

Matroids arisen from seeds

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ABSTRACT. This paper aims to shed light on new (sub)classes of matroids originating from cluster algebras and investigate their properties. We focus on what we call *cluster matroids* and build some results on them. Then, we point out a relationship between these kinds of matroids and uniform matroids and study their minors.

1. Introduction

The theory of matroids originated in 1935 by Whitney to abstract the notion of the linear independence of vector spaces. Since then, because of its interesting properties and applications, the matroid theory has formed one of the most active areas of algebraic combinatorics. On the other hand, cluster algebras were invented by Fomin and Zelevinsky in 2002 and have quickly received a lot of interest because of their significant applications and connections to different areas of mathematics. For instance, the applications of the theory of cluster algebras appear in representation theory, combinatorics, algebraic geometry, Poisson geometry, integrable systems, mathematical physics, and topology. Although both theories have the flavor of algebra and combinatorics and despite the existence of some works built on both theories, their immediate relationships have not yet been studied well.

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In this study, we provide an overview of matroids in Section 2 and cluster algebras in Section 3. We then introduce some connections between them in Section 4. Next, we take a close look at the behavior of the minors of the main class of matroids in this study in Section 5.

2. Matroid preliminaries

This section introduces the theory of matroids and the needed results. The reader who is interested in a deeper look is referred to Oxley's book [8].

Definition 2.1. A *matroid* M is a pair (E, \mathcal{B}) where E is a finite non-empty set called the *ground set* and \mathcal{B} is a subset of the power set of E in which

1. $\mathcal{B} \neq \emptyset$.
2. If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, then there is a $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

A member of \mathcal{B} is called a *basis* of the matroid M , while a subset I of E is called *independent* if it is a subset of a basis. Any subset of E that is not independent is called *dependent*. Sometimes, we may write $\mathcal{B}(M)$ instead of \mathcal{B} to emphasize that we are considering the set of bases of the matroid M .

Example 2.2. Let A be a matrix and E be the set of column labels of A . Let \mathcal{B} be the set of linearly independent sets of maximal size induced by the column labels of A . Then, (E, \mathcal{B}) is a matroid. A matroid is called *representable* if it can be formed by the linear independence relations of a matrix.

Remark 2.3. It is not hard to see that the definition of a matroid is a generalization of the properties of bases of a linear space V . A matroid can be defined in other equivalent ways, such as the independent sets or *circuits*, which are the minimal dependent sets.

Remark 2.4. Let M be a matroid whose ground set is E and set of bases is \mathcal{B} . The pair $M^* = (E, \mathcal{B}^*)$, in which $\mathcal{B}^* = \{E \setminus B \mid B \in \mathcal{B}\}$, forms a matroid whose set of bases is \mathcal{B}^* . This matroid is called the *dual matroid* of M . Bases, (in)dependent sets and circuits in the dual matroid are called cobases, co(in)dependent sets and cocircuits of the original matroid, respectively.

Now, we recall the following definition:

Definition 2.5. Let \mathbb{K} be an extension field of a field \mathbb{F} . Let $\alpha \in \mathbb{K}$ and let $E \subset \mathbb{K}$.

1. An element α is said to be *algebraic* over \mathbb{F} if there exists a nonzero polynomial $p \in \mathbb{F}[x]$ such that $p(\alpha) = 0$. If no such polynomial exists, α is called *transcendental* over \mathbb{F} .
2. A set E is said to be *algebraically independent* over \mathbb{F} if there is no nonzero polynomial $p \in \mathbb{F}[x_1, x_2, \dots, x_n]$, where $n = |E|$, such that p vanishes when evaluated at the elements of E . Otherwise, E is said to be *algebraically dependent* over \mathbb{F} .

Theorem 2.6. Let \mathbb{K} be an extension field of a field \mathbb{F} and $E \subset \mathbb{K}$ be finite. The collection \mathcal{I} of subsets of E that are algebraically independent over \mathbb{F} forms a matroid on E whose independent sets are the members of \mathcal{I} .

Definition 2.7. A matroid M is called *connected* if every two elements of it lie in a common circuit or cocircuit.

Definition 2.8. Let M_1 and M_2 be two matroids with disjoint ground sets. The *direct sum* $M_1 \oplus M_2$ is the matroid whose ground sets is the union of the ground sets of M_1 and M_2 and whose bases are the union of their bases.

Theorem 2.9. A matroid M is connected if and only if it cannot be written as a direct sum of two matroids.

Definition 2.10. Let $M = (E, \mathcal{B})$ be a matroid and $X \subset E$.

1. The *restriction* of M to X , denoted $M|X$ or $M \setminus (E \setminus X)$, is the matroid whose set of bases is the set of the maximal independent sets contained in X . In the case of singletons, we will denote $M|\{e\}$ and $M \setminus \{e\}$ by $M|e$ and $M \setminus e$, respectively. The operation of restricting M to X is called the *deletion* of $E \setminus X$.
2. The *contraction* of X is the matroid denoted by M/X and given by $(M^* \setminus X)^*$. Similarly to the restriction, we will omit the set brackets when we deal with singletons.
3. A *minor* of M is a matroid induced from M by a sequence of contractions and restrictions.

3. Cluster algebra overview

This section introduces the notion of cluster algebra. For a wide overview, the reader is referred to [2, 5]. We begin with the following sequence of definitions.

Definition 3.1. A (*labeled*) *seed* is a pair (\mathbf{x}, B) such that

$$\mathbf{x} = (x_1, \dots, x_n, x_{n+1}, \dots, x_m)$$

is a tuple of algebraically independent variables generating a field isomorphic to the field $\mathbb{C}(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$. Also, B is an $m \times n$ extended skew-symmetrizable matrix, that is, a matrix whose north $n \times n$ submatrix can be transformed to a skew-symmetric matrix by multiplying each row r_i by a nonzero integer d_i . The matrix B is called the *exchange matrix* and the tuple \mathbf{x} is called the *extended cluster*. The variables x_1, \dots, x_n are called *mutable*, while the variables x_{n+1}, \dots, x_m are called *frozen*.

Remark 3.2. In some cases, the field $\mathbb{C}(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ of the previous definition is replaced by the field $\mathbb{Q}(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$. We will mainly deal with the first in this paper, but all the results still make sense for the second.

Definition 3.3. Let k be an index of a mutable variable of a seed (\mathbf{x}, B) . A *mutation* at k is a transformation to a new seed (\mathbf{x}', B') in which B' is an $m \times n$ matrix whose entries are

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}, & \text{otherwise;} \end{cases} \quad (3.1)$$

and $\mathbf{x}' = (x'_1, \dots, x'_n, x'_{n+1}, \dots, x'_m)$ is a tuple such that $x'_i = x_i$ for $i \neq k$ and

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}.$$

The seed (\mathbf{x}', B') obtained by a mutation at k is denoted sometimes by $\mu_k(\mathbf{x}, B)$.

Remark 3.4. It is not hard to see that the mutation of a seed provides a new seed. Moreover, mutating twice at the same index brings the original seed back. In symbols,

$$\mu_k(\mu_k(\mathbf{x}, B)) = (\mathbf{x}, B).$$

Definition 3.5. Let (\mathbf{x}, B) be a seed. A *cluster algebra (of geometric type)* attached to (\mathbf{x}, B) is the polynomial algebra $\mathcal{A} = \mathbb{C}[x_{n+1}, \dots, x_m][\chi]$, where χ is the set of all possible mutable variables, that is, the mutable variables of the original seed or a seed obtained by a mutation or a sequence of mutations. The seed (\mathbf{x}, B) is called the *initial seed*.

Remark 3.6. By the properties of mutation and algebraically independent sets, it is not hard to see that the cluster algebra attached to some seed is the same cluster algebra attached to any seed mutation. Therefore, in the context of cluster algebras, we often fix an initial seed and describe the cluster algebra \mathcal{A} by means of it. In terms of notation, we sometimes write $\mathcal{A}(\mathbf{x}, B)$ instead of \mathcal{A} , if there is an emphasis on the initial seed (\mathbf{x}, B) .

Definition 3.7. The *rank* of a seed or a cluster algebra attached to it is the number of mutable variables of its initial seed. A cluster algebra is *of finite type* if it has finitely many seeds. Otherwise, it is *of infinite type*.

Remark 3.8. The finite type classification of cluster algebras is closely related to Lie Theory. In fact, the cluster algebras of finite type are classified by the Dynkin Diagrams, which are also the main objects classifying the semisimple complex Lie algebras.

Definition 3.9. Let B be an $n \times n$ square integer matrix. The *Cartan counterpart* of B is the matrix $A(B) = (a_{i,j})$ defined by:

$$a_{ii} := 2, \quad \text{and} \quad a_{ij} := -|b_{ij}| \text{ if } i \neq j.$$

Theorem 3.10. A cluster algebra \mathcal{A} is of finite type if and only if the Cartan counterpart of one of its seeds is a Cartan matrix of finite type, that is, of type $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$, or G_2 .

Definition 3.11. For a cluster algebra \mathcal{A} , a *cluster monomial* is a monomial consisting of variables from a single seed.

4. Cluster matroids

In this section, we investigate a matroid structure in the set of extended clusters and build a connection between the two topics from there. Throughout, the terms “cluster algebra” and “seed” mean a cluster algebra and a seed of finite type.

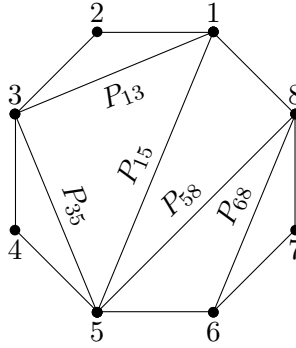


Figure 1: A triangulation of the octagon.

Example 4.1. This example produces a way to give the Grassmannian $\text{Gr}_{2,n}$ a cluster algebra. We skip some details here. For a wider overview, the reader is encouraged to see [2] Section 1.2. Consider the octagon of Figure 1. A *triangulation* of the octagon is a shape obtained by drawing a maximal number of pairwise non-crossing diagonals. It is easily seen that any triangulation of the octagon produces exactly 5 non-crossing diagonals, one of them is the one in Figure 1. More generally, a triangulation of an m -gon produces exactly $m - 3$ non-crossing diagonals. Now, this octagon forms a combinatorial way of describing a seed whose frozen variables are the sides of the octagon and whose mutable variables are the non-crossing diagonals. The mutation of seeds here corresponds to diagonal flipping. For instance, flipping the diagonal P_{58} to P_{16} corresponds to another triangulation that is a mutation of the first one at the variable P_{58} . Another example is to flip P_{68} to P_{57} and so on.

Naïvely, one might think that this gives rise to a matroid on the set of diagonals of an n -gon by taking as the bases those edges that are the edges of a triangulation, but this turns out to be false. In fact, consider the triangulation in Figure 2. If P_{14} is removed from this triangulation, then there is no diagonal from the triangulation of Figure 1 that can be inserted and give a new triangulation. Hence, the second axiom of matroid bases is not satisfied and this is not a matroid. However, we will be able to solve this issue in the next remark.

Remark 4.2. From Theorem 2.6, it is straightforward to see that the set of extended clusters of some cluster algebra induces a matroid. Indeed, define the bases of this matroid to be all maximal algebraically independent sets containing the frozen variables, where the ground set is

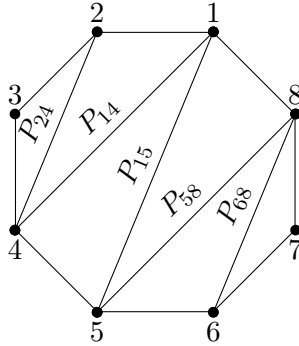


Figure 2: Another triangulation of the octagon.

the set of all possible variables generated by any sequence of mutations. We will call this a *cluster matroid*.

Remark 4.3. Since cluster algebras of type A_n are combinatorially described by the triangulations of an $(n+3)$ -gon and since the $(2n+3)$ -quantities attached to the triangulations are algebraically independent, one can see that this can give a matroid on any n -gon using the algebraic independence and the definition of cluster matroids of the previous remark.

Example 4.4 (Rank 1). (c.f. Example 3.2.2 of [2]) A cluster algebra of rank 1 has exactly two mutable variables, say x_1 and x'_1 , each of them is a mutation of the other. It can have any number of frozen variables. Any $m \times 1$ matrix B with top entry 0 is a possible extended exchange matrix for such a cluster algebra. Of course, x_1 and x'_1 are related by the mutation relation $x_1 x'_1 = \prod_i x_i^{b_i} + \prod_j x_j^{-b_j}$ where $i \neq j$ and $b_i > 0$ and $b_j < 0$ for all i and j . This cluster algebra is generated by the variables $x_1, x'_1, x_2, \dots, x_m$ and lives inside $\mathbb{C}(x_1, x_2, \dots, x_m) = \mathbb{C}(x'_1, x_2, \dots, x_m)$. The matroid attached to this cluster algebra is given by the ground set $E = \{x_1, x'_1, x_2, \dots, x_m\}$ and the bases $B_1 = \{x_1, x_2, \dots, x_m\}$ and $B_2 = \{x'_1, x_2, \dots, x_m\}$. A more concrete example is the coordinate ring of the subgroup of unipotent upper triangular matrices

$$U^+ = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \right\} \subset SL_3.$$

This coordinate ring is $\mathbb{C}[a, b, c]$ and it forms a cluster algebra of rank 1 whose extended clusters are $\{a, b, ac - b\}$ and $\{c, b, ac - b\}$. Clearly, the mutable variables are a and c and the frozen variables are b and $ac - b$.

Example 4.5. Let (x_1, x_2) be a seed in which both of its variables are mutable. Let

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

be the exchange matrix attached to this seed. It is not hard to see that the list of all possible cluster variables is: $x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}$, and $\frac{1+x_1}{x_2}$.

Any 2-set of elements of this list is a basis of the induced cluster matroid. This matroid is denoted by $U_{2,5}$. More generally, for $n \leq m$, the *uniform matroid* $U_{n,m}$ is the matroid whose ground set is $\{1, \dots, m\}$ such that any n -subset forms a basis.

Remark 4.6. In general, cluster matroids are not closed under duality. This is because the number of seeds is fully determined by the exchange matrix together with the cluster algebra rank. Thus, if the rank is changed, then the number of seeds, and hence the size of the matroid, will be different.

Remark 4.7. The matroids introduced in Theorem 2.6 are called *algebraic*. There are many open questions about them and their duals. We believe that the study of cluster matroids can help answer some of these open questions. One can see [8] for a deeper look at algebraic matroids.

Now, we note the following results of cluster matroids:

Theorem 4.8. *A cluster matroid of size greater than or equal to 2 is connected if and only if it has no frozen variables.*

Proof. Since the frozen variables appear in every basis, they are coloops, that is, codependent singletons. It is straight forward to verify that any matroid with (co)loops is disconnected. Now, assume that there is a disconnected cluster matroid M consisting merely of mutable variables. This implies the existence of two matroids M_1 and M_2 such that $M = M_1 \oplus M_2$. Assume that $(x_1, \dots, x_n, \dots, x_m)$ is an extended cluster

and suppose that $x_i \in M_1$. If x_i is mutable, then its mutation $x'_i \in M_1$. Otherwise, M has two bases $\{x_1, \dots, x_i, \dots, x_n\}$ and $\{x_1, \dots, x'_i, \dots, x_n\}$ such that the first has elements from M_1 more than the second, a contradiction. Similarly, if x_j is a mutable variable living in M_2 , then its mutation $x'_j \in M_2$. Now, assume without loss of generality that there exists a number r such that $x_1, \dots, x_r \in M_1$ and $x_{r+1}, \dots, x_n \in M_2$. Then, at the first level, any mutation at mutable indices of the first r spots induces a variable in M_1 . Likewise, any mutation at the rest spots induces a variable in M_2 . At the second level, a second mutation at $k \in [1, r]$ produces a variable in M_1 , no matter if the first mutation was at $[1, r]$ or $[r+1, m]$. Similarly, a second mutation at $k \in [r+1, m]$ induces a variable in M_2 . This continues to any level of mutations. Note that x_1 and x_{r+1} must not live in a same circuit; otherwise, there is a connected component containing both of them, which means that they must be both in M_1 or both in M_2 . However, $\{x_1, x'_1, x_2, \dots, x_r, x_{r+1}, \dots, x_n\}$ is dependent and has no dependent subset. Hence, it is a circuit, a contradiction. Since the mutation at an index produces a variable in the same original connected component, the previous argument can be generalized for any two variables by comparing their mutations at some certain level. \square

In Example 4.5, we have seen a connection between cluster matroids and uniform matroids. This guides us to the following theorem:

Theorem 4.9. *Let M be a cluster matroid whose seeds have no frozen variables and whose off-diagonal entries are nonzero. Then M is equal to $U_{n,|M|}$, where n is the size of the initial seed and $|M|$ is the ground set cardinality.*

Proof. First, note that any n -subset of the ground set is algebraically independent. It is not hard to see this, since the initial seed contains n algebraically independent variables and the mutation formula

$$x_k x'_k = \prod_{b_{ik} > 0} x_i^{b_{ik}} + \prod_{b_{ik} < 0} x_i^{-b_{ik}}$$

implies that the variable x'_k preserves the appearance of all the variables at the initial seed, no more no less.

Second, any $(n+1)$ -subset of the ground set is algebraically dependent. Indeed, since all cluster variables lie in the rational function field

$\mathbb{C}(x_1, \dots, x_n)$, whose transcendence degree over \mathbb{C} is n , no subset of mutable variables of size greater than n can be algebraically independent. Thus, any $(n + 1)$ -subset of cluster variables must be algebraically dependent.

Clearly, the combination of the results of the two paragraphs above implies that $M = U_{n, |M|}$. \square

We have seen that cluster algebras induce matroids via the property of algebraic independence. In fact, we can get another type of matroid from cluster algebras. This comes from the *Laurent phenomenon*, which is one of the most powerful phenomena of cluster algebras:

Theorem 4.10 (Laurent phenomenon). *Let \mathcal{A} be a cluster algebra. Any cluster variable of \mathcal{A} can be expressed as a Laurent polynomial in the variables of any extended cluster with integer coefficients. Moreover, the frozen variables do not appear in the denominator of any such Laurent polynomial.*

Corollary 4.11. *The cluster monomials of any cluster algebra are linearly independent over the ground field (\mathbb{Q} or \mathbb{C}).*

Corollary 4.12. *Cluster monomials of a cluster algebra of finite type form a representable matroid.*

5. Minors

To better understand the class of cluster matroids, it is important to see how it behaves after contraction. Since the cluster algebras of finite type are classified by Dynkin diagrams (or Cartan matrices), we choose one example of type A_2 to build some results from. Clearly, the mutable and frozen variables are the main ingredients to determine the seed pattern of a cluster algebra. Thus, we look closely to the effect of each of them.

We first make the following note regarding the contraction by frozen variables in a cluster matroid.

Theorem 5.1. *The contraction of a frozen variable from a cluster matroid is a cluster matroid.*

Proof. Assume that M is a cluster matroid and e is a frozen variable. We have the following:

$$\mathcal{B}(M/e) = \left\{ X \subset E \setminus e : M|e \text{ has a basis } B \text{ such that } X \cup B \in \mathcal{B}(M) \right\}$$

$$\begin{aligned}
&= \left\{ X \subset E \setminus e : X \cup e \in \mathcal{B}(M) \right\} \\
&= \left\{ B \setminus e : B \in \mathcal{B}(M) \right\},
\end{aligned}$$

where the last equality holds because of the facts that e is frozen and frozen variables appear in every cluster-matroid basis. Note that this is the same cluster matroid obtained by removing the frozen variable e from its original seed together with its corresponding row from the exchange matrix B . \square

Corollary 5.2. *If M is a cluster matroid and X is a set of frozen variables, then M/X is a cluster matroid.*

Remark 5.3. Based on the definition of a cluster matroid, it is straightforward to see that the deletion of a frozen variable results in the same as the contraction of it. Therefore, the deletion of a frozen variable from a cluster matroid is a cluster matroid as well.

Let us recall this theorem for the classification of type A_n :

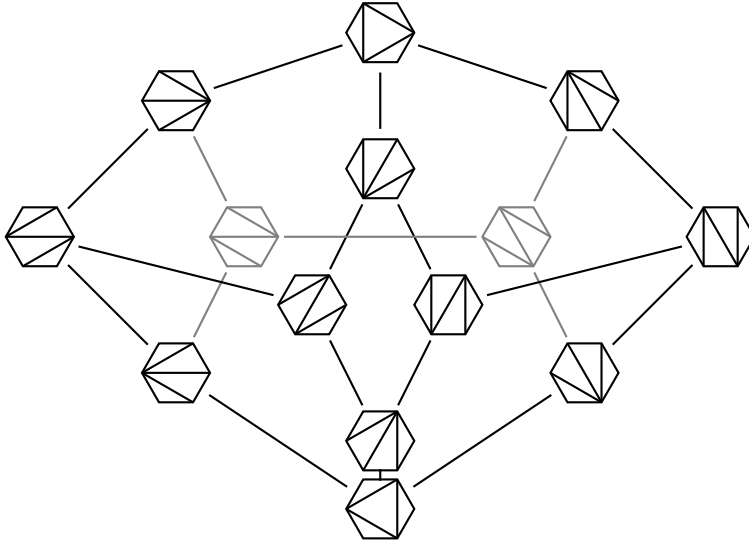
Theorem 5.4. *Let \mathcal{A} be a cluster algebra of type A_n . Then, its clusters can be labeled by the diagonals of a convex $(n+3)$ -gon so that*

- *clusters correspond to triangulations of the gon P_{n+3} by noncrossing diagonals,*
- *mutations correspond to flips, and*
- *exchange matrices are given such that their counterparts are Cartan matrices of type A_n .*

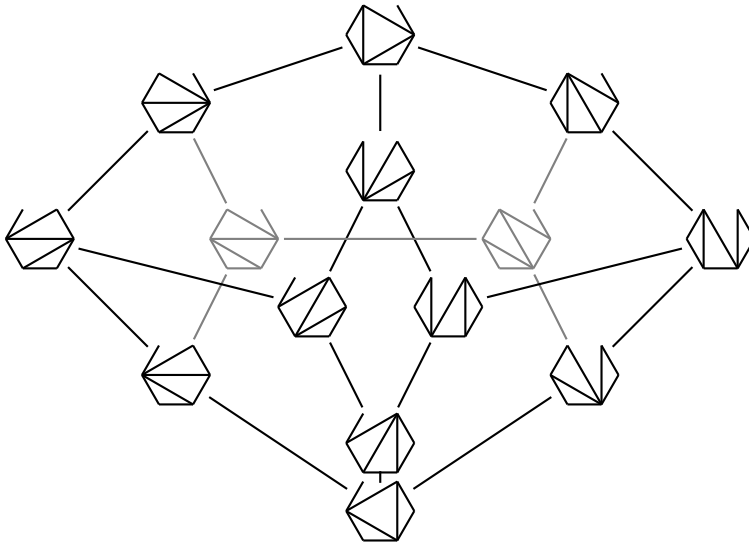
Cluster variables labeled by different diagonals are distinct, so there are altogether $\frac{n(n+3)}{2}$ cluster variables and $\frac{1}{n+2} \binom{2n+2}{n+1}$ seeds.

We use this theorem now to develop the following example:

Example 5.5. Let us focus on a cluster matroid of type A_2 . The set of extended clusters can be described combinatorially using the following graph, where the diagonals represent mutable variables and the sides represent the frozen ones:



It is easily seen that the contraction (or deletion) of a frozen variable, for example the top side of each hexagon, will produce bases that can be described using this graph, where again the sides are frozen and diagonals are mutable:



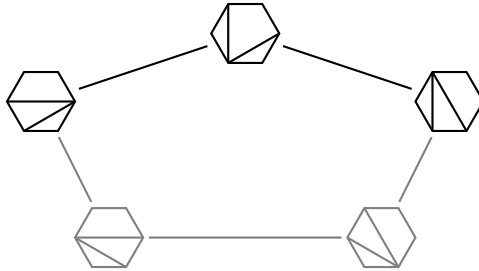
Example 5.6. Now, using the previous example, we investigate what happens if we contract a mutable variable. Let us start by mutating the variable represented by the dotted diagonal below:



Note that if e is a mutable variable, then we have

$$\begin{aligned}\mathcal{B}(M/e) &= \left\{ X \subset E \setminus e : M|e \text{ has a basis } B \text{ such that } X \cup B \in \mathcal{B}(M) \right\} \\ &= \left\{ X \subset E \setminus e : X \cup e \in \mathcal{B}(M) \right\} \\ &= \left\{ B \setminus e : B \in \mathcal{B}(M) \text{ and } e \in B \right\}.\end{aligned}$$

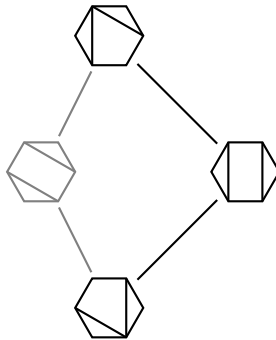
This will easily imply that the set generating the bases of the new matroid is the one that appears in the following graph:



Next, we mutate the variable represented by the dotted diagonal below:



Using the same logic, it is not hard to see that the graph representing the set that generates the bases is:



Interestingly, one can see here that the mutation of different mutable variables can give two different matroids. This is true because the size in the graph after the first mutation is 5, while it is 4 in the second case.

Remark 5.7. Note that the cluster matroids can be described combinatorially in a nice way, and this description can still be used to describe their minors. However, one can easily see that the class of cluster matroids is not closed under minors. One evidence is the difference of the numbers that we got after contracting two different mutable variables in the previous example.

6. Result summary and remarks

Remark 6.1. The study of cluster matroids reveals several key structural properties. While n -gon triangulations do not form a set of matroid bases, this limitation is resolved by leveraging the algebraic independence of extended clusters, defining a structure called a cluster matroid. Such a matroid is connected precisely when only mutable variables are present. Removing frozen variables, together with making the off-diagonal entries nonzero, results in a uniform matroid, and the Laurent phenomenon enables the construction of a representable matroid from cluster algebra seeds. Notably, deleting or contracting any set of frozen variables always yields another cluster matroid. However, contracting mutable variables does not guarantee analogous consistency, as it may produce distinct matroids. These results highlight the nuanced interplay between frozen and mutable variables in shaping the matroidal framework of cluster structures. The following table summarizes key characteristics and results of this paper.

7. Further destination

This paper is a starting point for a connection between matroids and cluster algebras that can be interestingly deepened. We believe that this can serve as a framework for a new shining path in an interesting area of mathematics that has the potential to grow significantly if it receives proper care from interested mathematicians. It would be very nice to see both theories' simultaneous development in algebraic and combinatorial ways. The work of this paper has the flavor of what matroid theorists usually do. Of course, much deeper work can be built on this, like studying the minors, duals, representability, etc. Other interesting questions

| Question/Property | Answer/Result |
|--|--|
| Do n -gon triangulations form matroid bases? | No |
| Can this issue be resolved? | Yes, define a <i>cluster matroid</i> via algebraic independence |
| Connectedness condition | Connected \iff presence of mutable variables only |
| Effect of removing frozen variables | Yields a uniform matroid if the off-diagonal entries are nonzero |
| Deletion/contraction of frozen variables | Always produces another cluster matroid |
| Contraction of mutable variables | Not guaranteed; may generate distinct matroids |
| What about representable matroids? | They can be produced using the Laurent phenomenon |

Table 1: Properties of cluster matroids

arise if we make changes to cluster algebras that build cluster matroids. For example, one might wonder what happens if the ground field of the cluster algebra is changed or if one of its exchange matrix entries is changed. Moreover, the classification of this new class of matroid can be studied and help in answering questions regarding algebraic matroids, which have many unsolved problems yet.

It is worth mentioning that this is not the first work that relates both matroid theory and cluster algebras together; there is another interesting path that has a more cluster-algebra flavor. For instance, the reader is referred to the work of Karp and Williams [7] that is followed by a sequence of other significant papers of the same direction.

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