

Variable lemma for polynomial rings

Daria Holik* and Marek Karaś

Communicated by I. Shestakov

ABSTRACT. Let k be a field of characteristic zero, and let $k[x_1, \dots, x_n]$ be a polynomial ring in n variables, where $n \geq 3$ is an arbitrary positive integer. Assume that we have $l \in \mathbb{N}$ algebraically independent polynomials $F_1, \dots, F_l \in k[x_1, \dots, x_n]$ with $2 \leq l < n$. In this paper, we prove that if linear parts of polynomials F_1, \dots, F_l are linearly independent and depend only on variables x_1, \dots, x_l and the polynomials F_1, \dots, F_l meet some weighted-differential criteria then actually $F_1, \dots, F_l \in k[x_1, \dots, x_l]$.

Introduction

In the landmarking paper [8] Shestakov and Umirbaev proved, in particular, that the following polynomial mapping, named Nagata automorphism

$$\begin{aligned} \sigma : \mathbb{C}^3 \ni (x, y, z) \mapsto & (x + 2y(y^2 + xz) - z(y^2 + xz)^2, \\ & y - z(y^2 + xz), z) \in \mathbb{C}^3. \end{aligned} \tag{1}$$

is so-called *wild* (polynomial) automorphism, that is, it is not a composition of linear/affine automorphisms and triangular automorphisms. Let us recall that a polynomial automorphism F is called *triangular* if (after

*Corresponding author.

2020 Mathematics Subject Classification: *Primary:* 13N05, 14R10. *Secondary:* 13B25, 13N15, 14L40.

Key words and phrases: *polynomial automorphism, tame (wild) automorphism, (weighted) multidegree, general algebraic group.*

a possible permutation of variables and/or possible permutation of coordinate functions F_1, \dots, F_n of F) the functions F_i have the following form $F_1 = x_1 + a$ with $a \in k$, $F_2 = x_2 + P_2$ with $P_2 \in k[x_1], \dots, F_i = x_i + P_i$ with $P_i \in k[x_1, \dots, x_{i-1}], \dots, F_n = x_n + P_n$ with $P_n \in k[x_1, \dots, x_{n-1}]$.

The motivation of the research presented in this paper is the application of the following so-called variable lemma (giving relationship between the degree of the Poisson bracket and the number of variables)

Lemma 1 ([5, Lemma 3.20]). *Let $f, g \in k[x_1, \dots, x_n]$ be such that*

$$f = x_1 + f_2 + \dots + f_r, \quad g = x_2 + g_2 + \dots + g_s,$$

where f_i, g_i are homogeneous forms of degree i . If $\deg[f, g] = 2$ then $f, g \in k[x_1, x_2]$, where

$$\deg[f, g] = 2 + \max_{1 \leq i < j \leq n} \deg \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right)$$

in the studying the following conjecture concerning the existence of tame automorphism with given degrees of its components.

Conjecture 1 ([3, Conj. 5.1]). *For any prime number $p \geq 2$ and $d_3 \geq d_2 \geq p$ there exists a tame automorphism of k^3 with multidegree equal to (p, d_2, d_3) if and only if $p|d_2$ or $d_3 \in p\mathbb{N} + d_2\mathbb{N}$.*

The conjecture, up to now, is proved for $p = 2, 3, 5$ (see [4, 5]). For $p = 5$ the most difficult part was to show that there is no tame automorphism $F : k^3 \rightarrow k^3$ such that $\text{mdeg } F = (5, 6, 9)$, where $\text{mdeg } F = (\deg F_1, \dots, \deg F_m)$ if $F = (F_1, \dots, F_m) : k^n \rightarrow k^m$ is a polynomial mapping and in proving this, Lemma 1 was used (for details see [5]).

The application of the weighted multidegrees in studying tameness of polynomial automorphisms can be found in [1]. For example, among other things, in [1] it was established that for $\mathbf{w} = (5, 4, 3)$ the \mathbf{w} -multidegree of the Nagata automorphism, $\text{mdeg}_{\mathbf{w}} \sigma = (19, 11, 3)$ is the so-called wild \mathbf{w} -multidegree, i.e. there is no tame automorphism with the same \mathbf{w} -multidegree.

In the present paper we establish a collection of generalizations of the above mentioned variable lemma (for more than two polynomials and for weighted degrees; see Section 2). The authors believe that the results of Section 2 can be used in the future in studying tameness in dimensions higher than three.

It should be emphasized that the question about existence of wild automorphism of k^n for $n \geq 4$ is still open. The situation is even worse:

there is no tools for researching tameness/wildness in dimension $n \geq 4$. Perhaps as one of such tools can be regarded the result of Kuroda [6] and as the second one the variable lemma presented in this paper (Theorem 1 below).

1. Differential forms and their degrees

In this section, for the convenience of the reader, we collect some necessary facts about differential forms. At the beginning we recall that for any polynomial $H \in k[x_1, \dots, x_n]$, i.e. $H = \sum_{\alpha=(\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\deg H$ denotes the usual total degree of H that is

$$\deg H := \max\{\alpha_1 + \dots + \alpha_n \mid a_\alpha \neq 0\}$$

or $\deg H := -\infty$ when $H = 0$. We remind that a homogeneous polynomial is a polynomial whose nonzero terms all have the same degree.

Let $A = k[x_1, \dots, x_n]$ be a polynomial ring and $\Omega_{A/k} = A \cdot \{dH \mid H \in k[x_1, \dots, x_n]\}$ an A -module of differential forms of $k[x_1, \dots, x_n]$ over k (for definition and more information about differential forms we refer to [7, Chapter 5.3] or [2, Chapter 2.8]). One can write $\Omega_{A/k} = \bigoplus_{i=1}^n A dx_i$, and then $\bigwedge^p \Omega_{A/k} = \bigoplus_{1 \leq i_1 < \dots < i_p \leq n} A dx_{i_1} \wedge \dots \wedge dx_{i_p}$, where the symbol

$\bigwedge^p \Omega_{A/k}$ denotes the p -th exterior power of the A -module $\Omega_{A/k}$. Arbitrary differential form $\omega \in \bigwedge^p \Omega_{A/k}$, which is said to be of order p , one can uniquely express as

$$\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} \varphi_{i_1, \dots, i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p},$$

where $\varphi_{i_1, \dots, i_p} \in A$ for all $1 \leq i_1 < \dots < i_p \leq n$.

We define standard total degree of a differential form $\omega \in \bigwedge^p \Omega_{A/k}$ by

$$\deg \omega := \max_{1 \leq i_1 < \dots < i_p \leq n} \{\deg \varphi_{i_1, \dots, i_p} + p\}.$$

Using the fact that the field k is of characteristic zero and $dH = \sum_{i=1}^n \frac{\partial H}{\partial x_i} dx_i$, we get the following equality

$$\deg dH = \max_{i=1, \dots, n} \left\{ \deg \frac{\partial H}{\partial x_i} + 1 \right\} = \deg H$$

for each $H \in k[x_1, \dots, x_n] \setminus k$.

It is natural to say that a differential form $\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} \varphi_{i_1, \dots, i_p} \cdot dx_{i_1} \wedge \dots \wedge dx_{i_p}$ is homogeneous of degree d if for all $1 \leq i_1 < \dots < i_p \leq n$ with $\varphi_{i_1, \dots, i_p} \neq 0$, the coefficient $\varphi_{i_1, \dots, i_p}$ is a homogeneous polynomial such that $\deg \varphi_{i_1, \dots, i_p} + p$ is equal to d .

Now we can formulate the following

Lemma 2. *If $H_1, \dots, H_p \in k[x_1, \dots, x_n]$ are nonzero, homogeneous polynomials in n variables x_1, \dots, x_n , then the differential form $dH_1 \wedge \dots \wedge dH_p$ is homogeneous of degree*

$$\begin{aligned} \deg(dH_1 \wedge \dots \wedge dH_p) &= \begin{cases} \deg H_1 + \dots + \deg H_p & \text{if } dH_1 \wedge \dots \wedge dH_p \neq 0, \\ -\infty & \text{if } dH_1 \wedge \dots \wedge dH_p = 0. \end{cases} \end{aligned} \tag{2}$$

The proof of the above lemma is a consequence of the definition of the degree of a differential form and the straightforward calculations. We omit it, since below we give a generalization of this lemma with a detailed proof (see Lemma 3).

Let Γ be any totally ordered additive group, and $\mathbf{w} = (w_1, \dots, w_n) \in \Gamma^n$ with $w_i \geq 0$, $i = 1, \dots, n$. The polynomial ring $k[x_1, \dots, x_n]$ in n variables over a field k of characteristic zero with $n \in \mathbb{N}$ can be equipped with the \mathbf{w} -weighted grading structure as follows $k[x_1, \dots, x_n] = \bigoplus_{\gamma \in \Gamma} k[x_1, \dots, x_n]_\gamma$, where $k[x_1, \dots, x_n]_\gamma$, for any $\gamma \in \Gamma$, denotes k -vector space spanned by $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\alpha_1, \dots, \alpha_n \in \mathbb{N}$ with $\sum_{i=1}^n \alpha_i w_i = \gamma$.

Now we can define \mathbf{w} -degree of the polynomial $H = \sum_{\gamma \in \Gamma} H_\gamma$, where for any $\gamma \in \Gamma$, $H_\gamma \in k[x_1, \dots, x_n]_\gamma$ is the homogeneous component of degree γ of the polynomial H , in the following way

$$\deg_{\mathbf{w}} H = \max\{\gamma \in \Gamma \mid H_\gamma \neq 0\}$$

or $\deg_{\mathbf{w}} H = -\infty$ while $H = 0$. It should be mentioned that if $\Gamma = \mathbb{Z}$ and $\mathbf{w} = (1, \dots, 1)$, then \mathbf{w} -degree is equal to the usual total degree. It is easy to see that for $H = \sum_{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and any $\gamma \in \Gamma$,

$$\text{we have } H_\gamma = \sum_{\alpha_1 w_1 + \dots + \alpha_n w_n = \gamma} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$

If for some $\gamma \in \Gamma$ the following equality holds $H = H_\gamma$, then H is called **w-homogeneous**. Analogously to the usual total degree of the differential form $\omega \in \bigwedge^p \Omega_{A/k}$, we define the **w-degree** of ω as follows

$$\text{deg}_{\mathbf{w}} \omega := \max_{1 \leq i_1 < \dots < i_p \leq n} \{ \text{deg}_{\mathbf{w}} \varphi_{i_1, \dots, i_p} + w_{i_1} + \dots + w_{i_p} \}.$$

As in the case of polynomials, if $\Gamma = \mathbb{Z}$ and $\mathbf{w} = (1, \dots, 1)$ then the **w-degree** of differential form is equal to its usual total degree, i.e $\text{deg}_{\mathbf{w}} \omega = \text{deg} \omega$ for all $\omega \in \bigwedge^p \Omega_{A/k}$. It is natural to say that a differential form $\omega = \sum_{1 \leq i_1 < \dots < i_p \leq n} \varphi_{i_1, \dots, i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p}$ is **w-homogeneous** of **w-degree** γ if for every $1 \leq i_1 < \dots < i_p \leq n$ with $\varphi_{i_1, \dots, i_p} \neq 0$, the polynomial $\varphi_{i_1, \dots, i_p}$ is **w-homogeneous** and $\text{deg}_{\mathbf{w}} \varphi_{i_1, \dots, i_p} + w_{i_1} + \dots + w_{i_p} = \gamma$.

The following lemma will be used in the proofs of the main results, in the next section.

Lemma 3. *Let $\mathbf{w} = (w_1, \dots, w_n) \in \Gamma^n$ be any n -tuple of elements of the group Γ with $w_i \geq 0$ for $i = 1, \dots, n$. If $H_1, \dots, H_p \in k[x_1, \dots, x_n]$ are nonzero, **w-homogeneous** polynomials in n variables x_1, \dots, x_n , then the differential form $dH_1 \wedge \dots \wedge dH_p$ is **w-homogeneous** of **w-degree***

$$\begin{aligned} \text{deg}_{\mathbf{w}}(dH_1 \wedge \dots \wedge dH_p) &= \begin{cases} \text{deg}_{\mathbf{w}} H_1 + \dots + \text{deg}_{\mathbf{w}} H_p & \text{if } dH_1 \wedge \dots \wedge dH_p \neq 0, \\ -\infty & \text{if } dH_1 \wedge \dots \wedge dH_p = 0. \end{cases} \end{aligned} \tag{3}$$

Proof. Since, for any $i \in \{1, \dots, p\}$, we have $dH_i = \sum_{j=1}^n \frac{\partial H_i}{\partial x_j} dx_j$, it follows from the multilinearity of the wedge product that

$$\begin{aligned} dH_1 \wedge \dots \wedge dH_p &= \left(\sum_{j_1=1}^n \frac{\partial H_1}{\partial x_{j_1}} dx_{j_1} \right) \wedge \dots \wedge \left(\sum_{j_p=1}^n \frac{\partial H_p}{\partial x_{j_p}} dx_{j_p} \right) \\ &= \sum_{j_1=1}^n \dots \sum_{j_p=1}^n \frac{\partial H_1}{\partial x_{j_1}} \dots \frac{\partial H_p}{\partial x_{j_p}} dx_{j_1} \wedge \dots \wedge dx_{j_p}. \end{aligned} \tag{4}$$

Because the wedge product is antisymmetric, the above sum can be written as

$$\begin{aligned} dH_1 \wedge \dots \wedge dH_p &= \sum_{1 \leq j_1 < \dots < j_p \leq n} \left(\sum_{\sigma \in S_p} \text{sgn} \sigma \frac{\partial H_1}{\partial x_{j_{\sigma(1)}}} \dots \frac{\partial H_p}{\partial x_{j_{\sigma(p)}}} \right) dx_{j_1} \wedge \dots \wedge dx_{j_p}, \end{aligned} \tag{5}$$

where S_p denotes the symmetric group of the set $\{1, \dots, p\}$ and the $\text{sgn } \sigma$ is the sign of the permutation σ . Thus, by the definition of \mathbf{w} -degree of a differential form and by (5), we have

$$\begin{aligned} & \text{deg}_{\mathbf{w}}(dH_1 \wedge \dots \wedge dH_p) \\ &= \max_{1 \leq j_1 < \dots < j_p \leq n} \left\{ \text{deg}_{\mathbf{w}} \left(\sum_{\sigma \in S_p} \text{sgn } \sigma \frac{\partial H_1}{\partial x_{j_{\sigma(1)}}} \dots \frac{\partial H_p}{\partial x_{j_{\sigma(p)}}} \right) + w_{j_1} + \dots + w_{j_p} \right\}. \end{aligned} \tag{6}$$

Using the fact that k is of characteristic zero, we obtain that the polynomial $\frac{\partial H_i}{\partial x_{j_{\sigma(m)}}}$ is \mathbf{w} -homogeneous and $\text{deg}_{\mathbf{w}} \frac{\partial H_i}{\partial x_{j_{\sigma(m)}}} = \text{deg}_{\mathbf{w}} H_i - w_{j_{\sigma(m)}}$ iff $\frac{\partial H_i}{\partial x_{j_{\sigma(m)}}} \neq 0$, for any $m = 1, \dots, p$, any $\sigma \in S_p$, and any $1 \leq j_1 < \dots < j_p \leq n$. Consequently, any summand of the expression $\sum_{\sigma \in S_p} \text{sgn } \sigma \frac{\partial H_1}{\partial x_{j_{\sigma(1)}}} \dots \frac{\partial H_p}{\partial x_{j_{\sigma(p)}}}$ is a \mathbf{w} -homogeneous polynomial which has \mathbf{w} -degree equal to $\text{deg}_{\mathbf{w}} H_1 - w_{j_{\sigma(1)}} + \dots + \text{deg}_{\mathbf{w}} H_p - w_{j_{\sigma(p)}}$ or it is equal to zero. Taking into consideration that σ is bijective function, the above \mathbf{w} -degree we can write as follows $\text{deg}_{\mathbf{w}} H_1 + \dots + \text{deg}_{\mathbf{w}} H_p - w_{j_1} - \dots - w_{j_p}$. Finally, for any $1 \leq j_1 < \dots < j_p \leq n$, the polynomial $\sum_{\sigma \in S_p} \text{sgn } \sigma \frac{\partial H_1}{\partial x_{j_{\sigma(1)}}} \dots \frac{\partial H_p}{\partial x_{j_{\sigma(p)}}}$ is \mathbf{w} -homogeneous, and its \mathbf{w} -degree equals $\text{deg}_{\mathbf{w}} H_1 + \dots + \text{deg}_{\mathbf{w}} H_p - w_{j_1} - \dots - w_{j_p}$. From the above calculations and (6) we obtain the assertion. \square

2. Main results – variable lemmas

In this section we present main results of the paper, that is Theorems 1 and 2. Both of them can be named variable lemma because they assert that some polynomials that a priori depend on the whole set of variables x_1, \dots, x_n under some assumptions actually appear to be dependent only on some proper subset of variables. One can also be interested in Corollaries 1 and 2 which can be considered, respectively, as a generalization of Theorems 2 and 1.

Before we give the results let us formulate the following two conditions:

- (A) Γ is an ordered additive group such that every finitely generated subgroup is embedded in a subgroup of Γ which is isomorphic to $(\mathbb{Z}, +)$, in the category of ordered groups.

(B) Γ is an ordered additive group and $\mathbf{w} = (w_1, \dots, w_n) \in \Gamma^n$ is such that the subgroup $\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n \subset \Gamma$ is embedded in a subgroup of Γ which is isomorphic to $(\mathbb{Z}, +)$, in the category of ordered groups.

One can easily see that condition (A) implies (B), and moreover $\Gamma = \mathbb{Z}$ satisfies condition (A). In the next section we show example of totally ordered group which does not satisfy condition (A) as well an example of Γ and $\mathbf{w} = (w_1, \dots, w_n) \in \Gamma^n$ such that the condition (B) is not satisfied. The example of group which is not isomorphic to $(\mathbb{Z}, +)$ but satisfies the condition (A) is also presented there. As well we present, in Section 3, examples showing that assumptions of the following theorem are essential.

Theorem 1. Assume that Γ is a totally ordered additive group and that $\mathbf{w} = (w_1, \dots, w_n) \in \Gamma^n$ is such that $w_i > 0$ for $i = 1, \dots, n$, and

$$\max\{w_1, \dots, w_l\} \leq \min\{w_{l+1}, \dots, w_n\}. \tag{7}$$

Assume also that at least one of the conditions (A) or (B) is satisfied (in particular, this is the case for $\Gamma = \mathbb{Z}$). Let $F_1, \dots, F_l \in k[x_1, \dots, x_n]$, where $2 \leq l < n$, be polynomials in n variables x_1, \dots, x_n such that their linear parts are linearly independent and depend only on variables x_1, \dots, x_l . If the \mathbf{w} -degree of the differential form $dF_1 \wedge \dots \wedge dF_l$ is the smallest possible, i.e.

$$\deg_{\mathbf{w}}(dF_1 \wedge \dots \wedge dF_l) = w_1 + \dots + w_l, \tag{8}$$

then we have $F_1, \dots, F_l \in k[x_1, \dots, x_l] \subsetneq k[x_1, \dots, x_n]$.

Let us notice that the minimality of $\deg_{\mathbf{w}}(dF_1 \wedge \dots \wedge dF_l)$, in the context of Theorem 1, is equivalent to equality (8). Indeed, since $\bigwedge^l \Omega_{A/k} = \bigoplus_{1 \leq j_1 < \dots < j_l \leq n} Adx_{j_1} \wedge \dots \wedge dx_{j_l}$, it follows that for any nonzero $\omega \in \bigwedge^l \Omega_{A/k}$, we have

$$\deg_{\mathbf{w}} \omega \geq \min\{w_{j_1} + \dots + w_{j_l} \mid 1 \leq j_1 < \dots < j_l \leq n\}.$$

When inequality (7) is satisfied, then we have

$$\min\{w_{j_1} + \dots + w_{j_l} \mid 1 \leq j_1 < \dots < j_l \leq n\} = w_1 + \dots + w_l,$$

and in the situation like in Corollary 2 we have

$$\begin{aligned} & \min\{w_{j_1} + \dots + w_{j_l} \mid 1 \leq j_1 < \dots < j_l \leq n\} \\ &= \min\{w_{j_1} + \dots + w_{j_l} \mid 1 \leq j_1 < \dots < j_l \leq r\}. \end{aligned}$$

The proof of the above theorem will be given at the end of this section, but now we present following particular/non-weighted version of it, i.e. when $\Gamma = \mathbb{Z}$ and $\mathbf{w} = (1, \dots, 1)$. One can notice that if $(w_1, \dots, w_n) = (1, \dots, 1)$, then the assumption (7) is obviously satisfied.

Theorem 2. *Let $F_1, \dots, F_l \in k[x_1, \dots, x_n]$, where $2 \leq l < n$, be polynomials in n variables x_1, \dots, x_n such that their linear parts are linearly independent and depend only on variables x_1, \dots, x_l .*

If the degree of the differential form $dF_1 \wedge \dots \wedge dF_l$ is the smallest possible, i.e.

$$\deg(dF_1 \wedge \dots \wedge dF_l) = l, \tag{9}$$

then the polynomials F_1, \dots, F_l involve only variables x_1, \dots, x_l . In other words $F_1, \dots, F_l \in k[x_1, \dots, x_l] \subsetneq k[x_1, \dots, x_n]$.

Before we give the proof of the above theorem let us explain why we present it, besides of the fact that it is a particular version of Theorem 1. The reason is the following. The proof of Theorem 1 is based of Proposition 1, and the proof of Proposition 1 is a modification of the proof of Theorem 2. Moreover, the situation of Theorem 2 makes that the idea of the proof is much easier to present.

Proof. First of all we show that, without loss of generality, we can assume that the polynomials F_1, \dots, F_l have the form

$$F_i = x_i + F_{i,2} + \dots + F_{i,d_i} \quad \text{for } i = 1, \dots, l, \tag{10}$$

where $F_{i,j} \in k[x_1, \dots, x_n]$ is a homogeneous component of degree j of the polynomial F_i . Of course, one can notice that we can assume that the constant terms of the polynomials F_1, \dots, F_l are zeros. Let us denote the linear parts of F_i by $L_i \in k[x_1, \dots, x_l]$ for $i = 1, \dots, l$, and consider the mapping L^{-1} , where $L = (L_1, \dots, L_l) : k^l \rightarrow k^l$. Now, by $h_i = a_{i,1}x_1 + \dots + a_{i,l}x_l$, for $i = 1, \dots, l$, we denote the coordinate functions of L^{-1} . Then, $\det(a_{i,j}) \neq 0$ and, as one can calculate using multilinearity and antisymmetry of the wedge product, that

$$dh_1(F_1, \dots, F_l) \wedge \dots \wedge dh_l(F_1, \dots, F_l) = \det(a_{i,j})dF_1 \wedge \dots \wedge dF_l. \tag{11}$$

This gives that the degrees of the differential forms $dh_1(F_1, \dots, F_l) \wedge \dots \wedge dh_l(F_1, \dots, F_l)$ and $dF_1 \wedge \dots \wedge dF_l$ are the same. One can also notice that the polynomials $h_1(F_1, \dots, F_l), \dots, h_l(F_1, \dots, F_l)$ have the desired form and that these polynomials do not depend on variables x_{l+1}, \dots, x_n if

and only if the same holds for the polynomials F_1, \dots, F_l . Thus, up to the end of the proof we assume that (10) holds, in other words, we have $F_{i,1} = x_i$ and $F_{i,0} = 0$ for $i = 1, \dots, l$.

Now, from multilinearity of the wedge product and linearity of the operator d , we have

$$\begin{aligned}
 dF_1 \wedge \dots \wedge dF_l &= \left(\sum_{j_1=1}^{d_1} dF_{1,j_1} \right) \wedge \dots \wedge \left(\sum_{j_l=1}^{d_l} dF_{l,j_l} \right) \\
 &= \sum_{j_1=1}^{d_1} \dots \sum_{j_l=1}^{d_l} dF_{1,j_1} \wedge \dots \wedge dF_{l,j_l} \tag{12} \\
 &= \sum_{d=0}^{d_1+\dots+d_l} \sum_{j_1+\dots+j_l=d} dF_{1,j_1} \wedge \dots \wedge dF_{l,j_l}.
 \end{aligned}$$

Observe that by Lemma 2 (or Lemma 3 for the weighted case) we have $(dF_i)_j = d(F_{i,j})$. Thus we can omit brackets in formulas like above. By Lemma 2, one can see that for any $d = 0, \dots, d_1 + \dots + d_l$, the sum $\sum_{j_1+\dots+j_l=d} dF_{1,j_1} \wedge \dots \wedge dF_{l,j_l}$ is the homogeneous component of the degree d of the differential form $dF_1 \wedge \dots \wedge dF_l$. In particular, based on the observation from the beginning of the proof, we see that the homogeneous components of the differential form $dF_1 \wedge \dots \wedge dF_l$ of degree $d = 0, 1, \dots, l - 1$ are equal to zero and the homogeneous component of degree l is

$$(dF_1 \wedge \dots \wedge dF_l)_l = dx_1 \wedge \dots \wedge dx_l \neq 0. \tag{13}$$

Notice that

$$\begin{aligned}
 (dF_1 \wedge \dots \wedge dF_l)_{l+1} &= dF_{1,1} \wedge \dots \wedge dF_{l-1,1} \wedge dF_{l,2} \\
 &\quad + dF_{1,1} \wedge \dots \wedge dF_{l-2,1} \wedge dF_{l-1,2} \wedge dF_{l,1} \\
 &\quad \vdots \\
 &\quad + dF_{1,2} \wedge dF_{2,1} \wedge \dots \wedge dF_{l,1} \\
 &= dx_1 \wedge \dots \wedge dx_{l-1} \wedge \left(\sum_{j=1}^n \frac{\partial F_{l,2}}{\partial x_j} dx_j \right) \\
 &\quad + dx_1 \wedge \dots \wedge dx_{l-2} \wedge \left(\sum_{j=1}^n \frac{\partial F_{l-1,2}}{\partial x_j} dx_j \right) \wedge dx_l \\
 &\quad \vdots \\
 &\quad + \left(\sum_{j=1}^n \frac{\partial F_{1,2}}{\partial x_j} dx_j \right) \wedge dx_2 \wedge \dots \wedge dx_l.
 \end{aligned}$$

Now, consider arbitrary $j > l$ and notice that the only summand of the above sum involving $dx_1 \wedge \dots \wedge dx_{l-1} \wedge dx_j$ is the following one $\frac{\partial F_{i,2}}{\partial x_j} dx_1 \wedge \dots \wedge dx_{l-1} \wedge dx_j$. Since $(dF_1 \wedge \dots \wedge dF_l)_{l+1} = 0$, it follows that $\frac{\partial F_{i,2}}{\partial x_j} = 0$ for all $j > l$. In other words the polynomial $F_{i,2}$ does not involve variables x_{l+1}, \dots, x_n .

Similarly, one can notice that for $j > l$ and for any $i = 1, \dots, l-1$, the only summand involving $dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_l \wedge dx_j$ in the above sum is the following one $(-1)^{l-i} \frac{\partial F_{i,2}}{\partial x_j} dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_l \wedge dx_j$. For the similar reason as above this means that $\frac{\partial F_{i,2}}{\partial x_j} = 0$ and so that $F_{i,2}$ does not involve x_j for any $j > l$ and any $i = 1, \dots, l-1$. Thus, we have proved that the homogeneous components of degree 2 of polynomials F_1, \dots, F_l do not involve variables x_{l+1}, \dots, x_n .

Now, assume that we have proved that the homogeneous components of degree $d = 2, \dots, s$ of the polynomials F_1, \dots, F_l do not involve variables x_{l+1}, \dots, x_n . We will show that this is also true for homogeneous components of degree $s+1$. To do this consider the homogeneous component of degree $s+l$ of the differential form $dF_1 \wedge \dots \wedge dF_l$. One can observe that

$$(dF_1 \wedge \dots \wedge dF_l)_{l+s} = \sum_{\substack{j_1 + \dots + j_l = l+s \\ j_1, \dots, j_l \leq s}} dF_{1,j_1} \wedge \dots \wedge dF_{l,j_l} \quad (14)$$

$$+ \sum_{i=1}^l dF_{1,1} \wedge \dots \wedge dF_{i-1,1} \wedge dF_{i,s+1} \wedge dF_{i+1,1} \wedge \dots \wedge dF_{l,1}. \quad (15)$$

Take any $j > l$. By induction hypothesis, one can easily see that the sum (14) does not involve dx_j . On the other hand one can see that in the sum (15) the only summand involving $(dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_l) \wedge dx_j$ is the following one

$$(-1)^{l-i} \frac{\partial F_{i,s+1}}{\partial x_j} (dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_l) \wedge dx_j. \quad (16)$$

From this, as before, it follows that $F_{i,s+1}$ does not involve x_j for any $j > l$ and any $i = 1, \dots, l$. Now, by induction (with respect to s) we obtain the thesis. \square

The result of the above theorem can be generalized as follows.

Corollary 1. *Let $F_1, \dots, F_l \in k[x_1, \dots, x_n]$, where $2 \leq l < n$, be polynomials in n variables x_1, \dots, x_n such that their linear parts are linearly independent and depend only on variables x_1, \dots, x_r for some $l \leq r < n$.*

If the degree of the differential form $dF_1 \wedge \dots \wedge dF_l$ is the smallest possible, i.e.

$$\deg(dF_1 \wedge \dots \wedge dF_l) = l, \tag{17}$$

then the polynomials F_1, \dots, F_l involve only variables x_1, \dots, x_r , i.e. $F_1, \dots, F_l \in k[x_1, \dots, x_r] \subsetneq k[x_1, \dots, x_n]$.

Proof. Let us denote the linear parts of F_i by $L_i \in k[x_1, \dots, x_r]$ for $i = 1, \dots, l$. There are linear forms $L_{l+1}, \dots, L_r \in k[x_1, \dots, x_r]$ such that $L_1, \dots, L_l, L_{l+1}, \dots, L_r$ are linearly independent. Now, one can check that polynomials $F_1, \dots, F_l, L_{l+1}, \dots, L_r$ satisfy the assumptions of Theorem 2 with l replaced by r . Thus, by Theorem 2, we obtain the thesis. \square

Similar arguments as in the proof of Corollary 1, but based on Theorem 1, give the following result.

Corollary 2. *Assume that Γ is a totally ordered additive group and that $\mathbf{w} = (w_1, \dots, w_n) \in \Gamma^n$ is such that $w_i > 0$ for $i = 1, \dots, n$, and*

$$\max\{w_1, \dots, w_r\} \leq \min\{w_{r+1}, \dots, w_n\}. \tag{18}$$

Assume also that at least one of the conditions (A) or (B) is satisfied. Let $F_1, \dots, F_l \in k[x_1, \dots, x_n]$, where $2 \leq l < n$ be polynomials in n variables x_1, \dots, x_n such that their linear parts are linearly independent and depend only on variables x_1, \dots, x_r , where $l \leq r < n$. If the \mathbf{w} -degree of the differential form $dF_1 \wedge \dots \wedge dF_l$ is the smallest possible then we have $F_1, \dots, F_l \in k[x_1, \dots, x_r] \subsetneq k[x_1, \dots, x_n]$.

The following technical proposition gives a weighted version of Theorem 2 (with some additional assumptions), and will be used in the proof of Theorem 1. Because in Theorem 1 we make an assumption (7), which is absent in the proposition below, then the following result can be interesting on its own right.

Proposition 1. *Assume that $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n$ is such that $w_i > 0$ for $i = 1, \dots, n$. Let $F_1, \dots, F_l \in k[x_1, \dots, x_n]$, where $2 \leq l < n$ be polynomials in n variables x_1, \dots, x_n such that*

$$F_i = x_i + F_{i,w_i+1} + \dots + F_{i,d_i} \quad \text{for } i = 1, \dots, l, \tag{19}$$

where $F_{i,j} \in k[x_1, \dots, x_n]$ is a \mathbf{w} -homogeneous component of \mathbf{w} -degree j of the polynomial F_i . If the \mathbf{w} -degree of the differential form $dF_1 \wedge \dots \wedge dF_l$ is the smallest possible, i.e.

$$\deg_{\mathbf{w}}(dF_1 \wedge \dots \wedge dF_l) = w_1 + \dots + w_l, \tag{20}$$

then the polynomials F_1, \dots, F_l involve only variables x_1, \dots, x_l , i.e. $F_1, \dots, F_l \in k[x_1, \dots, x_l] \subsetneq k[x_1, \dots, x_n]$.

Proof. The proof is the appropriate modification and extension of the proof of Theorem 2. First, observe that by the assumption, we have

$$F_{i,0} = F_{i,1} = \dots = F_{i,w_i-1} = 0 \tag{21}$$

and

$$F_{i,w_i} = x_i \tag{22}$$

for all $i = 1, \dots, l$.

Secondly, similarly as in the proof of Theorem 2, we have

$$dF_1 \wedge \dots \wedge dF_l = \sum_{d=0}^{d_1+\dots+d_l} \sum_{j_1+\dots+j_l=d} dF_{1,j_1} \wedge \dots \wedge dF_{l,j_l}, \tag{23}$$

where, this time, $F_{i,j}$ means the \mathbf{w} -homogeneous component of \mathbf{w} -degree j of the polynomial F_i .

Now, by Lemma 3, we have that for any number $d \in \mathbb{N}$ the sum $\sum_{j_1+\dots+j_l=d} dF_{1,j_1} \wedge \dots \wedge dF_{l,j_l}$ is the \mathbf{w} -homogeneous component of \mathbf{w} -degree d of the differential form $dF_1 \wedge \dots \wedge dF_l$. Without loss of generality, we can assume that $w_1 \leq \dots \leq w_l$.

Let us take any $d \in \{0, \dots, w_1 + \dots + w_l - 1\}$. Since for any non-negative integers j_1, \dots, j_l with $j_1 + \dots + j_l = d$ at least one of them satisfies $j_i < w_i$ (for some $i \in \{1, \dots, l\}$), we see that the \mathbf{w} -homogeneous components of the \mathbf{w} -degree $d = 0, 1, \dots, w_1 + \dots + w_l - 1$, of the differential form $dF_1 \wedge \dots \wedge dF_l$, are equal to zero. One can also notice that the \mathbf{w} -homogeneous component of the \mathbf{w} -degree $w_1 + \dots + w_l$ of $dF_1 \wedge \dots \wedge dF_l$ is

$$(dF_1 \wedge \dots \wedge dF_l)_{w_1+\dots+w_l} = dx_1 \wedge \dots \wedge dx_l \neq 0. \tag{24}$$

To imitate the proof of Theorem 2, now we consider \mathbf{w} -homogeneous component of \mathbf{w} -degree $w_1 + \dots + w_l + 1$. One can notice that

$$\begin{aligned} (dF_1 \wedge \dots \wedge dF_l)_{w_1+\dots+w_l+1} &= dF_{1,w_1} \wedge \dots \wedge dF_{l-1,w_{l-1}} \wedge dF_{l,w_{l+1}} \\ &+ dF_{1,w_1} \wedge \dots \wedge dF_{l-2,w_{l-2}} \wedge dF_{l-1,w_{l-1}+1} \wedge dF_{l,w_l} \\ &\vdots \\ &+ dF_{1,w_1+1} \wedge dF_{2,w_2} \wedge \dots \wedge dF_{l,w_l}. \end{aligned}$$

Similar analysis, as in the proof of Theorem 2, of the summands involving $dx_1 \wedge \dots \wedge dx_{l-1} \wedge dx_j$, $dx_1 \wedge \dots \wedge dx_{l-2} \wedge dx_l \wedge dx_j$, \dots , $dx_2 \wedge \dots \wedge dx_l \wedge dx_j$ with $j > l$ that appear in the above sum gives us that for any $i = 1, \dots, l$ and for any $j > l$ the polynomial F_{i,w_i+1} does not involve x_j .

The induction step presented in the proof of Theorem 2 needs following modifications. Assume that we have proved that the \mathbf{w} -homogeneous components of \mathbf{w} -degree $d_i = w_i + 1, \dots, w_i + s$ with $s \geq 1$ and for all $i = 1, \dots, l$, do not involve x_j for all $j > l$. Similarly, as in the proof of Theorem 2, one can notice that $(dF_1 \wedge \dots \wedge dF_l)_{w_1+\dots+w_l+s+1}$ can be decomposed into two sums (compare (14) and (15)). One of them with components such that $j_1 \leq w_1 + s, \dots, j_l \leq w_l + s$. This sum does not involve dx_j for $j > l$ by induction hypothesis. The second sum have the following form (compare (15))

$$\sum_{i=1}^l dF_{1,w_1} \wedge \dots \wedge dF_{i-1,w_{i-1}} \wedge dF_{i,w_i+s+1} \wedge dF_{i+1,w_{i+1}} \wedge \dots \wedge dF_{l,w_l}. \tag{25}$$

By the similar arguments, as in the proof of Theorem 2, we obtain that for all $i = 1, \dots, l$ the polynomial F_{i,w_i+s+1} does not involve x_j for all $j > l$. This completes the induction step and the proof of the proposition. \square

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. Since Γ and $\mathbf{w} = (w_1, \dots, w_n) \in \Gamma^n$ satisfy condition (B), there is subgroup $\Gamma_0 \subset \Gamma$ isomorphic to $(\mathbb{Z}, +)$ (in the category of ordered groups) such that $\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n \subset \Gamma_0$. All considerations will be done inside Γ_0 , because any \mathbf{w} -degree (of polynomial or differential form) is an element of $\mathbb{N}w_1 + \dots + \mathbb{N}w_n \subset \mathbb{Z}w_1 + \dots + \mathbb{Z}w_n$. Thus without loss of generality we can assume $\Gamma = \mathbb{Z}$.

As in the proof of Theorem 2 we can assume that the polynomials F_1, \dots, F_l have the form given by (10), with respect to the usual degree. By inequality (7), permutating if necessary variables in the two sets $\{x_1, \dots, x_l\}$ and $\{x_{l+1}, \dots, x_n\}$ separately, we can assume, without loss of generality, that

$$w_1 \leq \dots \leq w_l \leq w_{l+1} \leq \dots \leq w_n. \tag{26}$$

Let

$$F_i = F_{i,1} + \dots + F_{i,w_i-1} + \left(x_i + \tilde{F}_{i,w_i}\right) + F_{i,w_i+1} + \dots + F_{i,d_i}, \tag{27}$$

for $i=1, \dots, l$, be the decomposition of the polynomial F_i into the \mathbf{w} -homogeneous components. In particular, $x_i + \tilde{F}_{i,w_i}$ is the \mathbf{w} -homogeneous component of \mathbf{w} -degree w_i of F_i , while \tilde{F}_{i,w_i} is the \mathbf{w} -homogeneous component of \mathbf{w} -degree w_i of $\tilde{F}_i = F_i - x_i$.

The idea of the proof is to construct an automorphism Φ of k^n of the form

$$\Phi = \tilde{\Phi} \times \text{id}_{k^{n-l}}, \quad \Phi(0) = 0, \tag{28}$$

where $\tilde{\Phi}$ is an automorphism of k^l , and such that the polynomials $\hat{F}_1, \dots, \hat{F}_l$ satisfy the assumptions of Proposition 1, where $(\hat{F}_1, \dots, \hat{F}_n) = \Phi \circ (F_1, \dots, F_l, x_{l+1}, \dots, x_n) : k^n \rightarrow k^n$. In this situation, of course, we have $\hat{F}_{l+1} = x_{l+1}, \dots, \hat{F}_n = x_n$. Then, by Proposition 1 one obtain that the polynomials $\hat{F}_1, \dots, \hat{F}_l$ do not involve variables x_{l+1}, \dots, x_n . Since $(F_1, \dots, F_l, x_{l+1}, \dots, x_n) = \Phi^{-1} \circ (\hat{F}_1, \dots, \hat{F}_l, x_{l+1}, \dots, x_n)$ and because of the form of the automorphism Φ , it follows that also F_1, \dots, F_l do not involve variables x_{l+1}, \dots, x_n . Thus, to the end of the proof we will construct a desired polynomial automorphism.

First, notice that F_1 already has the form as in the assumptions of Proposition 1. Indeed, we have $w_1 \leq w_2 \leq \dots \leq w_n$ and $w_i > 0$ for all i . Thus, for any monomial m of usual degree greater than 1, we have $\text{deg}_{\mathbf{w}} m \geq 2w_1 > w_1$, but only such monomials appear in the polynomial $F_1 - x_1$.

The construction of Φ will be done by induction. Assume that we have already constructed an automorphism $\Phi_u = \tilde{\Phi}_u \times \text{id}_{k^{n-u}}$, where $\tilde{\Phi}_u$ is an automorphism of k^u , for some $u \in \{1, \dots, l\}$, such that for $(\hat{F}_1^{(u)}, \dots, \hat{F}_n^{(u)}) = \Phi_u \circ (F_1, \dots, F_l, x_{l+1}, \dots, x_n)$, we have

$$\hat{F}_i^{(u)} = x_i + \text{higher } \mathbf{w}\text{-degree components} \quad \text{for } i = 1, \dots, u \tag{29}$$

and

$$d\hat{F}_1^{(u)} \wedge \dots \wedge d\hat{F}_l^{(u)} = dF_1 \wedge \dots \wedge dF_l. \tag{30}$$

Let us notice that for $u = 1$, we can take $\Phi_u = \text{id}_{k^n}$.

Now, let a_1 be the \mathbf{w} -homogeneous component of $\hat{F}_{u+1}^{(u)} - x_{u+1}$ of the minimal \mathbf{w} -degree. If $\text{deg}_{\mathbf{w}} a_1 > w_{u+1}$, then $\hat{F}_{u+1}^{(u)}$ already has the desired form, and so we can take $\Phi_{u+1} = \Phi_u$. Thus, without loss of generality, we can assume that $\text{deg}_{\mathbf{w}} a_1 \leq w_{u+1}$. Since all monomials appearing in $\hat{F}_{u+1}^{(u)} - x_{u+1}$ have usual degree greater than 1, it follows that $a_1 \in k[x_1, \dots, x_u]$. Moreover, one can easily notice that the \mathbf{w} -homogeneous component of $a_1(\hat{F}_1^{(u)}, \dots, \hat{F}_u^{(u)})$ of the minimal \mathbf{w} -degree is equal to a_1 . Now, let a_2 be the \mathbf{w} -homogeneous component of $\hat{F}_{u+1}^{(u)} - x_{u+1} - a_1(\hat{F}_1^{(u)}, \dots, \hat{F}_u^{(u)})$ of the minimal \mathbf{w} -degree. By the above considerations, we have $\text{deg}_{\mathbf{w}} a_2 > \text{deg}_{\mathbf{w}} a_1$. If $\text{deg}_{\mathbf{w}} a_2 \leq w_{u+1}$, then as before $a_2 \in k[x_1, \dots, x_u]$ and, also as before, the \mathbf{w} -homogeneous component of $a_2(\hat{F}_1^{(u)}, \dots, \hat{F}_u^{(u)})$ of the minimal \mathbf{w} -degree is equal to a_2 . In a similar way we construct \mathbf{w} -homogeneous polynomials a_1, \dots, a_s, a_{s+1} until $\text{deg}_{\mathbf{w}} a_s \leq w_{u+1}$ and $\text{deg}_{\mathbf{w}} a_{s+1} > w_{u+1}$. By the construction, we obtain that the polynomial $\hat{F}_{u+1}^{(u)} - \sum_{i=1}^s a_i(\hat{F}_1^{(u)}, \dots, \hat{F}_u^{(u)})$ has the desired form. Moreover, by the fact that wedge product is multilinear and antisymmetric, one can notice that

$$\begin{aligned} & d\hat{F}_1^{(u)} \wedge \dots \wedge d\hat{F}_u^{(u)} \\ & \wedge d\left(\hat{F}_{u+1}^{(u)} - \sum_{i=1}^s a_i(\hat{F}_1^{(u)}, \dots, \hat{F}_u^{(u)})\right) \wedge d\hat{F}_{u+2}^{(u)} \wedge \dots \wedge d\hat{F}_l^{(u)} \\ & = d\hat{F}_1^{(u)} \wedge \dots \wedge d\hat{F}_u^{(u)} \wedge d\hat{F}_{u+1}^{(u)} \wedge d\hat{F}_{u+2}^{(u)} \wedge \dots \wedge d\hat{F}_l^{(u)} \\ & = dF_1 \wedge \dots \wedge dF_l. \end{aligned} \tag{31}$$

Summarizing, we can take $\Phi_{u+1} = \varphi_{u+1} \circ \Phi_u$, where the automorphism φ_{u+1} is given by

$$\begin{aligned} \varphi_{u+1} : k^n \ni (x_1, \dots, x_n) \mapsto \\ (x_1, \dots, x_u, x_{u+1} - \sum_{i=1}^s a_i, x_{u+2}, \dots, x_n) \in k^n. \end{aligned} \tag{32}$$

This completes the proof of the induction step, and the existence of the desired automorphism $\Phi = \Phi_l$. □

3. Examples

In this section we show by giving examples, that not all totally ordered groups satisfy condition (A) and (B), as well as that assumptions of Theorem 1 are essential. We start with the following lemma.

Lemma 4. *Assume that Γ is a totally ordered additive group and that $(w_1, \dots, w_n) \in \Gamma^n$ is such that the condition (B) is satisfied. Then*

1. *The set $(\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n) \cap \Gamma_{>0}$ has minimal element, where $\Gamma_{>0} = \{\gamma \in \Gamma : \gamma > 0\}$.*
2. *For any $a, b \in (\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n) \cap \Gamma_{>0}$ there is a natural number k such that $ka \geq b$.*

Proof. Let $\Gamma_0 \subset \Gamma$ be a subgroup that is isomorphic to $(\mathbb{Z}, +)$, in the category of ordered groups, such that $\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n \subset \Gamma_0$. To see assertion (1) observe that $(\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n) \cap \Gamma_{>0} = (\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n) \cap (\Gamma_0)_{>0}$. The last intersection can be viewed as a subset of $\mathbb{Z}_{>0}$. Thus one can use the fact that any nonempty subset of $\mathbb{Z}_{>0}$ has a minimal element.

Assertion (2) is a consequence of the fact that we can assume, by the above considerations, that $a, b \in (\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n) \cap \Gamma_{>0} \subset \mathbb{Z}_{>0}$. \square

Now, using the above lemma, we can present the following two examples of Γ and $\mathbf{w} \in \Gamma^n$ which does not satisfy conditions (A) and (B).

Example 1. Consider the following totally ordered additive group $\Gamma = (\mathbb{R}, +)$ and $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$, where $w_1 = 1$ and $w_2 \in \mathbb{R} \setminus \mathbb{Q}$. By the ergodic theorem [9, Theorem 0.14(i)] for any $\varepsilon > 0$ there exist $k \in \mathbb{N}$ such that $0 < kw_2 - [kw_2] < \varepsilon$. Indeed, we can consider the subgroup $G = \{e^{2\pi ikw_2} \mid k \in \mathbb{Z}\}$ of the unit circle K , with addition given by $K \times K \ni (e^{2\pi i\alpha_1}, e^{2\pi i\alpha_2}) \mapsto e^{2\pi i(\alpha_1 + \alpha_2)} \in K$. In this situation G is infinite. Moreover, one can check that the closure \overline{H} of any subgroup H of K is the closed subgroup of K . Thus, using [9, Theorem 0.14(i)] we obtain that the group G is dense in K , and this implies that the set $\{kw_2 - [kw_2] \mid k \in \mathbb{Z}\}$ is dense in $[0, 1]$. Taking $l = -[kw_2]$ we have $kw_2 - [kw_2] = lw_1 + kw_2 \in \mathbb{Z}w_1 + \mathbb{Z}w_2$. This shows that the set $(\mathbb{Z}w_1 + \mathbb{Z}w_2) \cap \mathbb{R}_{>0}$ does not have a minimal element. Thus, by Lemma 4(1) Γ and \mathbf{w} do not satisfy condition (B) and of course Γ does not satisfy condition (A).

Observe that if $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n \subset \Gamma^n$, then $\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n \subset \mathbb{Z} \subset \mathbb{R}$ and so Γ and \mathbf{w} satisfy condition (B).

Example 2. Let $\Gamma = \mathbb{Z}^2$ with the following operation $+$: $\mathbb{Z}^2 \times \mathbb{Z}^2 \ni ((a_1, a_2), (b_1, b_2)) \mapsto (a_1 + b_1, a_2 + b_2) \in \mathbb{Z}^2$ and with the order relation defined by $(a_1, a_2) \leq (b_1, b_2)$ if and only if $a_1 < b_1$ or $(a_1 = b_1$ and $a_2 \leq b_2)$. One can check that Γ is totally ordered additive group. Moreover we have $\Gamma_{>0} = \{(a_1, a_2) \in \mathbb{Z}^2 : a_1 > 0 \text{ or } (a_1 = 0, a_2 > 0)\}$ and so taking $a = (0, 1)$ and $b = (1, 0)$ we have $a, b \in (\mathbb{Z}a + \mathbb{Z}b) \cap \Gamma_{>0}$ and $ka < b$ for any $k \in \mathbb{N}$. Thus, by Lemma 4(2) Γ and $\mathbf{w} = (a, b)$ do not satisfy condition (B) and of course Γ does not satisfy condition (A).

The next example shows the group Γ which is not isomorphic to $(\mathbb{Z}, +)$ but satisfy the conditions (A) and (B).

Example 3. Let $\mathbb{Z}_{(2)}$ denotes the localization of \mathbb{Z} with respect to the set $\{1, 2, 4, \dots\}$ of powers of 2, i.e. $\mathbb{Z}_{(2)} = \{\frac{k}{2^l} \in \mathbb{Q} \mid k \in \mathbb{Z}, l \in \mathbb{N}\}$. The group $\Gamma = \mathbb{Z}_{(2)}$ with natural relation \leq is of course totally ordered additive group. Moreover the set $\Gamma_{>0}$ does not have minimal element but Γ satisfies condition (A). Indeed, if $w_1, \dots, w_n \in \Gamma$ are any elements, then taking $l = \max\{l_1, \dots, l_n\}$, where $w_i = \frac{k_i}{2^{l_i}}$, $k_i \in \mathbb{Z}$, $l_i \in \mathbb{N}$, for $i = 1, \dots, n$ one can check that $\mathbb{Z}w_1 + \dots + \mathbb{Z}w_n \subset \mathbb{Z}\frac{1}{2^l}$ and the last subgroup of Γ is isomorphic to $(\mathbb{Z}, +)$ (in the category of ordered group).

In the subsequent examples we show that assumptions of Theorem 1 are essential.

Example 4. Let $\Gamma = \mathbb{Z}$, $\mathbf{w} = (2, 2, 1) \in \mathbb{Z}^3$ and $F_1 = x_1 + x_3^2$, $F_2 = x_2 + x_3^2$. Then $dF_1 \wedge dF_2 = (dx_1 + 2x_3 dx_3) \wedge (dx_2 + 2x_3 dx_3) = dx_1 \wedge dx_2 + 2x_3 dx_1 \wedge dx_3 - 2x_3 dx_2 \wedge dx_3$. Thus, all assumptions of Theorem 1, except the inequality (7), are satisfied. Obviously, $F_1, F_2 \notin k[x_1, x_2]$.

Example 5. Let $\Gamma = \mathbb{Z}$, $\mathbf{w} = (1, 1, 1) \in \mathbb{Z}^3$ and $F_1 = x_1 + x_3^2$, $F_2 = x_2 + x_3^2$. Then $dF_1 \wedge dF_2 = dx_1 \wedge dx_2 + 2x_3 dx_1 \wedge dx_3 - 2x_3 dx_2 \wedge dx_3$. Thus, all assumptions of Theorem 1, except that $\deg_{\mathbf{w}}(dF_1 \wedge dF_2)$ is minimal possible, are satisfied. Obviously, $F_1, F_2 \notin k[x_1, x_2]$.

Example 6. Let $\Gamma = \mathbb{Z}$, $\mathbf{w} = (1, 1, 1, 1) \in \mathbb{Z}^4$, $F_1 = x_1 + x_2 + x_4^2$, $F_2 = x_1 + x_3 + x_4^2$ and $F_3 = F_1 + F_2$. Then, $dF_1 \wedge dF_2 \wedge dF_3 = 0$. Thus, all assumptions of Theorem 1, except linear independence of linear parts of F_1, F_2, F_3 , are satisfied. Obviously, $F_1, F_2, F_3 \notin k[x_1, x_2, x_3]$.

Acknowledgements

This research turned into supported by the AGH University of Krakow under grant no. 16.16.420.054, funded by the Polish Ministry of Science and Higher Education.

References

- [1] Edo, E., Kanehira, T., Karaś, M., Kuroda, S.: Separability of wild automorphisms of a polynomial ring. *Transform. Groups* **18**(1), 81–96 (2013). <https://doi.org/10.1007/s00031-013-9212-2>
- [2] Hartshorne, R.: Algebraic Geometry. In: Hersh, P., Vakil, R., Wunsch, J. (eds.) *Graduate Texts in Mathematics*, vol. **52**. Springer New York, NY (1977). <https://doi.org/10.1007/978-1-4757-3849-0>
- [3] Karaś, M.: There is no tame automorphism of \mathbb{C}^3 with multidegree $(3, 4, 5)$. *Proc. Amer. Math. Soc.* **139**(3), 769–775 (2011). <https://doi.org/10.1090/S0002-9939-2010-10779-7>
- [4] Karaś, M.: Tame automorphisms of \mathbb{C}^3 with multidegree of the form $(3, d_2, d_3)$. *J. Pure Appl. Algebra* **214**(12), 2144–2147 (2010). <https://doi.org/10.1016/j.jpaa.2010.02.017>
- [5] Karaś, M.: Multidegrees of tame automorphisms of \mathbb{C}^n . *Diss. Math.* **477** (2011)
- [6] Kuroda, S.: On the Karaś type theorems for the multidegrees of polynomial automorphisms. *J. Algebra* **423**, 441–465 (2015). <https://doi.org/10.1016/j.jalgebra.2014.10.024>
- [7] Shafarevich, I.R.: *Basic Algebraic Geometry 1. Varieties in Projective Space*. Springer Berlin, Heidelberg (2013). <https://doi.org/10.1007/978-3-642-37956-7>
- [8] Shestakov, I.P., Umirbaev, U.U.: The tame and the wild automorphisms of polynomial rings in three variables. *J. Amer. Math. Soc.* **17**, 197–227 (2004). <https://doi.org/10.1090/S0894-0347-03-00440-5>
- [9] Walters, P.: *An Introduction to Ergodic Theory*. In: Hersh, P., Wunsch, J., Vakil, R. (eds.) *Graduate Texts in Mathematics*, vol. **79**. Springer New York, NY (1982)

CONTACT INFORMATION

D. Holik,
M. Karaś

AGH University of Krakow, Faculty of Applied
Mathematics al. A. Mickiewicza 30, 30-059
Kraków, Poland
E-Mail: holikd@agh.edu.pl,
mkaras@agh.edu.pl

Received by the editors: 22.08.2025
and in final form 20.06.2026.