}; \frac{\gamma}{\delta}\right)$$

holds.

Lemma 2. *Let $\alpha, \beta \in G$, $\mathfrak{p}$ be a prime number from G. Then*

$$K(\alpha, \beta; \mathfrak{p}) = \begin{cases} -1 & \textit{if} \quad (\alpha, \mathfrak{p}) = 1,\ \beta \equiv 0 \pmod{\mathfrak{p}} \\ & \textit{or} \quad \alpha \equiv 0 \pmod{\mathfrak{p}},\ (\beta, \mathfrak{p}) = 1, \\ \theta N(\mathfrak{p})^{\frac{1}{2}},\ |\theta| \le 2 & \textit{on all occasions.} \end{cases}$$

For $m \ge 2$

$$K(\alpha, \beta; \mathfrak{p}^m) = \begin{cases} 0 & \textit{if} \quad \alpha\beta \textit{ is a quadratic nonresidue mod } \mathfrak{p}; \\ \theta N(\mathfrak{p})^{\frac{m}{2}} & \textit{if} \quad \alpha\beta \textit{ is a quadratic residue mod } \mathfrak{p}, \\ & \quad |\theta| \le 2. \end{cases}$$

For $m = 1$ this assertion was proved by A. Weil [4], and the case $m \geq 2$ is an analogue of the estimate for rational case (see, [1]).

Lemma 3 (generalized identity of Selberg-Kuznetsov)**.** *Let α, β, γ over the Gaussian integers. Then*

$$K(\alpha, \beta; \gamma) = \sum_{\delta|(\alpha,\beta,\gamma)} N(\delta) K\left(1, \frac{\alpha\beta}{\delta^2}; \frac{\gamma}{\delta}\right).$$

Proof. By the quasi-multiplicativity of the Kloosterman sum in the variable γ, it suffices to prove the assertion for $\gamma = \mathfrak{p}^m$, where p is a Gaussian prime.

If $(\alpha, \beta; \mathfrak{p}^m) = 1$, the proposition clearly holds.

If $(\alpha, \beta; \mathfrak{p}^m) = \mathfrak{p}^m$, we have $K(\alpha, \beta; \mathfrak{p}^m) = \widetilde{\varphi}(\mathfrak{p}^m) = N(\mathfrak{p}^m) \cdot (1 - N^{-1}(\mathfrak{p}))$.

On the other hand,

$$\begin{aligned} \sum_{\delta|\mathfrak{p}^m} N(\delta) K\left(1, \frac{\alpha\beta}{\delta^2}, \mathfrak{p}^m\right) &= \sum_{\ell=0}^{m} N(\mathfrak{p}^\ell) \times \\ &\times \sum_{x \in G^*_{\mathfrak{p}^{m-\ell}}} \exp\left(2\pi i \Re \frac{x + x^{-1}\alpha_0 \mathfrak{p}^{m-\ell}}{\mathfrak{p}^{m-\ell}}\right) = \\ &= \sum_{\ell=0}^{m} N(\mathfrak{p})^\ell \sum_{x \in G^*_{\mathfrak{p}^{m-\ell}}} \exp\left(2\pi i \Re \frac{x}{\mathfrak{p}^{m-\ell}}\right) = \\ &= \sum_{\ell=0}^{m} N(\mathfrak{p}^\ell) \mu(\mathfrak{p}^{m-\ell}) = \overline{\varphi}(\mathfrak{p}^m). \end{aligned}$$

Now let $(\alpha, \beta, \mathfrak{p}^m) = \mathfrak{p}^k$, where $0 < k < m$.

We have

$$\sum_{x \in G^*_{\mathfrak{p}^m}} \exp\left(2\pi i \Re \frac{\alpha x + \beta x^{-1}}{\mathfrak{p}^m}\right) = N(\mathfrak{p})^k \sum_{x \in G^*_{\mathfrak{p}^{m-k}}} \exp\left(2\pi i \Re \frac{\alpha_1 x + \beta_1 x^{-1}}{\mathfrak{p}^{m-k}}\right).$$

Hence,

$$K(\alpha, \beta; \mathfrak{p}^m) = N(\mathfrak{p}^k) \cdot K\left(1, \frac{\alpha\beta}{\mathfrak{p}^{2k}}; \mathfrak{p}^{m-k}\right).$$

Consider the sum

$$\sum_{\ell=0}^{k} N(\mathfrak{p}^{\ell}) K\left(1, \frac{\alpha\beta}{\mathfrak{p}^{2\ell}}; \mathfrak{p}^{m-\ell}\right) =$$

$$= \sum_{\ell=0}^{k} N(\mathfrak{p})^{\ell} \sum_{x \in G^*_{\mathfrak{p}^{m-\ell}}} \exp\left(2\pi i \Re \frac{x + x^{-1}\mathfrak{p}^{2(k-\ell)}\alpha_1\beta_1}{\mathfrak{p}^{m-\ell}}\right).$$

Using standard properties of Gauss sums, the inner sum is equal to zero whenever $\ell \neq k$. This completes the proof of the Selberg–Kuznetsov identity. □

Theorem 1. *Let $g(\omega)$ be a completely multiplicative function over G, and let the Dirichlet series*

$$\sum_{\omega \in G} g(\omega) N(\omega)^{-1}$$

converge absolutely in the half-plane $\Re s > 1$. Then, for every $\alpha, \gamma \in G$ such that $N(\gamma) > 1$ and $(\alpha, \gamma) = 1$, we have

$$\sum_{\omega \in G} \frac{f(\omega) K(1, \alpha\omega; \gamma)}{N(\omega)^s} = \sum_{\delta | \gamma_1} \mu(\delta) \sum_{\substack{t_1, t_2 \in G \\ t_1 t_2 | \frac{\gamma}{\delta}}} \frac{\mu(t_1)\mu(t_2)}{N(t_1)^s N(t_2)^s} \times$$
$$\times \sum_{S(C)} g(\delta) Z_g\left(s; 0; \frac{\alpha_1\delta_1\delta^{-1}}{\frac{\gamma}{\delta}}\right) Z_g\left(s; 0; \frac{\alpha_2\delta_2\delta^{-1}}{\frac{\gamma}{\delta}}\right), \tag{1}$$

where $f(\omega) = \sum_{\delta | \omega} g(\omega)$, γ_1 denotes a square-free part of γ,

$$C := \left\{\alpha_1, \alpha_2 \in G^*_{\frac{\gamma}{\delta}},\ \alpha_1\alpha_2 \equiv 1 \pmod{\frac{\gamma}{\delta}}\right\};\ \delta^{-1} \pmod{\frac{\gamma}{\delta}}.$$

From now on, we shall write $S(C)$ under the summation sign to indicate that the summation is performed over elements satisfying the condition C.

Proof. For every $\delta | \gamma_1$, we have $\left(\delta, \frac{\gamma_1}{\delta}\right) = 1$, and hence there exists δ^{-1} $\pmod{\frac{\gamma}{\delta}}$ such that $\delta\delta^{-1} \equiv 1 \pmod{\frac{\gamma}{\delta}}$. Using the completely multiplicative function $g(\omega)$ and corollary of Lemma 1, we deduce that for every $\delta | \gamma_1$, $\ell_1, \ell_2 \in G^*_{\gamma}$, $\gamma \in G$, and $z = \alpha_1\delta_1\omega_1\delta^{-1} + \alpha_2\delta_2\omega_2\delta^{-1}$,

$$\sum_{S(C_1)} \sum_{\delta_1\delta_2 = \delta} g(\delta_1) g(\delta_2) \times \sum \frac{g(\omega_1) g(\omega_2)}{N(\omega_1\delta_1)^s N(\omega_2\delta_2)^s} \exp\left(\frac{2\pi i z}{\frac{\gamma}{\delta}}\right) =$$

$$= \sum_{\substack{\omega\in G\\(\omega,\gamma)=\delta}} N(\omega)^{-s} \sum_{\delta_1\delta_2=\delta} g(\omega) \times \sum_{S(C_3)} g(\omega_1)g(\omega_2) \sum_{S(C_1)} \exp\left(\frac{z}{\frac{\gamma}{\delta}}\right) = \tag{2}$$

$$= \sum_{\substack{\omega\in G\\(\omega,\gamma)=\delta}} f(\omega)N(\omega)^{-s}K\left(1, \alpha\omega\delta^{-2}; \frac{\gamma}{\delta}\right),$$

where

$$\begin{aligned} &C_1 := \left\{\alpha_1, \alpha_2 \in G^*_{\frac{\gamma}{\delta}},\ \alpha_1\alpha_2 \equiv \alpha \pmod{\frac{\gamma}{\delta}}\right\}, \\ &C_2 := \left\{\omega_1, \omega_2 \in G,\ (\omega_1, \omega_2) = 1,\ \left(\omega_1, \frac{\gamma}{\delta}\right) = \left(\omega_2, \frac{\gamma}{\delta}\right) = 1\right\}, \\ &C_3 := \{\omega_1, \omega_2 \in G,\ \omega_1\delta_1 \cdot \omega_2\delta_2 = \omega\}. \end{aligned} \tag{3}$$

On the other hand, we have

$$\begin{aligned} &\sum_{S(C_1)} \sum_{\delta_1\delta_2=\delta} g(\delta_1)g(\delta_2)\times \\ &\qquad \times \sum_{\substack{\omega\\S(C_2)}} \frac{g(\omega_1)g(\omega_2)}{N(\omega_1)^s N(\omega_2)^s} \exp\left(\frac{\alpha_1\omega_1\delta_1\delta^{-1} + \alpha_2\omega_2\delta_2\delta^{-1}}{\frac{\gamma}{\delta}}\right) = \\ &= \sum_{S(C_1)} \sum_{\delta_1\delta_2=\delta} \sum_{t_1,t_2|\frac{\gamma}{\delta}} \frac{\mu(t_1)\mu(t_2)}{N(t_1)^s N(t_2)^s} g(t_1t_2)\times \\ &\qquad \times \sum_{\substack{\omega_1,\omega_2\in G\\S(C_3)}} \frac{g(\omega_1)\exp\left(\frac{\alpha_1\omega_1t_1\delta_1\delta^{-1}}{\frac{\gamma}{\delta}}\right)}{N(\omega_1)^s}\times \\ &\qquad\qquad \times \frac{g(\omega_2)\exp\left(\frac{\alpha_2\omega_2t_2\delta_2\delta^{-1}}{\frac{\gamma}{\delta}}\right)}{N(\omega_2)^s} = \\ &= \sum_{S(C_1)} \sum_{t_1,t_2|\frac{\gamma}{\delta}} \frac{\mu(t_1)}{N(t_1)^s} \cdot \frac{\mu(t_2)}{N(t_2)^s}\times \\ &\qquad \times \sum_{\delta_1\delta_2=\delta} Z_g\left(s; \frac{\alpha_1t_1\delta_1\delta^{-1}}{\frac{\omega}{\delta}}\right) Z_g\left(s; \frac{\alpha_2t_2\delta_2\delta^{-1}}{\frac{\omega}{\delta}}\right), \end{aligned} \tag{4}$$

where

$$Z_g\left(s; \frac{\beta}{\frac{\omega}{\delta}}\right) = \sum_{\omega\in G} \frac{g(\omega)\exp\left(2\pi i\frac{\beta\omega}{\frac{\gamma}{\delta}}\right)}{N(\omega)^s},\ \left(\alpha, \frac{\gamma}{\delta}\right) = 1,\ \Re s > 1.$$

Now, we multiply the identity in (2) and (3) by δ, apply the Selberg-Kuznetsov lemma, and after summing over δ, where $\delta|\gamma$, we obtain our statement. $\square$

Lemma 1 can be generalized as follows:

Lemma 4. *By the notation of Theorem 1, we have for $\ell, q \in G$ and $(\ell, q) = 1$ that*

$$\sum_{\substack{\omega\in G \\ \omega\equiv\ell \pmod q}} \frac{f(\omega)K(1,\alpha\omega;\gamma)}{N(\omega)^s} =$$

$$= \sum_{\delta|\gamma_1} \mu(\delta) \sum_{t_1,t_2|\frac{\gamma}{\delta}} \frac{\mu(t_1)\mu(t_2)}{N(t_1)^s N(t_2)^s} \sum_{S(C)} g(\delta)g(\delta) \times$$

$$\times Z_g\left(s; \frac{\ell_1}{q}, \frac{\alpha_1 t_1 \delta_1 \delta^{-1}}{\frac{\gamma}{\delta}}\right) \cdot Z_g\left(s; \frac{\ell_2}{q}, \frac{\alpha_2 t_2 \delta_2 \delta^{-1}}{\frac{\gamma}{\delta}}\right),$$

where

$$Z_g\left(s; \frac{\ell_i}{q}, \frac{\alpha_i \delta_i \delta^{-1}}{\frac{\gamma}{\delta}}\right) = \sum_{\omega\in G} \frac{g(\omega)\exp\left(2\pi i \frac{\alpha_i\delta_i\delta^{-1}}{\frac{\gamma}{\delta}}\right)}{N\left(\omega + \frac{\ell_i}{q}\right)^s},\ i = 1, 2.$$

The proof of this lemma is similar to the proof of Theorem 1 (with $Z_g\left(s; \frac{\beta}{\frac{\gamma}{\delta}}\right)$ replaced by $Z_g\left(s; \frac{\ell}{q}, \frac{\beta}{\frac{\gamma}{\delta}}\right)$).

Corollary 2. *Let γ be a Gaussian integer, and let $g(\omega) = e^{4mi\arg\omega}$. Then, for $\Re s > 1$, we deduce at once from Lemma 3 that for $\alpha \in G_\gamma^*$*

$$F_m(s) := \sum_{\omega\in G} \frac{\tau^{(m)}(\omega)K(1,\alpha\omega;\gamma)}{N(\omega)^s} = \sum_{\delta|\gamma} \mu(\delta) F_m(s;\delta,\gamma), \tag{5}$$

where

$$F_m(s;\delta,\gamma) = \sum_{t_1,t_2|\frac{\gamma}{\delta}} \frac{\mu(t_1)\mu(t_2)}{N(t_1,t_2)^s} \sum_{\delta_1\delta_2=\delta} \times$$

$$\times \sum_{\substack{\alpha_1,\alpha_2\in G^*_{\frac{\gamma}{\delta}} \\ \alpha_1\alpha_2\equiv\alpha \left(\mathrm{mod}\ \frac{\gamma}{\delta}\right)}} Z_m\left(s; 0, \frac{\alpha_1\delta_1 t_1 \delta^{-1}}{\frac{\gamma}{\delta}}\right) \cdot Z_m\left(s; 0, \frac{\alpha_2\delta_2 t_2 \delta^{-1}}{\frac{\gamma}{\delta}}\right). \tag{6}$$

Corollary 3. *Let q and γ be Gaussian integers, and let $\alpha \in G_\gamma^*$ with $\Re s > 1$. Then we have*

$$\sum_{\substack{\omega\in G \\ \omega\equiv\ell \pmod q}} \frac{\tau^{(m)}(\omega)K(1,\alpha\omega;\gamma)}{N(\omega)^s} = \sum_{\delta\mid\gamma} \mu(\delta)F_m(s;\delta,\ell,\gamma,q), \tag{7}$$

where

$$\begin{aligned} F_m(s,\delta,\ell,\gamma,q) = &\sum_{t_1,t_2\mid\frac{\gamma}{\delta}} \frac{\mu(t_1)\mu(t_2)}{N(t_1t_2)^s} \sum_{\substack{\alpha_1,\alpha_2\in G^*_{\frac{\gamma}{\delta}} \\ \alpha_1\alpha_2\equiv\alpha \pmod{\frac{\gamma}{\delta}}}} \sum_{\delta_1\delta_2=\delta} \times \\ &\times \sum_{\substack{\ell_1,\ell_2 \\ \ell_1\ell_2\equiv\ell \pmod q}} Z_m\left(s;\frac{\ell_1}{q},\frac{\alpha_1\delta_1t_1\delta^{-1}}{\frac{\gamma}{\delta}}\right)\cdot Z_m\left(s;\frac{\ell_2}{q},\frac{\alpha_2\delta_2t_2\delta^{-1}}{\frac{\gamma}{\delta}}\right). \end{aligned} \tag{8}$$

We now turn to the truncated functional equation and its estimates for $Z_m(s;\delta_1,\delta_2)$, with $\delta_1,\delta_2\in\mathbb{Q}(i)$. We now consider two Dirichlet series

$$f(s,m) = \sum_{n=1}^{\infty}\frac{a_n}{n^s},\ a_n = \sum_{\substack{\omega\in G \\ N(\omega)=n}} e^{4mi\arg\omega+\delta_1}e^{2\pi i\Re(\delta_2\omega)},\ \Re s>1, \tag{9}$$

$$\varphi(s,m) = \sum_{n=1}^{\infty}\frac{b_n}{n^s},\ b_n = \sum_{\substack{\omega\in G \\ N(\omega)=n}} e^{-4mi\arg\omega-\delta_2}e^{2\pi i\Re(\delta_1\omega)},\ \Re s>1, \tag{10}$$

which are linked by the functional equation

$$f(s,m) = \pi^{-(1-2s)}\frac{\Gamma(2|m|+1-s)}{\Gamma(2|m|+s)}e^{-2\pi i\Re(\delta_1\delta_2)}\varphi(1-s,m).$$

It is clear that

$$\begin{aligned} f(s,m) &= Z_m(s;\delta_1,\delta_2), \\ \varphi(s,m) &= Z_m(s;-\delta_2,\delta_1). \end{aligned}$$

Then, by the Lavrik theorem on the functional equation for the Dirichlet function [2], [3], we have:

Lemma 5. *Let $s=\sigma+it$, $-2\le\sigma\le 2$, and let τ be a complex number with $|\arg\tau|<\frac{\pi}{2}$. Let*

$$\begin{aligned} u &= \tfrac{\sqrt{t^2+4m^2}}{2\pi|\tau|}, & v &= \tfrac{\sqrt{t^2+4m^2}}{2\pi|\tau^{-1}|}, \\ U &= u(1+H\log u), & V &= v(1+H\log v), \end{aligned}$$

where H is a positive number.

Then there exists a constant $T_0 > 0$ *such that for* $|4m^2 + t^2| > T_0$

$$Z_m(s;\delta_1,\delta_2) = \sum_{\omega\in G} \frac{e^{4im\arg(\omega+\delta_1)}}{N(\omega+\delta_1)^s} e^{2\pi i\Re(\delta_1\omega)}\Gamma^*(s,\pi\tau N(\omega+\delta_1))+$$
$$+\pi^{-(1-2s)}e^{-2\pi i\Re(\delta_1\delta_2)}\times$$
$$\times\sum_{\omega\in G}\frac{e^{4mi\arg(\omega-\delta_2)}}{N(\omega-\delta_2)^{1-s}}e^{-2\pi i\Re(\delta_1\omega)}\Gamma^*(z,Z),$$

where

$$\Gamma^*(z,Z) := \Gamma^*(1-s,\pi\tau N(\omega+\delta_2)) = \frac{1}{2\pi i}\int\limits_{\Delta-i\infty}^{\Delta+i\infty}\Gamma(z+w)\frac{Z^{-w}}{w}dw.$$

Here, Δ *is such that for* $\Re s \geq \Delta$ *there are no singularities of the integrand.*

Moreover, if $\arg\tau = \operatorname{arctg}\frac{\mathrm{t}}{2|\mathrm{m}|+\sigma}$*, then*

$$Z_m(s;\delta_1,\delta_2) = \sum_{\substack{\omega\in G\\ N(\omega)\leq U}}\frac{e^{4im\arg\omega+\delta_1}}{N(\omega+\delta_1)^s}e^{2\pi i\Re(\delta_2\omega)}\Gamma^*(s+2|m|,\pi N(\omega+\delta_1))+$$
$$+\pi^{-(1-2s)}e^{-2\pi i\Re(\delta_1\delta_2)}\frac{\Gamma(2|m|+1-s)}{\Gamma(2|m|+s)}\times$$
$$\times\sum_{\substack{\omega\in G\\ N(\omega)\leq V}}\frac{e^{-4mi\arg(\omega-\delta_2)}}{N(\omega-\delta_2)^{1-s}}e^{-2\pi i\Re(\delta_1\omega)}\Gamma^*(2|m|+s,\pi N(\omega+\delta_2)),$$

where H *is an arbitrary positive constant* ≥ 5,

$$\Gamma^*(z,Z) = \varepsilon_0 + O\left(\left(-C_1\left|\frac{Z}{z}\right|\right)\left|\frac{Z}{Im(z)}\right|^{\Re(z)}\times\right.$$
$$\left.\times\left(1+|Im(z)|^{\frac{1}{2}}-C_2\frac{|Z|}{|Im(z)|^{\frac{1}{2}}}\right)^{-1}\right)$$

and

$$\varepsilon_0 = \begin{cases} 1 & if \quad |Z| \leq |z|, \\ 0 & if \quad |Z| > |z| \end{cases}$$

(here, C_1 *and* C_2 *are absolute positive constants, and the implied* O*-constant is independent of* s, τ, m, H*).*

It follows immediately that

Theorem 2. *Let $f(s,m)$, H and T_0 be as in Lemma 5. Then for $T > T_0$, $s = \frac{1}{2} + it$, $|t| \geq T_0$, $\arg\tau = \arg s$, and $|\tau| = N(\gamma)^{\frac{1}{2}}$, we have*

$$\begin{aligned} f(s,m) = &\sum_{\substack{\omega\in G\\ N(\omega)\leq\frac{|s|+2|m|}{\pi}}} \frac{e^{4mi\arg\left(\omega+\frac{\ell_1}{\delta}\right)}}{N\left(\omega+\frac{\ell_1}{\gamma}\right)^{\frac{1}{2}-it}} e^{2\pi i\Re\left(\frac{\ell_2\omega}{\gamma}\right)}+\\ &+\pi^{-2it}\frac{\Gamma\left(2|m|+\frac{1}{2}-it\right)}{\Gamma\left(2|m|+\frac{1}{2}+it\right)}\times\\ &\times\sum_{N(\omega)\leq\frac{|s|+2|m|}{\pi}N(\gamma)} \frac{e^{-4mi\arg\left(\omega-\frac{\ell_2}{\gamma}\right)}}{N\left(\omega-\frac{\ell_2}{\gamma}\right)^{\frac{1}{2}+it}} e^{-2\pi i\Re\left(\frac{\omega\ell_1}{\gamma}\right)}+\\ &+O(\log(H(t^2+4m^2)N(\gamma)))+O(|t|^{-L+2}). \quad\square \end{aligned} \tag{11}$$

Lemma 5 and Theorem 2 will be applied in the case $\delta_1 = \frac{\ell_1}{\gamma}$, $\delta_2 = 0$. We now prove a statement essential for the main results.

We introduce the following notation

$$\widetilde{Z}_m\left(s;\frac{\ell_1}{\gamma},0\right) = Z_m\left(s;\frac{\ell_1}{\gamma},0\right) - \sum_{\omega\in B}\frac{e^{4mi\arg\left(\omega+\frac{\ell_1}{\gamma}\right)}}{N\left(\omega+\frac{\ell_1}{\gamma}\right)^{s}}.$$

Lemma 6. *As $T\to\infty$, for any $\varepsilon>0$*

$$\int\limits_{\substack{-T\\ \Re s=\frac{1}{2}}}^{T} \left|\widetilde{Z}_m\left(s;\frac{\ell_1}{\gamma},0\right)\right|^2 ds = O\left(x^{\frac{1}{2}}(T+|m|)^{1+\varepsilon}\right)N(\gamma)^{\frac{1}{2}+\varepsilon},$$

where the implied O-constant depends only on ε.

Proof. On the line $\Re s = \frac{1}{2}$, we apply the truncated functional equation for $Z_m\left(s;\frac{\ell_1}{\gamma},0\right)$. Using the notations (9) and (10), we have

$$\begin{aligned} \int\limits_{\substack{-T\\ \Re s=\frac{1}{2}}}^{T} \left|\widetilde{Z}_m\left(s;\frac{\ell_1}{\gamma},0\right)\right|^2 dt \ll \int\limits_{-T}^{T} N(\omega)\left|\sum_{\substack{\omega\\ N(\gamma)\leq N(\omega)\leq KN(\gamma)}}\frac{e^{4mi\arg(\omega+\ell_1)}}{N(\omega)^s}\right|^2 dt\\ +\int\limits_{-T}^{T}\left|\frac{\Gamma\left(2|m|+\frac{1}{2}-it\right)}{\Gamma(2|m|+it)}\cdot\frac{1}{\pi}\sum_{\substack{\omega\in G\\ N(\omega)\leq K}}\frac{e^{-2\pi i\Re(\omega\ell_1)}}{N(\omega)^{\frac{1}{2}-it}}\right|^2 dt \end{aligned}$$

$$
\begin{aligned}
&+ O\big(T \log^2\!\big((T^2+m^2)N(\omega)\big)\big) \\
&= I_1 + I_2 + O\big(T \log^2\!\big((T^2+m^2)N(\omega)\big)\big)\,, \qquad (12)
\end{aligned}
$$

where $K = \frac{T+2|m|}{\pi}$.

The integrals I_1 and I_2 can be estimated in a similar way.

We calculate I_1. Using the estimates for $\widetilde{Z}_m\left(s, \frac{\ell_1}{\gamma}, 0\right)$ on the lines $\Re s = -\varepsilon$ and $\Re s = 1+\varepsilon$, by the Phragmen-Lindelöf principle we deduce

$$
Z_m\left(\frac{1}{2}+it, \frac{\ell_1}{\gamma}, \frac{\ell_2}{\gamma}\right) << N(\gamma)^{\frac{1}{2}}(t^2+m^2)^{\frac{1+\varepsilon}{2}}
$$

for arbitrary $\ell_1, \ell_2 \in G$, $\ell_i \not\equiv 0 \pmod{\gamma}$ or $\ell_2 \not\equiv 0 \pmod{\gamma}$.

In particular,

$$
\widetilde{Z}_m\left(\frac{1}{2}+it, \frac{\ell_1}{\gamma}, 0\right) << N(\gamma)^{-\frac{1}{2}}(t^2+m^2)^{\frac{1+\varepsilon}{2}} \qquad (13)
$$

for $\ell_1 \not\equiv 0 \pmod{\gamma}$.

Let $I_1 = \int\limits_{-T_0}^{T_0} + \int\limits_{T_0}^{T} + \int\limits_{-T}^{T_0} = I_{11} + I_{12} + I_{13}$. From (13), we have

$$
I_{11} \ll N(\gamma)^{-1}(T_0^2+m^2)^{\frac{1}{2}} \ll N(\gamma)^{-1+\varepsilon}(|m|+1). \qquad (14)
$$

The estimates for I_{12} and I_{13} are similar; we detail I_{12} below.

Using the previous bounds, we obtain

$$
\begin{aligned}
I_{12} \ll &\int\limits_{T_0}^{T} \left| \sum_{\substack{\omega \equiv x \pmod{\gamma} \\ N(\gamma) < N(\omega) \le U_0}} e^{4mi\arg\omega} N^{-\frac{1}{2}-it}(\omega) \right|^2 dt + \\
&+ \frac{1}{N(\gamma)} \int\limits_{T_0}^{T} \left| \sum_{N(\omega)\le V} \exp\left(-\pi i \Re \frac{2\pi i \omega}{\gamma}\right) N^{-\frac{1}{2}+it}(\omega) \right|^2 dt + \\
&+ TN^{-1}(\gamma)\log^2(MTN(\gamma)) + T_0^{-H+3}.
\end{aligned} \qquad (15)
$$

The integrals on the right-hand side of (15) can be estimated using standard mean value results for Dirichlet series. Using the notations (9) and

(10), we have

$$\int_{T_0}^{T} \left| \sum_{\substack{\omega\equiv\alpha \pmod{\gamma} \\ N(\gamma)<N(\omega)\le U_0}} N^{-\frac{1}{2}-it}(\omega) \right|^2 dt = \int_{T_0}^{T} \left| \sum_{N(\gamma)<n\le U_0} \frac{a_n n^{-\frac{1}{2}}}{n^{it}} \right|^2 dt \ll$$

$$\ll ((T^2+M^2)^{\frac{1}{2}} + N_0) \sum_{N(\gamma)<n\le U_0} \frac{a_n^2}{n},$$

where

$$a_n = \sum_{\substack{\omega\equiv\alpha \pmod{\gamma} \\ N(\omega)=n}} 1, \quad U_0 = \frac{N^{-\frac{1}{2}}(\gamma)\left|\frac{1}{2}+iT\right|}{\pi|\tau|} \ll N(\gamma)T,$$

$$N_0 = \sum_{\substack{N(\gamma)<n\le U_0 \\ a_n\neq 0}} 1 = \sum_{\substack{\omega\equiv\alpha \pmod{\gamma} \\ N(\gamma)<N(\omega)\le U_0}} 1 \le \frac{U_0}{N(\gamma)} \ll T.$$

Moreover,

$$\sum_{N(\gamma)<n\le U_0} \frac{a_n^2}{n} \ll \sum_{\substack{\omega\equiv\alpha \pmod{\gamma} \\ N(\gamma)<N(\omega)\le U_0}} \frac{\tau^(\omega)}{N(\omega)} \ll \frac{U_0^{\varepsilon}}{N(\gamma)} \sum_{\substack{\omega\equiv\alpha \pmod{\gamma} \\ N(\gamma)<N(\omega)\le U_0}} \frac{1}{N(\omega)}.$$

Therefore,

$$\int_{T_0}^{T} \left| \sum_{\substack{\omega\equiv\alpha \pmod{\gamma} \\ N(\gamma)<N(\omega)\le u}} N^{-\frac{1}{2}-it}(\omega) \right|^2 dt \ll T^{1+\varepsilon} N^{-1+\varepsilon}(\gamma). \tag{16}$$

Similarly,

$$\int_{T_0}^{T} \left| \sum_{N(\omega)\le v} \exp\left(-\pi i \Re \frac{2\pi i \alpha\omega}{\gamma}\right) N^{-\frac{1}{2}+it}(\omega) \right|^2 dt$$

$$= \int_{T_0}^{T} \left| \sum_{n\le v} \frac{b_n n^{-\frac{1}{2}}}{n^{-it}} \right| dt \ll (T+V_0) \sum_{n\le V_0} \frac{|b_n|^2}{n} \ll (T+V_0) \sum_{n\le V_0} \frac{r^2(n)}{n}$$

with

$$b_n = \sum_{N(\omega)=n} \exp\left(-\pi i \Re \frac{2\pi i \alpha\omega}{\gamma}\right), \quad V_0 = \frac{N^{\frac{1}{2}}(\gamma)\left|\frac{1}{2} - it\right|}{\pi|\tau|^{-1}} \ll T,$$

where $r(n)$ denotes the number of representations of n as a sum of two squares.

Since

$$\sum_{n \le V_0} \frac{r^2(n)}{n} \ll \log^2 V_0 \ll \log^2 T$$

it follows that

$$\int_{T_0}^{T} \left| \sum_{N(\omega)\le v} \exp\left(-\pi i \Re \frac{2\pi i \omega}{\gamma}\right) N^{-\frac{1}{2}+it}(\omega)\right|^2 dt \ll T \log^2 T.$$

Hence,

$$I_2 \ll \frac{T}{N(\gamma)}((TN(\gamma))^2 + \log^2(MTN(\gamma)) + T_0^{-H+3}). \tag{17}$$

The statement of the lemma follows from the choice $|\tau| = N^{-\frac{1}{2}}(\gamma)$ and (17) with $H = 5 + \frac{1}{\varepsilon}$. □

Theorem 3. *Let $T \to \infty$. Then, for any $\varepsilon > 0$, $\ell, \gamma \in G$, $N(\ell) < N(\varphi)$*

$$\int_{\substack{-T\\ \Re s = \frac{1}{2}}}^{T} \left| Z_m\left(s; 0, \frac{\ell}{\gamma}\right)\right|^2 dt \ll (T + |m|)^{1+\varepsilon}.$$

Proof. By Theorem 2, we have

$$\int_{\substack{-T\\ \Re s = \frac{1}{2}}}^{T} \left| Z_m\left(s; 0, \frac{\ell}{\gamma}\right)\right|^2 dt \ll \int_{-T_0}^{T_0} + \int_{T_0}^{T} + \int_{-T}^{-T_0} = I_1 + I_2 + I_3. \tag{18}$$

For I_1, we obtain

$$I_1 \ll N(\gamma)^{\frac{1}{2}} \cdot T_0^{1+\varepsilon}. \tag{19}$$

The estimates for I_2 and I_3 proceed in a similar manner.

By Theorem 2, $\left|\frac{\Gamma^2(2|m|+\frac{1}{2}-it)}{\Gamma^2(2|m|+\frac{1}{2}+it)}\right| = 1$ and therefore

$$I_2 \ll \int\limits_{T_0}^{T} \left\{ \left| \left(\sum_{\substack{\omega\equiv\alpha \pmod{\gamma} \\ N(\omega)<N(\gamma)}} + \sum_{\substack{\omega\equiv\alpha \pmod{\gamma} \\ N(\gamma)<N(\omega)\leq\frac{|s|+2|m|}{\pi}N(\gamma)}} \right) \times \right.\right.$$

$$\left. \times \frac{e^{4mi\arg\omega} N(\gamma)^{\frac{1}{2}-it}}{N(\omega)^{\frac{1}{2}-it}} dt \right|^2$$

$$\left. + \left| \sum_{N(\omega)\leq\frac{|s|+2|m|}{\pi}} \frac{e^{4mi\arg\omega}\cdot e^{2\pi i\Re\left(\frac{\omega\ell}{\gamma}\right)}}{N(\omega)^{\frac{1}{2}+it}} \right|^2 \right\} dt \ll$$

$$\ll N(\gamma)^{\frac{1}{2}} T_0^{1+\varepsilon} + \int\limits_{T_0}^{T} \left| \sum_{\substack{\omega\equiv\ell \pmod{\gamma} \\ N(\gamma)<N(\omega)\leq\frac{|s|+2|m|}{\pi}N(\gamma)}} \frac{e^{4mi\arg\omega}}{N(\omega)^{\frac{1}{2}-it}} \right| \cdot N(\gamma)^{\frac{1}{2}} dt +$$

$$+ \int\limits_{T_0}^{T} \left| \sum_{\substack{\omega\in G \\ N(\omega)\leq\frac{|s|2|m|}{\pi}}} \frac{e^{4mi\arg\omega}}{N(\omega)^{\frac{1}{2}+it}} e^{2\pi i\Re\left(\frac{\omega\ell}{\gamma}\right)} \right|^2 dt \ll$$

$$\ll N(\gamma)^{\frac{1}{2}} (T_0^2+m^2)^{\frac{1+\varepsilon}{2}} + I_{11} + I_{12}.$$

Next, using the mean value estimates for segments of Dirichlete series, we estimate I_{11} and I_{12}.

Let

$$a_n = \sum_{\substack{\omega\equiv\ell \pmod{\gamma} \\ N(\omega)=n}} 1,\ U_0 = \frac{N(\gamma)^{\frac{1}{2}}\left(\frac{1}{2}-iT\right)}{\pi\tau} \ll N(\gamma)(T^2+m^2)^{\frac{1}{2}},$$

$$N_0 = \sum_{\substack{N(\gamma)<n\leq U_0 \\ a_n\neq 0}} 1 \ll \frac{U_0}{N(\gamma)} \ll (T^2+m^2)^{\frac{1}{2}}.$$

Taking into account that

$$\sum_{N(\gamma)<n\leq U_0} \frac{a_n^2}{n} \ll \sum_{\substack{\omega\equiv\ell \pmod{\gamma} \\ N(\gamma)<N(\omega)\leq U_0}} \frac{\tau^2(\omega)}{N(\omega)} \ll {U_0}^{\varepsilon} \sum_{\substack{\omega\equiv\ell \pmod{\gamma} \\ N(\gamma)<N(\omega)\leq U_0}} \frac{1}{N(\omega)}$$

we obtain

$$\int_{T_0}^{T} \left| \sum_{\substack{\omega \equiv \ell \pmod{\gamma} \\ N(\gamma) < N(\omega) \le U_0}} \frac{N(\gamma)^{\frac{1}{2}-it}}{N(\omega)^{\frac{1}{2}-it}} \right|^2 dt \ll N(\gamma)^{\varepsilon} (T^{2+\varepsilon} + m^2)^{\frac{1}{2}}. \tag{20}$$

Similarly,

$$\int_{T_0}^{T} \left| \sum_{N(\omega) \le U_0} e^{2\pi i \Re\left(\frac{\ell\omega}{\gamma}\right)} N(\omega)^{-\frac{1}{2}+it} \right|^2 dt \ll (T + |m|)^{1+\varepsilon} \tag{21}$$

and hence

$$I_2 \ll (T + |m|)^{1+\varepsilon}. \tag{22}$$

The statement of the theorem follows from (18)–(22). □

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