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Construction of local nearrings using GAP

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Abstract. Using a programming language of GAP, we constructed a general algorithm which allows us to classify local nearrings of order 32 and higher. Moreover, some functions for local nearrings from the LocalNR package are provided.

Introduction

Nearrings arise naturally in the study of systems of nonlinear mappings, and have been studied for many decades. Basic definitions and many results concerning nearrings can be for instance found in Pilz's book [9].

Nearrings are generalization of associative rings, in which the additive group can be non-abelian, and addition is connected with multiplication by only one distributive law, left or right. In this sense local nearrings are generalization of local rings.

Clearly every associative ring is a nearring and each group is the additive group of a nearring, but not necessarily of a nearring with identity. The question what group can be the additive group of a nearring with identity is far from solution.

The problem of finding groups that can be additive groups for the nearrings with identity is studied from the late 1960s. One of the first results in this direction was obtained in [3], where it was shown that there

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exists a unique nearring with identity whose additive group is cyclic and that, in fact, this nearring is a commutative ring.

The classification of all nearrings up to certain orders is an open problem. It requires extensive computations, and the most suitable platform for their implementation is the computational algebra system GAP [4].

1. Preliminaries

A natural example is the nearring Map(G) of all mappings of a group G into itself with addition and multiplication defined by $x^{f+g} = x^f x^g$ and $x^{fg} = (x^f)^g$ for all $x \in G$. In particular, for any non-trivial group G in $Map(G)$ only the left distributive law $f(g+h) = fg + fh$ is valid.

If e is the neutral element of G, then the mappings o, i with $x^o = e$ and $x^i = x$ for all $x \in G$ are zero and the identity element of $Map(G)$.

The additive group $Map(G)^+$ of $Map(G)$ is isomorphic to the cartesian product of $|G|$ copies of G and the multiplicative group $Map(G)^*$ coincides with the symmetric group $Sym(G)$ on G.

The sets $\{f \in Map(G) | e^f = e\}$ and $\{g \in Map(G) | x^g = const\}$ are subnearrings of $Map(G)$ called its zero-symmetric and constant parts and denoted by $Map_0(G)$ and $Map_c(G)$, respectively.

As the additive groups, $Map_0(G)^+$ is normal in $Map(G)^+$, $Map_c(G)$ is isomorphic to G and $Map(G) = Map_0(G) + Map_c(G)$ with $Map_0(G) \cap Map_c(G) = \{o\}$, so that $Map(G)$ is the semidirect sum of $Map_0(G)$ and $Map_c(G)$.

It is well-known that every ring is embedded into the ring $End(A)$ of all endomorphisms of an abelian group A. Similarly every nearring can be embedded into $Map(G)$ for some group G. From this point of view the ring theory describes "linear" mappings on groups, while nearrings handle the general non-linear case.

Definition 1. A nearring R with an identity $1 \neq 0$ is called a nearfield, if every non-zero element of R is invertible in R.

Historically, first examples of nearrings arose as nearfields. Dickson [5] constructed "fields with only one distributive law" by "distorting" the multiplication in usual fields called later as Dickson nearfields. The smallest of them is the nearfield $N = (GF(9), +, \circ)$ on the Galois field $GF(9)$ with a new operation $\circ$ such that $x \circ y = xy$ if y is a square in $GF(9)$ and $x \circ y = xy^3$ otherwise. Its multiplicative group is the quaternion group.

The finite nearfields were classified by H. Zassenhaus whose famous paper [17] contains one of the first deep group theoretical classifications which can be for instance found in chapter 20 of the book of M. Hall “The theory of groups” [6].

A nearfield is essentially the same as a group with an automorphism group acting sharply transitive on its non-trivial elements. A group G acts sharply transitive (sharply 2-transitive) on a set S if for each two different elements $s, t \in S$ (each two pairs s_1, s_2 and t_1, t_2 of different elements of S) there exists exactly one $g \in G$ with $s^g = t$ (${s^g}_i = t_i$ for $i = 1, 2$).

If N is a nearfield, then the semidirect product $N^+ \rtimes N^*$ admits a sharply 2-transitive action on N, in which N^+ acts by addition and N^* by multiplication. A long standing conjecture says that the converse is also true remains. In the finite case it was confirmed by Zassenhaus in the above-mentioned paper. The general case remains still open.

An account of results concerning nearfields can also be found in the book of H. Waehling “Theorie der Fastkoerper” [16].

Definition 2. Following Maxson [8], a nearring R with an identity $1 \neq 0$ is called local, if the set $L_R = R \setminus R^*$ of all non-invertible elements of R forms a subgroup of the additive group R^+. In particular, if $L_R = 0$, then R a nearfield.

Examples

- Every local ring is local as a nearring and every division ring is a nearfield.
- If G is a group of order 2, then $Map(G)$ is a local nearring of order 4 which is not a ring.
- If G is a finite p-group and A is a p-group of automorphisms of G containing the inner automorphism group $Inn(G)$, then the subnearring of $Map(G)$ generated by A is local [7].

Lemma 1 ([2, Lemma 3.2])**.** *Let R be a local nearring. Then the following statements hold*:

- the elements of L_R do not have neither left nor right inverses in R and $R = L_R \cup R^*$;
- $RL_RR = L_R$, i.e. L_R is an (R, R)-subgroup of R;

- L_R contains a subgroup M such that $RM = M$, M is normal in R^+ and the factor group $R^+ = M$ is either a p-group of prime exponent p or a divisible torsion-free group;
- the set $1 + L_R$ is a subgroup of R^* an so L_R is a "construction subgroup" in sense of P. Hubert;
- the semidirect product $G = L_R \rtimes (1 + L_R)$ has subgroups A and B isomorphic to $1 + L_R$ such that $G = AB = L_R \rtimes A = L_R \rtimes B$ and $A \cap B = 1$.

For a nearring R, its ideal I is a normal subgroup of R^+ such that $(r + x)s - rs \in I$ for each $x \in I$ and every $r, s \in R$. It is unknown at present whether for each local nearring R the subgroup L_R is an ideal of R or even a normal subgroup of R^+. But in the finite case we have the following (see, for example, [2]).

Lemma 2. *If R is a finite local nearring, then the following statements hold*:

- R^+ *is a p-group and so* $|R| = p^n$ *for some* $n \geq 1$;
- L_R *is an ideal of* R;
- *the factor nearring* R/L_R *is a nearfield*;
- $1 + L_R$ *is a normal Sylow p-subgroup of* R^*;
- $R^* = (1 + L_R) \rtimes K$ *for a p'-subgroup K isomorphic to* $(R/L_R)^*$.

Main results accumulated for local nearrings can be found in the surveys [11, 14].

2. A construction of nearrings with identity

The list of all local nearrings of order at most 31 can be extracted from the libraries of the package SONATA [1] of the computer system algebra GAP. In particular, there exist 9 non-isomorphic groups of order 16 which are the additive groups of local nearrings. Using a programming language of GAP, we constructed a general algorithm which allows us to classify local nearrings of the order 32 and more.

Our algorithm for a construction of nearrings with identity is based on the following theorem.

Theorem 1. *Let G be a group and let the set* $\operatorname{End} G$ *of all endomorphisms of G be regarded as a monoid under the composition of endomorphisms. Then G is the additive group of a nearring with identity if and only if there exists an element $g \in G$ and a submonoid F of* $\operatorname{End} G$ *such that $G = g^F$ and $g^\phi = g^\psi$ only if $\phi = \psi$ for any $\phi,\ \psi \in F$.*

Proof. Indeed, if we put $g^\phi \star g^\psi = g^{\phi\psi}$, then $Nr = (G, \cdot, \star_F)$ is a left nearring with identity g and the multiplicative monoid of Nr is isomorphic to F. In particular, in this case the exponent of G coincides with the order of g. □

Moreover, if F_1 and F_2 are submonoids of $\operatorname{End} G$ satisfying Theorem 1, then the (left) nearrings $Nr_1 = (G, \cdot, \star_{F_1})$ and $Nr_2 = (G, \cdot, \star_{F_2})$ are isomorphic if and only if there exists an automorphism α of G such that $F_1^\alpha = F_2$.

Thus, to determine all non-isomorphic nearrings with identity on a group G, it is necessary to find all submonoids of $\operatorname{End} G$ satisfying the conclusion of the latter theorem. Clearly, this is possible only if the monoid $\operatorname{End} G$ is small in some sense. For instance, if G is of order at most 31, then all such nearrings can be found.

If G is a simple group, then $\operatorname{End} G = \operatorname{Aut} G \cup \{0\}$ and so $F \setminus \{0\}$ is a group of automorphisms acting sharply transitive on G. It is known that in this case G is abelian of prime order.

The following theorem describes conditions under which the nearring $Nr = (G, \cdot, \star_F)$ constructed on G by the monoid F is local.

Theorem 2. *Let the group G, its element g and a submonoid F of* $\operatorname{End} G$ *satisfy the conclusion of Theorem 1 and let $A = F \cap \operatorname{Aut} G$. The nearring $Nr = (G, \cdot, \star_F)$ is local if an only if there exists a subgroup L of G such that $G = g^A \cup L$ and $g^A \cup L = \emptyset$.*

Proof. If G is finite, then the set $\operatorname{End} G \setminus \operatorname{Aut} G$ is a subsemigroup of $\operatorname{End} G$. Therefore all non-automorphisms of $F \subset \operatorname{End} G$ form an A-invariant subsemigroup X such that $F = A \cup X$ and $A \cap X = \emptyset$. Since $G = g^F = g^A \cup L$, it follows that $L = g^X$. □

3. An algorithm for construction of finite local nearrings

Let G be a finite group contained in the library of the computer system algebra GAP. We consider as an example the group $G = (\langle a \rangle \rtimes$

$\langle b\rangle) \rtimes \langle c\rangle$ of order 32 with $a^8 = b^2 = c^2 = 1$, $[a, c] = b$, $[a, b] = a^4$ and $[b, c] = 1$.

Let $[n, i]$ be the i-th group of order n in the SmallGroups library in GAP.

$gap > G := SmallGroup(32, 7);$

$< pc\ group\ of\ size\ 32\ with\ 5\ generators >$

$gap > StructureDescription(G);$

"$(C8 : C2) : C2$"

$gap > A := AutomorphismGroup(G);$

$< group >$

$gap > Size(A);$

128

$gap > OrbitLengths(A, G);$

$[1, 16, 8, 2, 4, 1]$

$gap > Or := OrbitsDomain(A, G)[2];;$

$gap > Size(Or);$

16

$gap > L := Difference(G, Or);;$

$gap > L := Subgroup(G; L);$

$Group([< identity > of \ldots, f2, f3, f4, f5, f2{*}f3, f2{*}f4, f2{*}f5, f3{*}f4, f3*f5, f4*f5, f2*f3*f4, f2*f3*f5, f2*f4*f5, f3*f4*f5, f2*f3*f4*f5])$

$gap > Size(L);$

16

$gap > StructureDescription(L);$

"$C2 \times D8$"

$gap > S := SubgroupsSolvableGroup(A);;$

$gap > T := Filtered(S, x- > Size(x) = 16);;$

$gap > U := Filtered(T, x- > Maximum(OrbitLengths(x, G)) = 16);;$

$gap > Size(U);$

6

$gap > List(U, IdGroup);$

$[[16, 3], [16, 11], [16, 11], [16, 13], [16, 8], [16, 7]]$

$gap > En := AllEndomorphisms(G);;$

$gap > Size(En);$

320
$gap > o := En[1];;$
$gap > Nil := [];$
$[]$
$gap > for\ x\ in\ En\ do;$
$> if\ x^5 = o\ then\ Add(Nil, x); fi;$
$> od;$
$gap > Size(Nil);$
64
$gap > \ldots$
$gap > StructureDescription(U[1]);$
"$(C4 \times C2) : C2$"
$gap > StructureDescription(U[2]);$
"$C2 \times D8$"
$gap > StructureDescription(U[3]);$
"$C2 \times D8$"
$gap > StructureDescription(U[4]);$
"$(C4 \times C2) : C2$"
$gap > \ldots$

This means in particular that we have only the following triply factorized groups $M \rtimes A = M \rtimes B = AB$ with $A \cap B = 1$ and $M \cong C_2 \times D_8$, namely the groups in which $A \cong B$ is isomorphic to one of the groups $C_2 \times D_8$, $(C_4 \times C_2) \rtimes C_2$, D_{16} or QD_{16}.

The algorithm for construction of finite local nearrings on the additive group $[32, 7]$ can be extracted from [15] via GAP using the packages LocalNR [13] and SONATA.

4. Local nearrings of small orders

The classification of all nearrings up to certain orders is an open problem. It requires extensive computations, and the most suitable platform for their implementation is the computational algebra system GAP. The current version of the LocalNR package (not yet redistributed with GAP) contains all local nearrings of order at most 361, except those of orders 32, 64, 128, 243, and 256.

There are 51 non-isomorphic groups of order 32 from which 19 are the additive groups (see, for example, [10]).

Theorem 3. *The following groups and only they are the additive groups of local nearrings of order* 32*:*

IdGroup	*Structure Description*	*Number of zero-symmetric local nearrings*
[32,1]	C_{32}	*1*
[32,2]	$(C_4 \times C_2) \rtimes C_4$	*1396*
[32,3]	$C_8 \times C_4$	*879*
[32,4]	$C_8 \rtimes C_4$	*797*
[32,5]	$(C_8 \times C_2) \rtimes C_2$	*1944*
[32,6]	$((C_4 \times C_2) \rtimes C_2) \rtimes C_2$	*432*
[32,7]	$(C_8 \rtimes C_2) \rtimes C_2$	*224*
[32,8]	$C_2.((C_4 \times C_2) \rtimes C_2)$	*208*
[32,12]	$C_4 \rtimes C_8$	*2406*
[32,16]	$C_{16} \times C_2$	*128*
[32,17]	$C_{16} \rtimes C_2$	*128*
[32,21]	$C_4 \times C_4 \times C_2$	*135500*
[32,22]	$C_2 \times ((C_4 \times C_2) \rtimes C_2)$	*149205*
[32,23]	$C_2 \times (C_4 \rtimes C_4)$	*157845*
[32,24]	$(C_4 \times C_4) \rtimes C_2$	*262480*
[32,36]	$C_8 \times C_2 \times C_2$	*177110*
[32,37]	$C_2 \times (C_8 \rtimes C_2)$	*527226*
[32,45]	$C_4 \times C_2 \times C_2 \times C_2$	> 6684533
[32,51]	$C_2 \times C_2 \times C_2 \times C_2 \times C_2$	> 7007053

The following theorem describes such groups of order $81 = 3^4$. In particular, among 15 non-isomorphic groups of this order only 9 are these additive groups [12].

Theorem 4. *The following groups and only they are the additive groups of local nearrings of order* 81*.*

IdGroup	*Structure Description*	*Number of zero-symmetric local nearrings*
$[81, 1]$	C_{81}	*1*
$[81, 2]$	$C_9 \times C_9$	*268*
$[81, 3]$	$(C_9 \times C_3) \rtimes C_3$	*46*
$[81, 5]$	$C_{27} \times C_3$	*388*
$[81, 6]$	$C_{27} \rtimes C_3$	*10*
$[81, 11]$	$C_9 \times C_3 \times C_3$	*22989*
$[81, 12]$	$C_3 \times ((C_3 \times C_3) \rtimes C_3)$	*807*
$[81, 13]$	$C_3 \times (C_9 \rtimes C_3)$	*337*
$[81, 15]$	$C_3 \times C_3 \times C_3 \times C_3$	*12322*

5. The LocalNR package

The LocalNR package contains the library of local nearrings up to order 361 and some functions to analyze finite nearrings. The current version of this package is version 1.0.4, released on 2024-03-15.

We give some functions for local nearrings from the LocalNR package.

TheAdditiveGroupsOfLibraryOfLNRsOfOrder(n) (function)

Returns: information

The argument is n. The output a list of IdGroup of the additive groups of local nearrings from Library of order n.

Example

gap>List(TheAdditiveGroupsOfLibraryOfLNRsOfOrder(81), IdGroup);
[[81, 1], [81, 2], [81, 3], [81, 5], [81, 6], [81, 11], [81, 12], [81, 13], [81, 15]].

TheLibraryOfLNRsOnGroup(G) (function)

Returns: information

The argument is a group G. The output a list of the catalogues of local nearrings from Library on G.

Example

gap> G:=SmallGroup(81,2);
<pc group of size 81 with 4 generators>
gap> TheLibraryOfLNRsOnGroup(G);
[“AllLocalNearRings(81,2,54,3)”, “AllLocalNearRings(81,2,54,6)”,
“AllLocalNearRings(81,2,54,9)”, “AllLocalNearRings(81,2,54,10)”,
“AllLocalNearRings(81,2,54,11)”, “AllLocalNearRings(81,2,54,15)”,
“AllLocalNearRings(81,2,72,14)”, “AllLocalNearRings(81,2,72,19)”,
“AllLocalNearRings(81,2,72,24)”, “AllLocalNearRings(81,2,72,26)”]

LocalNearRing(k, l, m, n, w) (operation)

Returns: a nearring

The arguments are k, l, m, n, w. The output is the local nearring from Library without check. The arguments k, l, m, n are from IdGroup of the additive group and the multiplicative group, respectively, w is the position in the list.

Example

gap> L:=LocalNearRing(81,12,54,8,3);

ExplicitMultiplicationNearRing (<pc group of size 81 with 4 generators> , multiplication)

AllLocalNearRings(k, l, m, n) (operation)

Returns: a list

The arguments are k, l, m, n. The output are all local nearrings from Library without check. The arguments k, l, m, n are as above.

Example

gap> L:=AllLocalNearRings(81,12,54,8);

gap> Size(L);

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IsAdditiveGroupOfLibraryOfLNRs(G) (function)

Returns: a boolean

The argument is a group G. The output is true if in Library there exists a local nearring whose additive group is isomorphic to G otherwise the output is false.

Example

gap> G:=SmallGroup(25,2);

<pc group of size 25 with 2 generators>

gap> IsAdditiveGroupOfLibraryOfLNRs(G);

true

gap> IsAdditiveGroupOfLibraryOfLNRs(SmallGroup(81,14));

false

InfoLocalNearRing(G) (function)

Returns: information

The argument is a group G. The output some information about local nearrings from Library on G.

Example

gap> InfoLocalNearRing(SmallGroup(361,2));

The local nearrings are sorted by their multiplicative groups.

["AllLocalNearRings(361,2,342,1) (2)",

"AllLocalNearRings(361,2,342,2) (2)",

"AllLocalNearRings(361,2,342,4) (1)",
"AllLocalNearRings(361,2,342,6) (1)",
"AllLocalNearRings(361,2,342,7) (7)",
"AllLocalNearRings(361,2,342,8) (6)",
"AllLocalNearRings(361,2,360,4) (1)",
"AllLocalNearRings(361,2,360,15) (1)"]

References

[1] Aichinger, E., Binder, F., Ecker, Ju., Mayr, P., Noebauer, C.: SONATA — system of near-rings and their applications. GAP package, Version 2.9.1. https://gap-packages.github.io/sonata/ (2018)

[2] Amberg, B., Hubert, P., Sysak, Ya.: Local nearrings with dihedral multiplicative group. J. Algebra **273**(2), 700–717 (2004). https://doi.org/10.1016/j.jalgebra.2003.10.007

[3] Clay, J.R., Malone, Jr.: The near-rings with identities on certain finite groups. Math. Scand. **19**, 146–150 (1966). https://doi.org/10.7146/math.scand.a-10803

[4] The GAP Group, GAP — Groups, Algorithms, and Programming. Version 4.14.0 https://www.gap-system.org

[5] Dickson, L.: Definitions of a group and a field by independent postulates. Trans. Amer. Math. Soc. **6**, 198–204 (1905). https://doi.org/10.1090/S0002-9947-1905-1500706-2

[6] Hall, M.Jr.: The theory of groups. MacMillan Company, New York (1959)

[7] Lyons, C.G., Peterson, G.L.: Local endomorphism near-rings. Proc. Edinburgh Math. Soc. **31**(3), 409–414 (1988). https://doi.org/10.1017/S0013091500006805

[8] Maxon, C.J.: On local near-rings. Math. Z. **106**, 197–205 (1968). https://doi.org/10.1007/BF01110133

[9] Pilz, G.: Near-rings, the theory and its applications. Math. Studies **23** (1977)

[10] Raievska, I., Raievska, M.: Skinchenni lokalni mayzhe-kiltsya (Finite local nearrings). Mohylyanskyy Mat. Zh. **1**, 38–48 (2018) (in Ukrainian). https://doi.org/10.18523/2617-7080i2018p38-48

[11] Raievska, I., Raievska, M.: Local nearrings, their structure, and the GAP system. Ukr. Math. J. **76**(11), 1831–1848 (2025). https://doi.org/10.1007/s11253-025-02426-y

[12] Raievska, I., Raievska, M., Sysak, Ya.: Local nearrings of order 81. Paper presented at the Xth international algebraic conference in Ukraine, Odessa, 20–27 August 2015

[13] Raievska, I., Raievska, M., Sysak, Y.: LocalNR, Package of local nearrings. Version 1.0.4 (GAP package). https://gap-packages.github.io/LocalNR (2024)

[14] Sysak, Ya.P.: Products of groups and local nearrings. Note Mat. **28**(2), 177–211 (2008). https://doi.org/10.1285/i15900932v28n2supplp177

[15] Sysak, Ya., Raievska, I., Raievska, M.: Construction-of-local-nearrings. https://github.com/raemarina/Construction-of-local-nearrings/ (2025)

[16] Wähling, H.: Theorie der Fastkörper. Thales Verlag, Essen (1987)

[17] Zassenhaus, H.: Über endliche Fastkörper. Abh. Math. Semin. Univ. Hambg. **11**, 187–220 (1935). https://doi.org/10.1007/BF02940723

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