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Natural partial order on semigroups of partial transformations with invariant set

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Abstract. Let X be a non-empty set, and let $P(X)$ denote the semigroup of partial transformations on X. For a non-empty subset Y of X, define

$$\overline{PT}(X,Y) = \{\, \alpha \in P(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y \,\}.$$

The semigroup $\overline{PT}(X,Y)$ generalizes $P(X)$ and consists of all partial transformations on X that leave Y invariant. In this paper, we investigate the natural partial order on $\overline{PT}(X,Y)$ and characterize its left-compatible, right-compatible, minimal, and maximal elements. The results obtained extend and unify several known properties of $P(X)$.

Introduction

The concept of the natural partial order in semigroups has evolved progressively over time. Within the framework introduced by Clifford and Preston [2], a *band* B is defined as a semigroup in which every element is idempotent. On such a semigroup, a partial order can be naturally defined by

$$e \le f \text{ if and only if } \quad e = ef = fe.$$

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When this order is compatible with the semigroup operation, that is, when $e \leq f$ implies $eg \leq fg$ and $ge \leq gf$ for all $g \in B$, the band is referred to as a *naturally ordered band*. The structural properties of such bands were studied in detail by Howie [6].

Earlier, in 1952, Vagner [18] introduced the natural partial order for inverse semigroups S, defining

$$a \leq b \text{ if and only if } a = eb \text{ for some } e \in E(S),$$

where $E(S)$ denotes the set of idempotents of S. Later, Hartwig [4] and Nambooripad [11] independently extended this notion to regular semigroups, defining

$$a \leq b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E(S). \tag{1}$$

In general, this order is not compatible with multiplication in a regular semigroup.

Mitsch [10] generalized the concept of the natural partial order to arbitrary semigroups S by defining

$$a \leq b \text{ if and only if } a = xb = by \text{ and } xa = a \text{ for some } x, y \in S^1. \tag{2}$$

Equivalently,

$$a \leq b \text{ if and only if } a = ub = bv \text{ and } av = a \text{ for some } u, v \in S^1. \tag{3}$$

For a non-empty set X, the *full transformation semigroup* on X, denoted by $T(X)$, is defined as the semigroup of all functions from X into itself under the operation of composition. Throughout this paper, function composition is written from right to left; that is, for $\alpha\beta$, the map α is applied first. It is well known that every semigroup is isomorphic to some suitable subsemigroup of $T(X)$. For this reason, the study of $T(X)$ is of particular importance, and its algebraic properties have been extensively investigated. Doss [3] proved that $T(X)$ is a regular semigroup and characterized Green's relations on $T(X)$ in terms of images and kernels. Later, in 1986, Kowol and Mitsch [7] investigated the behavior of the natural partial order on $T(X)$. In their work, using (1), they characterized this order in terms of images and kernels and described the minimal and maximal elements with respect to it.

The well-known supersemigroup of $T(X)$ is the *partial transformation semigroup* on X, denoted by $P(X)$, and defined as the set of all functions from a subset of X into X, that is,

$$P(X) = \{\, \alpha \mid \alpha : A \to X, \text{ where } A \subseteq X \,\}.$$

The semigroup $P(X)$ is also regular under composition, and $T(X)$ is properly contained in $P(X)$. The semigroup $P(X)$ is also regular under composition, and $T(X)$ is properly contained in $P(X)$. In 2003, Marques-Smith and Sullivan [9] extended the study of the natural partial order to the semigroup $P(X)$. They characterized this order on $P(X)$, determined its minimal and maximal elements, and described the left- and right-compatible elements of $T(X)$ and $P(X)$ with respect to this order.

Motivated by the algebraic significance of $T(X)$ and $P(X)$, several generalizations have been introduced to extend their algebraic properties. One notable generalization of $T(X)$ is the semigroup $\overline{T}(X,Y)$, where $\emptyset \neq Y \subseteq X$, defined by

$$\overline{T}(X,Y) = \{\, \alpha \in T(X) \mid Y\alpha \subseteq Y \,\}.$$

The semigroup $\overline{T}(X,Y)$ was first studied by Magill [8]. Subsequently, in 2005, Nenthein *et al.* [12] characterized the regular elements of $\overline{T}(X,Y)$ and showed that $\overline{T}(X,Y)$ is regular if and only if $X = Y$ or Y is a singleton set. In 2011, Honyam and Sanwong [5] determined the conditions under which $\overline{T}(X,Y)$ is isomorphic to $T(Z)$ for some set Z, and fully described Green's relations and ideals on $\overline{T}(X,Y)$. Later, in 2013, Sun and Wang [16] investigated the natural partial order on this semigroup: they determined when two elements of $\overline{T}(X,Y)$ are related, identified the left- and right-compatible elements, and described its minimal and maximal elements. However, errors were later found in Sun and Wang's descriptions of the left-compatible and maximal elements; these were corrected by Sun and Sun [17] and by Baka and Chaiya [1], respectively.

For a non-empty subset Y of X, and in analogy with $\overline{T}(X,Y)$, we consider the following generalization of $P(X)$:

$$\overline{PT}(X,Y) = \{\, \alpha \in P(X) \mid (\operatorname{dom}\alpha \cap Y)\alpha \subseteq Y \,\},$$

where $\operatorname{dom}\alpha$ is the domain of α. In 2023, Pantarak and Chaiya [13] showed that $\overline{PT}(X,Y)$ is not regular in general and established necessary and sufficient conditions under which an element of $\overline{PT}(X,Y)$ is regular. In 2024, Srisawat and Chaiya [14] described the ideals of this semigroup, and more recently, its Green's relations were completely characterized in [15].

In this paper, we focus on the natural partial order on $\overline{PT}(X,Y)$. We characterize the left-compatible, right-compatible, minimal, and maximal elements of $\overline{PT}(X,Y)$ with respect to this order. In particular, when $X = Y$, our results coincide with those obtained by Marques-Smith and Sullivan [9].

1. Preliminaries and notation

We begin with some basic preliminaries that will be used throughout the paper.

For $\alpha \in P(X)$, we use $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$ to denote the *domain* and *image* of α, respectively. For each $x \in \operatorname{dom}\alpha$, we write $x\alpha$ to denote the image of x under α. In particular, for $A \subseteq X$, we simply write $A\alpha$ to denote the image of A under α, that is,

$$A\alpha = \{\, a\alpha \mid a \in \operatorname{dom}\alpha \cap A \,\}.$$

For each $x \in X$,

$$x\alpha^{-1} = \{z \in \operatorname{dom}\alpha \mid z\alpha = x\}$$

denotes the set of inverse images of x under α, and for any $A \subseteq X$, we define

$$A\alpha^{-1} = \{a\alpha^{-1} \mid a \in A\} = \{z \in \operatorname{dom}\alpha \mid z\alpha \in A\}.$$

In particular, for any, $\alpha, \beta \in P(X)$, $\operatorname{dom}(\alpha\beta) = (\operatorname{dom}\beta)\alpha^{-1}$. Moreover, the *kernel* of α is the equivalence relation on $\operatorname{dom}\alpha$ defined by

$$\ker\alpha = \{\, (a, b) \in \operatorname{dom}\alpha \times \operatorname{dom}\alpha \mid a\alpha = b\alpha \,\}.$$

We usually express $\alpha \in P(X)$ in the two-line notation

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix},$$

where the index i ranges over some (unspecified) index set I. Here, $\{\, a_i : i \in I \,\} = \operatorname{im}\alpha$, each $a_i\alpha^{-1} = X_i \subseteq \operatorname{dom}\alpha$, and $\bigcup_{i\in I} X_i = \operatorname{dom}\alpha$. Moreover, each X_i is an equivalence class with respect to $\ker\alpha$.

2. The characterization of $\leq$ on $\overline{PT}(X, Y)$

In this section, we provide a characterization of the natural partial order $\leq$ on $\overline{PT}(X, Y)$. Since $\overline{PT}(X, Y)$ is not, in general, a regular semigroup, we adopt the definition given in (3). Note that $\overline{PT}(X, Y)$ is actually a monoid, with the identity map as multiplicative identity. The following lemmas will be required in our subsequent results.

Lemma 1 ([13])**.** *Let $\alpha,\beta \in \overline{PT}(X,Y)$. Then $\alpha = \gamma\beta$ for sone $\gamma \in \overline{PT}(X,Y)$ if and only if $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ and $Y\alpha \subseteq Y\beta$.*

Lemma 2 ([13])**.** *Let $\alpha,\beta \in \overline{PT}(X,Y)$. Then $\alpha = \beta\gamma$ for sone $\gamma \in \overline{PT}(X,Y)$ if and only if all the following conditions hold:*

(1) $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$;

(2) $\ker\beta \cap (\operatorname{dom}\beta \times \operatorname{dom}\alpha) \subseteq \ker\alpha$;

(3) $\forall x \in (\operatorname{dom}\alpha \cap \operatorname{dom}\beta) \setminus Y$, $x\beta \in Y \Rightarrow x\alpha \in Y$.

The following theorem provides a characterization of the natural partial order $\leq$ on $\overline{PT}(X,Y)$.

Theorem 1. *Let $\alpha,\beta \in \overline{PT}(X,Y)$. Then $\alpha \leq \beta$ if and only if all the following conditions hold:*

(1) $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ *and* $Y\alpha \subseteq Y\beta$;

(2) $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ *and* $\ker\beta \cap (\operatorname{dom}\beta \times \operatorname{dom}\alpha) \subseteq \ker\alpha$;

(3) $\forall x \in (\operatorname{dom}\alpha \cap \operatorname{dom}\beta) \setminus Y$, $x\beta \in Y \Rightarrow x\alpha \in Y$;

(4) $\forall x \in \operatorname{dom}\beta$, $x\beta \in \operatorname{im}\alpha \Rightarrow (x \in \operatorname{dom}\alpha \wedge x\alpha = x\beta)$.

Proof. Assume that $\alpha \leq \beta$. Then there exist $\gamma,\mu \in \overline{PT}(X,Y)$ such that $\alpha = \gamma\beta = \beta\mu$ and $\alpha = \alpha\mu$. By Lemma 1 and Lemma 2, we have that the first three conditions are satisfied. Let $x \in \operatorname{dom}\beta$ be such that $x\beta \in \operatorname{im}\alpha$. Then there exists $z \in \operatorname{dom}\alpha$ such that $z\alpha = x\beta$. Since $z \in \operatorname{dom}\alpha = \operatorname{dom}(\alpha\mu) = (\operatorname{dom}\mu)\alpha^{-1}$, it follows that $x\beta = z\alpha \in \operatorname{dom}\mu$. Hence, $x\beta \in \operatorname{dom}\mu$ and this implies $x \in (\operatorname{dom}\mu)\beta^{-1} = \operatorname{dom}(\beta\mu) = \operatorname{dom}\alpha$. Moreover, $x\beta = z\alpha = z\alpha\mu = x\beta\mu = x\alpha$. Therefore, $x \in \operatorname{dom}\alpha$ and $x\beta = x\alpha$.

Conversely, assume that the conditions hold. By Lemma 1, we get that $\alpha = \gamma\beta$ for some $\gamma \in \overline{PT}(X,Y)$. By Lemma 2, we have that $\alpha = \beta\mu$ for some $\mu \in \overline{PT}(X,Y)$. To show $\alpha = \alpha\mu$, we first show that $\operatorname{im}\alpha \subseteq \operatorname{dom}\mu$. Let $z \in \operatorname{im}\alpha$. Then there exists $x \in \operatorname{dom}\alpha$ such that $x\alpha = z$. Since $x \in \operatorname{dom}\alpha = \operatorname{dom}(\gamma\beta) = (\operatorname{dom}\beta)\gamma^{-1}$, we get that $x\gamma \in \operatorname{dom}\beta$. Thus, $x\gamma\beta = x\alpha = z \in \operatorname{im}\alpha$. By the condition (4), we conclude that $x\gamma \in \operatorname{dom}\alpha$ and $x\gamma\alpha = x\gamma\beta$. Since $x\gamma \in \operatorname{dom}\alpha = \operatorname{dom}(\beta\mu) = (\operatorname{dom}\mu)\beta^{-1}$, we obtain that $z = x\gamma\beta \in \operatorname{dom}\mu$. Therefore, $\operatorname{im}\alpha \subseteq \operatorname{dom}\mu$ and thus $\operatorname{dom}(\alpha\mu) = (\operatorname{dom}\mu)\alpha^{-1} = (\operatorname{im}\alpha)\alpha^{-1} = \operatorname{dom}\alpha$. Let $x \in \operatorname{dom}\alpha$. Since $\alpha = \gamma\beta$, we have that $x \in \operatorname{dom}(\gamma\beta) = (\operatorname{dom}\beta)\gamma^{-1}$.

Then $x\gamma \in \operatorname{dom}\beta$ and $x\gamma\beta \in \operatorname{im}\gamma\beta = \operatorname{im}\alpha$. By the condition (4), we have that $x\gamma \in \operatorname{dom}\alpha$ and $x\gamma\alpha = x\gamma\beta$ and thus

$$x\alpha = x\gamma\beta = x\gamma\alpha = x\gamma\beta\mu = x\alpha\mu.$$

Therefore, $\alpha = \alpha\mu$ and $\alpha \leq \beta$, as required. □

For $\alpha \in P(X)$, we denote by α^{-1} the inverse relation of α. Here, α is regarded as a relation, and $\alpha\alpha^{-1}$ denotes the composition of relations between α and its inverse relation α^{-1}. Observe that the relation composition $\alpha\alpha^{-1}$ is equal to $\ker\alpha$. Moreover, condition (4) in Theorem 1 is equivalent to $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$. Therefore, Theorem 1 can be restated as the following corollary:

Corollary 1. *Let $\alpha, \beta \in \overline{PT}(X, Y)$. Then $\alpha \leq \beta$ if and only if all the following conditions hold:*

(1) $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$ *and* $Y\alpha \subseteq Y\beta$;

(2) $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ *and* $\beta\beta^{-1} \cap (\operatorname{dom}\beta \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}$;

(3) $\forall x \in (\operatorname{dom}\alpha \cap \operatorname{dom}\beta) \setminus Y$, $x\beta \in Y \Rightarrow x\alpha \in Y$;

(4) $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$.

As a consequence of Theorem 1, we obtain the following corollary.

Corollary 2. *Let $\alpha, \beta \in \overline{PT}(X, Y)$ such that $\alpha \leq \beta$. If $\operatorname{im}\alpha = \operatorname{im}\beta$, then $\alpha = \beta$.*

Proof. Assume that $\operatorname{im}\alpha = \operatorname{im}\beta$. By Theorem 1(2), we have $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$. Let $x \in \operatorname{dom}\beta$. Then $x\beta \in \operatorname{im}\beta = \operatorname{im}\alpha$. By Theorem 1(4), it follows that $x \in \operatorname{dom}\alpha$ and $x\alpha = x\beta$. Since this holds for all $x \in \operatorname{dom}\beta$, we conclude that $\alpha = \beta$. □

3. Compatibility

Let $\leq$ be a partial order on a semigroup S. An element $c \in S$ is said to be *left-compatible* (respectively, *right-compatible*) if $ca \leq cb$ (respectively, $ac \leq bc$) for all $a, b \in S$ with $a \leq b$.

In this section, we characterize the elements of $\overline{PT}(X, Y)$ that are left- and right-compatible with respect to the natural partial order $\leq$. Since the empty map $\emptyset$ is both left- and right-compatible, we restrict our attention to nonempty elements in the discussion that follows.

Lemma 3. *Let $\emptyset \neq \gamma \in \overline{PT}(X,Y)$ be left compatible. Then the following statements hold:*

(1) $Y \subseteq \operatorname{im}\gamma$;

(2) *If* $\operatorname{im}\gamma \cap (X \setminus Y) \neq \emptyset$, *then* $\operatorname{im}\gamma = X$;

(3) $Y\gamma = \emptyset$ *or* $Y\gamma = Y$.

Proof. (1) Assume that $Y \nsubseteq \operatorname{im}\gamma$. Then there exists $y \in Y \setminus \operatorname{im}\gamma$. Since $\gamma \neq \emptyset$, there exists $x \in \operatorname{im}\gamma$. Define $\alpha, \beta \in \overline{PT}(X,Y)$ by

$$\alpha = \begin{pmatrix} \{x,y\} \\ y \end{pmatrix}, \qquad \beta = \begin{pmatrix} x & y \\ x & y \end{pmatrix}.$$

Thus, $\alpha \leq \beta$, and $\operatorname{im}(\gamma\alpha) = \{y\} \nsubseteq \{x\} = \operatorname{im}(\gamma\beta)$. By Theorem 1, it follows that $\gamma\alpha \nleq \gamma\beta$, which contradicts the assumption that γ is left compatible. Therefore, $Y \subseteq \operatorname{im}\gamma$.

(2) Assume that $\operatorname{im}\gamma \cap (X \setminus Y) \neq \emptyset$ and $\operatorname{im}\gamma \subsetneq X$. Then there exist $z \in \operatorname{im}\gamma \cap (X \setminus Y)$ and $x \in X \setminus \operatorname{im}\gamma$. By (1), we have $Y \subseteq \operatorname{im}\gamma$, which implies that $x \notin Y$. Define $\alpha, \beta \in \overline{PT}(X,Y)$ by

$$\alpha = \begin{pmatrix} \{x,z\} \\ x \end{pmatrix}, \qquad \beta = \begin{pmatrix} x & z \\ x & z \end{pmatrix}.$$

Thus, $\alpha \leq \beta$, and $\operatorname{im}(\gamma\alpha) = \{x\} \nsubseteq \{z\} = \operatorname{im}(\gamma\beta)$. By Theorem 1, it follows that $\gamma\alpha \nleq \gamma\beta$, a contradiction. Therefore, $\operatorname{im}\gamma = X$.

(3) Assume that $Y\gamma \neq \emptyset$ and $Y\gamma \subsetneq Y$. Then there exist $y \in Y\gamma$ and $w \in Y \setminus Y\gamma$. Hence, there exists $y' \in Y \cap \operatorname{dom}\gamma$ such that $y'\gamma = y$. Define $\alpha, \beta \in \overline{PT}(X,Y)$ by

$$\alpha = \begin{pmatrix} \{y,w\} \\ w \end{pmatrix}, \qquad \beta = \begin{pmatrix} y & w \\ y & w \end{pmatrix}.$$

Thus, $\alpha \leq \beta$, and $w = y'\gamma\alpha \in Y\gamma\alpha$. However, since $w \notin Y\gamma$, it follows that $w \notin Y\gamma\beta$. By Theorem 1, we have $\gamma\alpha \nleq \gamma\beta$, a contradiction. Therefore, $Y\gamma = \emptyset$ or $Y\gamma = Y$. □

The following theorem characterizes the left-compatible elements of $\overline{PT}(X,Y)$ with respect to the natural partial order $\leq$.

Theorem 2. *Let $\emptyset \neq \gamma \in \overline{PT}(X,Y)$. Then γ is left compatible if and only if*

$$(Y\gamma = \emptyset \text{ or } Y\gamma = Y) \quad \text{and} \quad (\operatorname{im}\gamma = Y \text{ or } \operatorname{im}\gamma = X).$$

Proof. Assume that γ is left compatible. By Lemma 3(1), we have $Y \subseteq \operatorname{im}\gamma$, and by Lemma 3(3), we have $Y\gamma = \emptyset$ or $Y\gamma = Y$. To show that $\operatorname{im}\gamma = Y$ or $\operatorname{im}\gamma = X$, suppose that $Y \subsetneq \operatorname{im}\gamma$. Then $\operatorname{im}\gamma \cap (X \setminus Y) \neq \emptyset$; therefore, by Lemma 3(2), $\operatorname{im}\gamma = X$.

Conversely, assume that the given conditions hold, and let $\alpha, \beta \in \overline{PT}(X,Y)$ be such that $\alpha \leq \beta$. Then α and β satisfy all the conditions of Theorem 1. To show that γ is left compatible, it suffices to prove that $\gamma\alpha \leq \gamma\beta$. For this purpose, we verify that $\gamma\alpha$ and $\gamma\beta$ satisfy all the conditions stated in Theorem 1.

Verification of condition (1). By the property of $Y\gamma$, it is clear that $Y\gamma\alpha \subseteq Y\gamma\beta$. To show that $\operatorname{im}(\gamma\alpha) \subseteq \operatorname{im}(\gamma\beta)$, we consider two cases.

Case 1: $\operatorname{im}\gamma = Y$. Then

$$\operatorname{im}(\gamma\alpha) = (Y \cap \operatorname{dom}\alpha)\alpha = Y\alpha \subseteq Y\beta = (Y \cap \operatorname{dom}\beta)\beta = \operatorname{im}(\gamma\beta).$$

Case 2: $\operatorname{im}\gamma = X$. Then

$$\begin{aligned}\operatorname{im}(\gamma\alpha) \subseteq \operatorname{im}\alpha \subseteq \operatorname{im}\beta =& (\operatorname{dom}\beta)\beta = (X \cap \operatorname{dom}\beta)\beta \\ =& (\operatorname{im}\gamma \cap \operatorname{dom}\beta)\beta = \operatorname{im}(\gamma\beta).\end{aligned}$$

Verification of condition (2). First note that

$$\operatorname{dom}(\gamma\alpha) = (\operatorname{dom}\alpha)\gamma^{-1} \subseteq (\operatorname{dom}\beta)\gamma^{-1} = \operatorname{dom}(\gamma\beta).$$

Next, let $(a,b) \in \ker(\gamma\beta) \cap (\operatorname{dom}(\gamma\beta) \times \operatorname{dom}(\gamma\alpha))$. Then $a\gamma\beta = b\gamma\beta$ with $a \in \operatorname{dom}(\gamma\beta)$ and $b \in \operatorname{dom}(\gamma\alpha)$. Hence $a\gamma \in \operatorname{dom}\beta$ and $b\gamma \in \operatorname{dom}\alpha$. It follows that $(a\gamma, b\gamma) \in \ker\beta \cap (\operatorname{dom}\beta \times \operatorname{dom}\alpha) \subseteq \ker\alpha$, and therefore $a\gamma\alpha = b\gamma\alpha$. Thus $(a,b) \in \ker(\gamma\alpha)$, and we conclude that $\ker(\gamma\beta) \cap (\operatorname{dom}(\gamma\beta) \times \operatorname{dom}(\gamma\alpha)) \subseteq \ker(\gamma\alpha)$.

Verification of condition (3). Let $x \in (\operatorname{dom}(\gamma\alpha) \cap \operatorname{dom}(\gamma\beta)) \setminus Y$ be such that $x\gamma\beta \in Y$. Then $x\gamma \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta$. If $x\gamma \in Y$, it follows that $x\gamma\alpha \in Y$. If $x\gamma \notin Y$, then since α and β satisfy condition (3), we also have $x\gamma\alpha \in Y$. Hence, condition (3) holds.

Verification of condition (4). Let $x \in \operatorname{dom}(\gamma\beta)$ be such that $x\gamma\beta \in \operatorname{im}(\gamma\alpha) \subseteq \operatorname{im}\alpha$. Then $x\gamma \in \operatorname{dom}\beta$. Since α and β satisfy condition (4), it follows that $x\gamma \in \operatorname{dom}\alpha$ and $x\gamma\alpha = x\gamma\beta$.

Hence, all conditions of Theorem 1 are satisfied, and we conclude that $\gamma\alpha \leq \gamma\beta$. Therefore, γ is left compatible. □

To determine the right-compatible elements, we first establish some necessary conditions.

Lemma 4. *Let $\emptyset \neq \gamma \in \overline{PT}(X,Y)$ be right compatible. Then the following statements hold:*

(1) *γ is injective;*

(2) $\operatorname{dom}\gamma = X$*;*

(3) $Y\gamma^{-1} \subseteq Y$ *or* $\operatorname{im}\gamma \subseteq Y$*.*

Proof. (1) Assume that γ is not injective. Then there exist distinct $a, b \in \operatorname{dom}\gamma$ such that $a\gamma = b\gamma$. Define $\alpha, \beta \in \overline{PT}(X,Y)$ by

$$\alpha = \begin{pmatrix} a \\ a \end{pmatrix}, \qquad \beta = \begin{pmatrix} a & b \\ a & b \end{pmatrix}.$$

Then $\alpha \leq \beta$, and $(b,a) \in \ker(\beta\gamma) \cap (\operatorname{dom}(\beta\gamma) \times \operatorname{dom}(\alpha\gamma))$, but $(b,a) \notin \ker(\alpha\gamma)$. By Theorem 1, it follows that $\alpha\gamma \not\leq \beta\gamma$, which contradicts the assumption that γ is right compatible. Therefore, γ is injective.

(2) Assume that $\operatorname{dom}\gamma \subsetneq X$. Then there exists $z \in X \setminus \operatorname{dom}\gamma$. Since $\gamma \neq \emptyset$, there exists $x \in \operatorname{dom}\gamma$. Define $\alpha, \beta \in \overline{PT}(X,Y)$ by

$$\alpha = \begin{pmatrix} \{x,z\} \\ x \end{pmatrix}, \qquad \beta = \begin{pmatrix} x & z \\ x & z \end{pmatrix}.$$

Then $\alpha \leq \beta$, and $\operatorname{dom}(\alpha\gamma) = \{x,z\} \nsubseteq \{x\} = \operatorname{dom}(\beta\gamma)$. By Theorem 1, it follows that $\alpha\gamma \not\leq \beta\gamma$, a contradiction.

(3) Assume that $Y\gamma^{-1} \nsubseteq Y$ and $\operatorname{im}\gamma \nsubseteq Y$. Then there exist $x \in Y\gamma^{-1} \setminus Y$ and $z \in \operatorname{im}\gamma \setminus Y$. It follows that $x \in \operatorname{dom}\gamma \setminus Y$ with $x\gamma \in Y$, and there exists $z' \in \operatorname{dom}\gamma \setminus Y$ such that $z'\gamma = z$. Define $\alpha, \beta \in \overline{PT}(X,Y)$ by

$$\alpha = \begin{pmatrix} \{x,z'\} \\ z' \end{pmatrix}, \qquad \beta = \begin{pmatrix} x & z' \\ x & z' \end{pmatrix}.$$

Then $\alpha \leq \beta$, and $x \in (\mathrm{dom}\,(\alpha\gamma) \cap \mathrm{dom}\,(\beta\gamma)) \setminus Y$. However, $x\beta\gamma = x\gamma \in Y$ while $x\alpha\gamma = z'\gamma = z \notin Y$. It follows from Theorem 1 that $\alpha\gamma \not\leq \beta\gamma$, a contradiction. □

Theorem 3. *Let $\emptyset \neq \gamma \in \overline{PT}(X, Y)$. Then γ is right compatible if and only if γ is an injection with $\mathrm{dom}\,\gamma = X$ and ($Y\gamma^{-1} \subseteq Y$ or $\mathrm{im}\,\gamma \subseteq Y$).*

Proof. The necessity follows directly from Lemma 4. It remains to prove the sufficiency. Assume that the given conditions hold, and let $\alpha, \beta \in \overline{PT}(X, Y)$ be such that $\alpha \leq \beta$. Then α and β satisfy all the conditions of Theorem 1. To show that γ is right compatible, it suffices to verify that $\alpha\gamma$ and $\beta\gamma$ satisfy all the conditions stated in Theorem 1.

Verification of condition (1).

$$\mathrm{im}\,(\alpha\gamma) = (\mathrm{im}\,\alpha \cap \mathrm{dom}\,\gamma)\gamma \subseteq (\mathrm{im}\,\beta \cap \mathrm{dom}\,\gamma)\gamma = \mathrm{im}\,(\beta\gamma),$$

$$Y\alpha\gamma = (Y\alpha)\gamma \subseteq (Y\beta)\gamma = Y\beta\gamma.$$

Verification of condition (2). First note that

$$\mathrm{dom}\,(\alpha\gamma) = X\alpha^{-1} = \mathrm{dom}\,\alpha \subseteq \mathrm{dom}\,\beta = X\beta^{-1} = \mathrm{dom}\,(\beta\gamma).$$

Next, let $(a, b) \in \ker(\beta\gamma) \cap (\mathrm{dom}\,(\beta\gamma) \times \mathrm{dom}\,(\alpha\gamma))$. Then $a\beta\gamma = b\beta\gamma$, with $a \in \mathrm{dom}\,(\beta\gamma) \subseteq \mathrm{dom}\,\beta$ and $b \in \mathrm{dom}\,(\alpha\gamma) \subseteq \mathrm{dom}\,\alpha$. Since γ is injective, it follows that $a\beta = b\beta$. Hence $(a, b) \in \ker\beta \cap (\mathrm{dom}\,\beta \times \mathrm{dom}\,\alpha) \subseteq \ker\alpha$, and therefore $a\alpha = b\alpha$. Thus $a\alpha\gamma = b\alpha\gamma$, implying $(a, b) \in \ker(\alpha\gamma)$. Therefore, $\ker(\beta\gamma) \cap (\mathrm{dom}\,(\beta\gamma) \times \mathrm{dom}\,(\alpha\gamma)) \subseteq \ker(\alpha\gamma)$.

Verification of condition (3). Let $x \in (\mathrm{dom}\,(\alpha\gamma) \cap \mathrm{dom}\,(\beta\gamma)) \setminus Y$ be such that $x\beta\gamma \in Y$. Then $x \in (\mathrm{dom}\,\alpha \cap \mathrm{dom}\,\beta) \setminus Y$. To show that $x\alpha\gamma \in Y$, we consider two cases.

Case 1: $Y\gamma^{-1} \subseteq Y$. Since $x\beta\gamma \in Y$, it follows that $x\beta \in Y\gamma^{-1} \subseteq Y$. Because α and β satisfy condition (3), we conclude that $x\alpha \in Y$, and hence $x\alpha\gamma \in Y$.

Case 2: $\mathrm{im}\,\gamma \subseteq Y$. In this case, we also have $x\alpha\gamma \in \mathrm{im}\,\gamma \subseteq Y$.

Therefore, condition (3) holds.

Verification of condition (4). Let $x \in \mathrm{dom}\,(\beta\gamma)$ be such that $x\beta\gamma \in \mathrm{im}\,(\alpha\gamma)$. Then there exists $z \in \mathrm{dom}\,(\alpha\gamma)$ such that $x\beta\gamma = z\alpha\gamma$. Since γ is injective, it follows that $x\beta = z\alpha \in \mathrm{im}\,\alpha$. Hence $x \in \mathrm{dom}\,\alpha$ and

$x\alpha = x\beta$. Thus,

$$x \in \operatorname{dom}\alpha = X\alpha^{-1} = (\operatorname{dom}\gamma)\alpha^{-1} = \operatorname{dom}(\alpha\gamma),$$

and $x\alpha\gamma = x\beta\gamma$. Therefore, condition (4) holds.

Hence, all conditions of Theorem 1 are satisfied, and we conclude that $\alpha\gamma \leq \beta\gamma$. Therefore, γ is right compatible. □

4. The minimal elements and maximal elements

In this section, we characterize the minimal elements of $\overline{PT}(X,Y)$ with respect to the natural partial order. Note that the empty map $\emptyset$ is the minimum element and the zero element of $\overline{PT}(X,Y)$. Therefore, we are interested in elements that are minimal within the set of non-zero elements. For convenience, we write $\alpha < \beta$ to denote $\alpha \leq \beta$ with $\alpha \neq \beta$.

Theorem 4. *Let $\emptyset \neq \alpha \in \overline{PT}(X,Y)$. Then α is minimal within the set of non-zero elements if and only if α is a constant map.*

Proof. Assume that α is not a constant map. Then $|\operatorname{im}\alpha| \geq 2$. Let $a \in \operatorname{im}\alpha$ and set $a\alpha^{-1} = A \subseteq \operatorname{dom}\alpha$. Define $\beta \in \overline{PT}(X,Y)$ by

$$\beta = \begin{pmatrix} A \\ a \end{pmatrix}.$$

Then $\beta < \alpha$, which shows that α is not minimal.

Conversely, assume that α is a constant map with $\operatorname{im}\alpha = \{a\}$. Let $\beta \in \overline{PT}(X,Y)$ be such that $\emptyset \neq \beta \leq \alpha$. Then $\emptyset \neq \operatorname{im}\beta \subseteq \operatorname{im}\alpha = \{a\}$, and hence $\operatorname{im}\beta = \{a\} = \operatorname{im}\alpha$. By Corollary 2, it follows that $\beta = \alpha$. Therefore, α is minimal. □

To determine the maximal elements, we first present some necessary conditions for an element to be maximal.

Lemma 5. *Let $\alpha \in \overline{PT}(X,Y)$ be maximal but not surjective. Then the following statements hold:*

(1) $\ker\alpha \cap (Y \times (X \setminus Y)) = \emptyset$;

(2) $X \setminus Y \subseteq \operatorname{dom}\alpha$.

Proof. Since α is not surjective, there exists $a \in X \setminus \operatorname{im}\alpha$.

(1) Assume that there exists $(y, z) \in \ker\alpha \cap (Y \times (X \setminus Y))$. Then $y\alpha = z\alpha$ with $y \in Y$ and $z \in X \setminus Y$. Define $\beta \in \overline{PT}(X, Y)$ by

$$\beta = \begin{pmatrix} z & x \\ a & x\alpha \end{pmatrix}_{x \in \operatorname{dom}\alpha \setminus \{z\}}.$$

Then $\alpha < \beta$, which contradicts the maximality of α.

(2) Assume that there exists $z \in (X \setminus Y) \setminus \operatorname{dom}\alpha$. Define $\beta \in \overline{PT}(X, Y)$ as in (1). Then again $\alpha < \beta$, which contradicts the maximality of α. □

Theorem 5. *Let $\alpha \in \overline{PT}(X, Y)$. Then α is maximal if and only if it satisfies one of the following conditions:*

(1) *α is surjective;*

(2) *α is injective and* $\operatorname{dom}\alpha = X$*;*

(3) $\ker\alpha \cap (Y \times (X \setminus Y)) = \emptyset$, $X \setminus Y \subseteq \operatorname{dom}\alpha$, *and one of the following holds:*

   (i) *$Y \subseteq \operatorname{im}\alpha$ and $\alpha|_{X\setminus Y}$ is injective;*

   (ii) *$X \setminus Y \subseteq \operatorname{im}\alpha$,* $\operatorname{dom}\alpha = X$, *and $\alpha|_{Y\alpha^{-1}}$ is injective.*

Proof. Let α be maximal and assume that it does not satisfy conditions (1) and (2). We will show that α satisfies condition (3). Since α does not satisfy condition (1), it follows that α is not surjective, and hence, by Lemma 5, we obtain that $\ker\alpha \cap (Y \times (X \setminus Y)) = \emptyset$ and $X \setminus Y \subseteq \operatorname{dom}\alpha$. Since α does not satisfy condition (2), it follows that α is not injective or $\operatorname{dom}\alpha \subsetneq X$.

Case 1: α is not injective. Since $\ker\alpha \cap (Y \times (X \setminus Y)) = \emptyset$, it follows that $\alpha|_Y$ is not injective or $\alpha|_{X\setminus Y}$ is not injective.

Subcase 1.1: $\alpha|_Y$ is not injective. Then there exist distinct elements $y_1, y_2 \in Y \cap \operatorname{dom}\alpha$ such that $y_1\alpha = y_2\alpha$. We claim that α satisfies condition (i). To prove $Y \subseteq \operatorname{im}\alpha$, suppose that there exists $y \in Y \setminus \operatorname{im}\alpha$. Define $\beta \in \overline{PT}(X, Y)$ by

$$\beta = \begin{pmatrix} y_1 & x \\ y & x\alpha \end{pmatrix}_{x \in \operatorname{dom}\alpha \setminus \{y_1\}}.$$

Then $\alpha < \beta$, which contradicts the maximality of α. Therefore, $Y \subseteq \operatorname{im}\alpha$. This implies that $a \in X \setminus Y$. To show $\alpha|_{X\setminus Y}$ is injective, suppose that there exist distinct $x_1, x_2 \in (X \setminus Y) \cap \operatorname{dom}\alpha$ such that $x_1\alpha = x_2\alpha$. Define $\beta \in \overline{PT}(X,Y)$ by

$$\beta = \begin{pmatrix} x_1 & x \\ a & x\alpha \end{pmatrix}_{x \in \operatorname{dom}\alpha \setminus \{x_1\}}.$$

Then $\alpha < \beta$, a contradiction. Hence, $\alpha|_{X\setminus Y}$ is injective. Therefore, α satisfies condition (3)(i).

Subcase 1.2: $\alpha|_{X\setminus Y}$ is not injective. Then there exist distinct $x_1, x_2 \in (X \setminus Y) \cap \operatorname{dom}\alpha$ such that $x_1\alpha = x_2\alpha$. We claim that α satisfies condition (ii). To show $X \setminus Y \subseteq \operatorname{im}\alpha$, suppose that there exists $z \in (X \setminus Y) \setminus \operatorname{im}\alpha$. Define $\beta \in \overline{PT}(X,Y)$ by

$$\beta = \begin{pmatrix} x_1 & x \\ z & x\alpha \end{pmatrix}_{x \in \operatorname{dom}\alpha \setminus \{x_1\}}.$$

Then $\alpha < \beta$, a contradiction. Therefore, $X \setminus Y \subseteq \operatorname{im}\alpha$. This implies that $a \in Y$. To show $\operatorname{dom}\alpha = X$, suppose that there exists $z \in X \setminus \operatorname{dom}\alpha$. Define $\beta \in \overline{PT}(X,Y)$ by

$$\beta = \begin{pmatrix} z & x \\ a & x\alpha \end{pmatrix}_{x \in \operatorname{dom}\alpha \setminus \{z\}}.$$

Then $\alpha < \beta$, a contradiction. Hence, $\operatorname{dom}\alpha = X$. Finally, to show $\alpha|_{Y\alpha^{-1}}$ is injective, suppose that there exist distinct $z_1, z_2 \in Y\alpha^{-1}$ such that $z_1\alpha = z_2\alpha$. Define $\beta \in \overline{PT}(X,Y)$ by

$$\beta = \begin{pmatrix} z_1 & x \\ a & x\alpha \end{pmatrix}_{x \in \operatorname{dom}\alpha \setminus \{z_1\}}.$$

Then $\alpha < \beta$, a contradiction. Therefore, $\alpha|_{Y\alpha^{-1}}$ is injective. Hence, α satisfies condition (3)(ii).

Case 2: $\operatorname{dom}\alpha \subsetneq X$. Since $X \setminus Y \subseteq \operatorname{dom}\alpha$, it follows that there exists $y_1 \in Y \setminus \operatorname{dom}\alpha$. We claim that α satisfies condition (i). To show $Y \subseteq \operatorname{im}\alpha$, assume that there exists $y \in Y \setminus \operatorname{im}\alpha$. Define $\beta \in \overline{PT}(X,Y)$

as in Subcase 1.1. Then, as before, $\alpha < \beta$, a contradiction. Hence, $Y \subseteq \operatorname{im}\alpha$. By the same argument as in Subcase 1.1, we can conclude that $\alpha|_{X\setminus Y}$ is injective. Therefore, α also satisfies condition (3)(i).

Conversely, assume that α satisfies condition (1), (2), or (3), and let $\alpha \leq \beta$ for some $\beta \in \overline{PT}(X,Y)$. By Theorem 1(1), we have $\operatorname{im}\alpha \subseteq \operatorname{im}\beta$. Hence, by Lemma 2, it suffices to show that $\operatorname{im}\beta \subseteq \operatorname{im}\alpha$ in order to conclude that $\alpha = \beta$.

Let $a \in \operatorname{im}\beta$. Then there exists $z \in \operatorname{dom}\beta$ such that $z\beta = a$.

Case 1: α satisfies condition (1). Since α is surjective, we immediately have $\operatorname{im}\alpha = \operatorname{im}\beta$.

Case 2: α satisfies condition (2). Since $\operatorname{dom}\alpha = X$, we have $z \in \operatorname{dom}\alpha$. Then $z\alpha \in \operatorname{im}\alpha \subseteq \operatorname{im}\beta$, and hence there exists $z' \in \operatorname{dom}\beta$ such that $z'\beta = z\alpha$. It follows that $z'\alpha = z'\beta = z\alpha$. Since α is injective, we obtain $z = z'$, and therefore $a = z\beta = z'\beta = z'\alpha \in \operatorname{im}\alpha$.

Case 3: α satisfies condition (3).

Subcase 3.1: α satisfies condition (i). If $a \in Y$, then clearly $a \in \operatorname{im}\alpha$. If $a \in X \setminus Y$, then $z \in X \setminus Y \subseteq \operatorname{dom}\alpha$, and hence $z\alpha \in \operatorname{im}\alpha \subseteq \operatorname{im}\beta$. As in Case 2, there exists $z' \in \operatorname{dom}\beta$ such that $z'\alpha = z'\beta = z\alpha$. Since $\ker\alpha \cap (Y \times (X \setminus Y)) = \emptyset$, it follows that $z' \in X \setminus Y$. Because $\alpha|_{X\setminus Y}$ is injective, we have $z = z'$. Thus, as before, $a \in \operatorname{im}\alpha$.

Subcase 3.2: α satisfies condition (ii). If $a \in X \setminus Y$, then $a \in \operatorname{im}\alpha$ immediately. Now consider the case $a \in Y$. Since $\operatorname{dom}\alpha = X$, we have $z \in \operatorname{dom}\alpha$. As in Case 2, there exists $z' \in \operatorname{dom}\beta$ such that $z'\alpha = z'\beta = z\alpha$. Since $z\beta = a \in Y$, we obtain $z\alpha \in Y$, and hence $z, z' \in Y\alpha^{-1}$. Because $\alpha|_{Y\alpha^{-1}}$ is injective, it follows that $z = z'$. Therefore, as before, $a \in \operatorname{im}\alpha$.

In all cases, $\operatorname{im}\beta \subseteq \operatorname{im}\alpha$, and hence by Lemma 2, we conclude that $\alpha = \beta$. Therefore, α is maximal. □

Conclusion

In this paper, we studied the semigroup $\overline{PT}(X,Y)$, which generalizes the partial transformation semigroup $P(X)$. We provided a characterization of the natural partial order on $\overline{PT}(X,Y)$ and classified its elements that are left-compatible, right-compatible, minimal, and maximal with respect to this order.

Noting that $\overline{PT}(X,X) = P(X)$, by setting $Y = X$ in Corollary 1, we recover the characterization of the natural partial order on $P(X)$, originally presented in [9, Theorem 2]. Similarly, Theorems 2 and 3 generalize [9, Theorem 9], and Theorems 4 and 5 generalize [9, Theorems 13 and 14], respectively.

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