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RESEARCH ARTICLE

Structure of algebra of derivation of some non-nilpotent Leibniz algebras

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Abstract. Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a left Leibniz algebra if it satisfies the left Leibniz identity $[[a,b],c]=[a,[b,c]]-[b,[a,c]]$ for all $a,b,c \in L$. We study algebras of derivations of some non-nilpotent Leibniz algebras of low dimensions.

Introduction

Let V be a vector space over a field F. Denote by

$$End_F(V)$$

the set of all linear transformations of V. Then $End_F(V)$ is an associative algebra by the operations $+$ and $\circ$. As usual, $End_F(V)$ is a Lie algebra by the operations $+$ and $[,]$ where

$$[f,g]=f\circ g-g\circ f$$

for all $f,g \in End_F(V)$.

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Let L be an algebra over a field F with the operations $+$ and $[,]$. Recall that a linear transformation f of an algebra L is called a *derivation* if

$$f([a,b]) = [f(a),b] + [a,f(b)] \text{ for all } a,b \in L.$$

Derivations play a very important role in studying the structure of many types of non-associative algebras. In particular, such is especially true for Lie and Leibniz algebras.

Let L be an algebra over a field F with the binary operations $+$ and $[,]$. Then L is called a *left Leibniz algebra* if it satisfies the left Leibniz identity,

$$[[a,b],c] = [a,[b,c]] - [b,[a,c]]$$

for all $a,b,c \in L$. We will also use another form of this identity:

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]].$$

Leibniz algebras were introduced in the paper of A. Blokh [2], but the term "Leibniz algebra" appears in the book of J.-L. Loday [16] and his article [17]. Certain aspects of this theory have been explored in the books [1,6].

Let $Der(L)$ be the subset of all derivations of a Leibniz algebra L. It can be proven that $Der(L)$ is a subalgebra of the Lie algebra $End_F(L)$. $Der(L)$ is called the *algebra of derivations* of the Leibniz algebra L.

The impact of the algebra of derivations on the structure of a Leibniz algebra is evident in the following result: If A is an ideal of a Leibniz algebra, then the factor-algebra of L by the annihilator of A is isomorphic to some subalgebra of $Der(A)$ [4, Proposition 3.2].

The structure of the algebra of derivations of finite-dimensional one-generator Leibniz algebras was described in the papers [11,18], its association with infinite-dimensional, one-generator Leibniz algebras was examined in the paper [15]. The study of derivation algebras for small-dimensional Leibniz algebras is a natural and intriguing question. Unlike Lie algebras, Leibniz algebras of dimension 3 exhibit considerable diversity. A classification of these algebras has been established, with the most comprehensive account available in [5].

The derivation algebras of Leibniz algebras of dimension 3 have been investigated in various papers, providing valuable insights into their structural properties [6–10,12,13].

The final consideration concerns derivation algebras of non-nilpotent Leibniz algebras of dimension 3 with a one-dimensional Leibniz kernel. In particular, the paper [10] focuses on the case where the center of the Leibniz algebra contains the Leibniz kernel.

1. Some preliminaries and remarks

We will need some general properties of an algebra of derivations of a Leibniz algebra. Here, we show some basic elementary properties of derivations that have been proven in the paper [13]. First, let us recall some definitions.

Every Leibniz algebra L has a specific ideal. Denote by $Leib(L)$ the subspace generated by the elements $[a, a]$, $a \in L$. It is possible to prove that $Leib(L)$ is an ideal of L. The ideal $Leib(L)$ is called the *Leibniz kernel* of algebra L. By the definition, factor-algebra $L/Leib(L)$ is a Lie algebra. Conversely, if K is an ideal of L such that L/K is a Lie algebra, then K includes the Leibniz kernel.

Let L be a Leibniz algebra. Define the *lower central series* of L,

$$L = \gamma_1(L) \geq \gamma_2(L) \geq \ldots \gamma_\alpha(L) \geq \gamma_{\alpha+1}(L) \geq \ldots \gamma_\delta(L),$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, recursively, $\gamma_{\alpha+1}(L) = [L, \gamma_\alpha(L)]$ for every ordinal α, and $\gamma_\lambda(L) = \bigcap_{\mu<\lambda} \gamma_\mu(L)$ for every limit ordinal λ. The last term $\gamma_\delta(L) = \gamma_\infty(L)$ is called the *lower hypocenter* of L. We have: $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

As usual, we say that a Leibniz algebra L is called *nilpotent* if a positive integer k exists, such that $\gamma_k(L) = \langle 0 \rangle$. More precisely, L is said to be *nilpotent of nilpotency class c* if $\gamma_{c+1}(L) = \langle 0 \rangle$ but $\gamma_c(L) \neq \langle 0 \rangle$.

The *left* (respectively *right*) *center* $\zeta^{\text{left}}(L)$ (respectively $\zeta^{\text{right}}(L)$) of a Leibniz algebra L is defined by the rule below:

$$\zeta^{\text{left}}(L) = \{x \in L \mid [x, y] = 0 \text{ for each element } y \in L\}$$

(respectively

$$\zeta^{\text{right}}(L) = \{x \in L \mid [y, x] = 0 \text{ for each element } y \in L\}).$$

It is not hard to prove that the left center of L is an ideal, but this is not true for the right center. Moreover, $Leib(L) \leq \zeta^{\text{left}}(L)$, so that $L/\zeta^{\text{left}}(L)$ is a Lie algebra. The right center is a subalgebra of L; the left and right centers are generally different; they may even have different dimensions (see [4]).

The center of L is defined by the rule below:

$$\zeta(L) = \{x \in L \mid [x, y] = 0 = [y, x] \text{ for each element } y \in L\}.$$

The center is an ideal of L.

Lemma 1. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\zeta^{\text{left}}(L)) \leq \zeta^{\text{left}}(L)$, $f(\zeta^{\text{right}}(L)) \leq \zeta^{\text{right}}(L)$ and $f(\zeta(L)) \leq \zeta(L)$.*

Corollary 1. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\zeta_\alpha(L)) \leq \zeta_\alpha(L)$ for every ordinal α.*

Lemma 2. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f([L, L]) \leq [L, L]$.*

Corollary 2. *Let L be a Leibniz algebra over a field F and f be a derivation of L. Then $f(\gamma_\alpha(L)) \leq \gamma_\alpha(L)$ for every ordinal α.*

Denote by Ξ the classic monomorphism of $End(L)$ in $M_3(F)$ (i.e., the mapping, assigning to each endomorphism its matrix concerning the basis $\{a_1, a_2, a_3\}$).

2. On the Leibniz algebras $Lei_{13}(3,F)$ and $Lei_{14}(3,F)$

Proposition 1. *Let L be a non-nilpotent Leibniz algebra of dimension 3 having the Leibniz kernel of dimension 1, $S = Ann_L(Leib(L))$. Suppose that $L \neq S$ and $L/Leib(L)$ is a non-abelian Lie algebra. Then $L = Lei_{13}(3, F)$ is a Leibniz algebra of the following type*

$$Lei_{13}(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1, a_1] = a_3 = [a_1, a_3], [a_1, a_2] = -a_2, [a_2, a_1] = a_2,$$
$$[a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

In this case, $Leib(L) = Fa_3 = \zeta^{\text{left}}(L)$, $[L, L] = S = Fa_2 \oplus Fa_3$ is an abelian ideal, Fa_2 is also an ideal of L, $Fa_1 \oplus Fa_3$ is a non-nilpotent subalgebra of L, $Fa_1 \oplus Fa_2$ is a non-abelian Lie subalgebra of L.

Proof. Let $K = Leib(L)$, then L/K is a Lie algebra of dimension 2. Then $Ann_L^{\text{left}}(K)$ is an ideal of L such that $L/Ann_L^{\text{left}}(K)$ is isomorphic to some subalgebra of the algebra of derivation of K [4, Proposition 3.2]. The inclusion $Leib(L) \leq \zeta^{\text{left}}(L)$ implies that, $Ann_L^{\text{left}}(K) = Ann_L(K) = S$. Since $S \neq L$, $dim_F(L/S) = 1$. The inclusion $Leib(L) \leq \zeta^{\text{left}}(L)$ shows that $S = Ann_L(K)$. As we have noted above, S/K is an ideal of L/K of dimension 1. It follows that S is nilpotent subalgebra of dimension 2. The factor-algebra L/K is a non-abelian Lie algebra of dimension 2.

Since S/K is an ideal of L/K, it follows that L/K has an element $b + K \notin S/K$ and S/K has an element $a + K$ such that $[a + K, b + K] = a + K$ (see, for example, the book [3, Chapter 1, § 4]).

Suppose initially that S is non-abelian, then $[a, a] = c \neq 0$. The equality $a + K = [a + K, b + K]$ implies that $[a, b] = a + \kappa_1 c$ for some scalar $\kappa_1 \in F$. If $\kappa_1 = 0$, then put $b_1 = b$. If not, then put $b_1 = b - \kappa_1 a$. In this case, we have

$$[a, b_1] = [a, b - \kappa_1 a] = [a, b] - [a, \kappa_1 a] = a + \kappa_1 c - \kappa_1 c = a.$$

Let x be an element such that $x \notin S$. Since S is an ideal of L, $[a, x] = \alpha a + \nu c$ for some scalars $\alpha, \nu \in F$. Since L/K is not nilpotent, $\alpha \neq 0$. The fact that L/K is a Lie algebra implies that $[x, a] = -\alpha a + \nu_1 c$ for some $\nu_1 \in F$. Since $x \notin S = Ann_L(K)$, $[x, c] = \xi c$ for some scalar $0 \neq \xi \in F$. We have

$$[x, [a, x]] = [[x, a], x] + [a, [x, x]] = [[x, a], x].$$

Then we obtain

$$\alpha[x, a] + \nu[x, c] = [x, \alpha a + \nu c] = [x, [a, x]]$$
$$= [[x, a], x] = [-\alpha a + \nu_1 c, x] = [-\alpha a, x] = -\alpha[a, x].$$

Since $\alpha \neq 0$ we obtain that $[x, a] = -[a, x] - \alpha^{-1}\nu[x, c]$. In particular, for $x = b_1$ we have $[b_1, a] = -[a, b_1] = -a$.

The fact that $b_1 \notin Ann_L(K)$ implies that $[b_1, c] = \alpha c$ where $0 \neq \alpha \in F$. We have also that $[b_1, b_1] = \gamma c$ for some scalar $\gamma \in F$. If $\gamma = 0$, then put $a_1 = b_1$. If not, then put $a_1 = b_1 - \alpha^{-1}\gamma c$. For this case, we have

$$[a_1, a_1] = [b_1 - \alpha^{-1}\gamma c, b_1 - \alpha^{-1}\gamma c]$$
$$= [b_1, b_1] - \alpha^{-1}\gamma[b_1, c] = \gamma c - \alpha^{-1}\gamma\alpha c = 0.$$

Moreover,

$$[a, a_1] = [a, b_1 - \alpha^{-1}\gamma c] = [a, b_1] - \alpha^{-1}\gamma[a, c] = [a, b_1] = a,$$
$$[a_1, a] = [b_1 - \alpha^{-1}\gamma c, a] = [b_1, a] - \alpha^{-1}\gamma[c, a] = [b_1, a] = -a.$$

Put now $a_2 = a$, $a_3 = c$. Then we come to the following Leibniz algebra

$$L = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1, a_1] = 0, [a_1, a_2] = -a_2, [a_2, a_1] = a_2,$$
$$[a_2, a_2] = a_3, [a_3, a_1] = [a_2, a_3] = [a_3, a_2] = 0, [a_1, a_3] = \gamma a_3$$

for some scalar $0 \neq \gamma \in F$.

Let's verify that an algebra with such defining relations is indeed a Leibniz algebra.

Let x, y, z be arbitrary elements of L,

$$\begin{aligned} x &= \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \\ y &= \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, \\ z &= \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3, \end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \sigma_1, \sigma_2, \sigma_3$ are arbitrary scalars. Then

$$\begin{aligned} [x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\ &= -\xi_1\eta_2 a_2 + \xi_1\eta_3\gamma a_3 + \xi_2\eta_1 a_2 + \xi_2\eta_2 a_3 \\ &= (\xi_2\eta_1 - \xi_1\eta_2)a_2 + (\xi_1\eta_3\gamma + \xi_2\eta_2)a_3, \\ [x, z] &= (\xi_2\sigma_1 - \xi_1\sigma_2)a_2 + (\xi_1\sigma_3\gamma + \xi_2\sigma_2)a_3 \\ [y, z] &= (\eta_2\sigma_1 - \eta_1\sigma_2)a_2 + (\eta_1\sigma_3\gamma + \eta_2\sigma_2)a_3. \end{aligned}$$

Thus

$$\begin{aligned} [x, [y, z]] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, (\eta_2\sigma_1 - \eta_1\sigma_2)a_2 + (\eta_1\sigma_3\gamma + \eta_2\sigma_2)a_3] \\ &= \xi_1(\eta_2\sigma_1 - \eta_1\sigma_2)[a_1, a_2] + \xi_1(\eta_1\sigma_3\gamma + \eta_2\sigma_2)[a_1, a_3] \\ &+ \xi_2(\eta_2\sigma_1 - \eta_1\sigma_2)[a_2, a_2] \\ &= -\xi_1(\eta_2\sigma_1 - \eta_1\sigma_2)a_2 + \xi_1(\eta_1\sigma_3\gamma + \eta_2\sigma_2)\gamma a_3 + \xi_2(\eta_2\sigma_1 - \eta_1\sigma_2)a_3 \\ &= (\xi_1\eta_1\sigma_2 - \xi_1\eta_2\sigma_1)a_2 + (\xi_1\eta_1\sigma_3\gamma^2 + \xi_1\eta_2\sigma_2\gamma + \xi_2\eta_2\sigma_1 - \xi_2\eta_1\sigma_2)a_3, \\ [[x, y], z] &= [(\xi_2\eta_1 - \xi_1\eta_2)a_2 + (\xi_1\eta_3\gamma + \xi_2\eta_2)a_3, \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3] \\ &= \sigma_1(\xi_2\eta_1 - \xi_1\eta_2)[a_2, a_1] + \sigma_2(\xi_2\eta_1 - \xi_1\eta_2)[a_2, a_2] \\ &= \sigma_1(\xi_2\eta_1 - \xi_1\eta_2)a_2 + \sigma_2(\xi_2\eta_1 - \xi_1\eta_2)a_3, \\ [y, [x, z]] &= [\eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, (\xi_2\sigma_1 - \xi_1\sigma_2)a_2 + (\xi_1\sigma_3\gamma + \xi_2\sigma_2)a_3] \\ &= \eta_1(\xi_2\sigma_1 - \xi_1\sigma_2)[a_1, a_2] + \eta_1(\xi_1\sigma_3\gamma + \xi_2\sigma_2)[a_1, a_3] \\ &+ \eta_2(\xi_2\sigma_1 - \xi_1\sigma_2)[a_2, a_2] \\ &= -\eta_1(\xi_2\sigma_1 - \xi_1\sigma_2)a_2 + \eta_1(\xi_1\sigma_3\gamma + \xi_2\sigma_2)\gamma a_3 + \eta_2(\xi_2\sigma_1 - \xi_1\sigma_2)a_3 \\ &= (\eta_1\xi_1\sigma_2 - \eta_1\xi_2\sigma_1)a_2 + (\eta_1\xi_1\sigma_3\gamma^2 + \eta_1\xi_2\sigma_2\gamma + \eta_2\xi_2\sigma_1 - \eta_2\xi_1\sigma_2)a_3. \end{aligned}$$

Then

$$\begin{gathered} [[x, y], z] + [y, [x, z]] \\ = (\xi_2\eta_1\sigma_1 - \xi_1\eta_2\sigma_1)a_2 + (\xi_2\eta_1\sigma_2 - \xi_1\eta_2\sigma_2)a_3 + (\eta_1\xi_1\sigma_2 - \eta_1\xi_2\sigma_1)a_2 \\ +(\eta_1\xi_1\sigma_3\gamma^2 + \eta_1\xi_2\sigma_2\gamma + \eta_2\xi_2\sigma_1 - \eta_2\xi_1\sigma_2)a_3 = (\eta_1\xi_1\sigma_2 - \xi_1\eta_2\sigma_2)a_2 \\ +(\xi_2\eta_1\sigma_2 - 2\xi_1\eta_2\sigma_2 + \eta_1\xi_1\sigma_3\gamma^2 + \eta_1\xi_2\sigma_2\gamma + \eta_2\xi_2\sigma_1)a_3. \end{gathered}$$

We have now

$$\begin{aligned}&[[x,y],z]+[y,[x,z]]-[x,[y,z]]\\&=(\eta_1\xi_1\sigma_2-\xi_1\eta_2\sigma_2)a_2\\&+(\xi_2\eta_1\sigma_2-2\xi_1\eta_2\sigma_2+\eta_1\xi_1\sigma_3\gamma^2+\eta_1\xi_2\sigma_2\gamma+\eta_2\xi_2\sigma_1)a_3\\&-(\xi_1\eta_1\sigma_2-\xi_1\eta_2\sigma_1)a_2-(\xi_1\eta_1\sigma_3\gamma^2+\xi_1\eta_2\sigma_2\gamma+\xi_2\eta_2\sigma_1-\xi_2\eta_1\sigma_2)a_3\\&=(\xi_1\eta_2\sigma_1-\xi_1\eta_2\sigma_2)a_2+(\eta_1\xi_2\sigma_2\gamma-2\xi_1\eta_2\sigma_2-\xi_1\eta_2\sigma_2\gamma)a_3.\end{aligned}$$

Thus we can see that it is possible to find the elements x, y, z such that $[x,[y,z]] \neq [[x,y],z]+[y,[x,z]]$, so that we came to a contradiction. This contradiction shows that S must be abelian.

Since $b \notin S = Ann_L(K)$, a subalgebra $\langle b, K\rangle$ is not nilpotent. The equality $[K, b] = \langle 0\rangle$ implies that $\langle b, K\rangle$ is not Lie algebra. Taking into account the information about the structure, the following conclusions can be drawn about Leibniz algebra of dimension 2 (see, for example, a survey [14]) we can suppose that $[b, b] = [b, c] = c$ where $K = Fc$.

Let x be an element such that $x \notin S$. Since S is an ideal of L, $[a, x] = \alpha a + \nu c$ for some scalars $\alpha, \nu \in F$. Since L/K is not nilpotent, $\alpha \neq 0$. The fact that L/K is a Lie algebra implies that $[x, a] = -\alpha a+\nu_1 c$ for some $\nu_1 \in F$. Since $x \notin S = Ann_L(K)$, $[x, c] = \xi c$ for some scalar $0 \neq \xi \in F$. We have

$$[x,[a,x]] = [[x,a],x]+[a,[x,x]] = [[x,a],x].$$

Then we obtain

$$\begin{aligned}\alpha[x,a]+\nu[x,c] &= [x,\alpha a+\nu c] = [x,[a,x]]\\ &= [[x,a],x] = [-\alpha a+\nu_1 c,x] = [-\alpha a,x] = -\alpha[a,x].\end{aligned}$$

Since $\alpha \neq 0$, we obtain that $[x, a] = -[a, x] - \alpha^{-1}\nu[x, c]$.

The equality $[a + K, b + K] = a + K$ implies that $[a, b] = a + \kappa_1 c$ for some scalar $\kappa_1 \in F$. From the results established above, we conclude that the following holds

$$[b,a] = -[a,b]-\kappa_1[b,c] = -a-\kappa_1 c-\kappa_1 c = -a-2\kappa_1 c.$$

If $\kappa_1 = 0$, then put $a_2 = a$. If not, then put $a_2 = a + \kappa_1 c$. For this case, we have

$$\begin{aligned}[b,a_2] &= [b,a+\kappa_1 c] = [b,a]+[b,\kappa_1 c] = -a-2\kappa_1 c+\kappa_1 c\\ &= -a-\kappa_1 c = -a_2,\end{aligned}$$

$$[a_2, b] = [a + \kappa_1 c, b] = [a, b] + [\kappa_1 c, b] = a + \kappa_1 c = a_2.$$

Put $a_1 = b$, $a_3 = c$. Then we come to the following Leibniz algebra

$$\begin{gathered} Lei_{13}(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where} \\ [a_1, a_1] = a_3 = [a_1, a_3], [a_1, a_2] = -a_2, [a_2, a_1] = a_2, \\ [a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0. \end{gathered}$$

Let us verify that an algebra with such defining relations is indeed a Leibniz algebra. Let x, y, z be arbitrary elements of L,

$$\begin{aligned} x &= \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \\ y &= \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, \\ z &= \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3, \end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \sigma_1, \sigma_2, \sigma_3$ are arbitrary scalars. Then

$$\begin{aligned} [x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\ &= \xi_1 \eta_1 a_3 - \xi_1 \eta_2 a_2 + \xi_1 \eta_3 a_3 + \xi_2 \eta_1 a_2 \\ &= (\xi_2 \eta_1 - \xi_1 \eta_2) a_2 + (\xi_1 \eta_1 + \xi_1 \eta_3) a_3, \\ [x, z] &= (\xi_2 \sigma_1 - \xi_1 \sigma_2) a_2 + (\xi_1 \sigma_1 + \xi_1 \sigma_3) a_3 \\ [y, z] &= (\eta_2 \sigma_1 - \eta_1 \sigma_2) a_2 + (\eta_1 \sigma_1 + \eta_1 \sigma_3) a_3. \end{aligned}$$

Thus

$$\begin{aligned} [x, [y, z]] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, (\eta_2 \sigma_1 - \eta_1 \sigma_2) a_2 + (\eta_1 \sigma_1 + \eta_1 \sigma_3) a_3] \\ &= \xi_1 (\eta_2 \sigma_1 - \eta_1 \sigma_2)[a_1, a_2] + \xi_1 (\eta_1 \sigma_1 + \eta_1 \sigma_3)[a_1, a_3] \\ &= -\xi_1 (\eta_2 \sigma_1 - \eta_1 \sigma_2) a_2 + \xi_1 (\eta_1 \sigma_1 + \eta_1 \sigma_3) a_3 \\ &= (\xi_1 \eta_1 \sigma_2 - \xi_1 \eta_2 \sigma_1) a_2 + (\xi_1 \eta_1 \sigma_1 + \xi_1 \eta_1 \sigma_3) a_3, \\ [[x, y], z] &= [(\xi_2 \eta_1 - \xi_1 \eta_2) a_2 + (\xi_1 \eta_1 + \xi_1 \eta_3) a_3, \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3] \\ &= (\xi_2 \eta_1 - \xi_1 \eta_2) \sigma_1 [a_2, a_1] = (\xi_2 \eta_1 - \xi_1 \eta_2) \sigma_1 a_2 \\ &= (\xi_2 \eta_1 \sigma_1 - \xi_1 \eta_2 \sigma_1) a_2, \\ [y, [x, z]] &= [\eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, (\xi_2 \sigma_1 - \xi_1 \sigma_2) a_2 + (\xi_1 \sigma_1 + \xi_1 \sigma_3) a_3] \\ &= \eta_1 (\xi_2 \sigma_1 - \xi_1 \sigma_2)[a_1, a_2] + \eta_1 (\xi_1 \sigma_1 + \xi_1 \sigma_3)[a_1, a_3] \\ &= -\eta_1 (\xi_2 \sigma_1 - \xi_1 \sigma_2) a_2 + \eta_1 (\xi_1 \sigma_1 + \xi_1 \sigma_3) a_3 \\ &= (\eta_1 \xi_1 \sigma_2 - \eta_1 \xi_2 \sigma_1) a_2 + (\eta_1 \xi_1 \sigma_1 + \eta_1 \xi_1 \sigma_3) a_3. \end{aligned}$$

Then

$$\begin{aligned}&[[x,y],z]+[y,[x,z]]\\&=(\xi_2\eta_1\sigma_1-\xi_1\eta_2\sigma_1)a_2+(\eta_1\xi_1\sigma_2-\eta_1\xi_2\sigma_1)a_2+(\eta_1\xi_1\sigma_1+\eta_1\xi_1\sigma_3)a_3\\&=(\xi_1\eta_1\sigma_2-\xi_1\eta_2\sigma_1)a_2+(\xi_1\eta_1\sigma_1+\xi_1\eta_1\sigma_3)a_3=[x,[y,z]].\end{aligned}$$

Thus, we indeed have a Leibniz algebra. □

Proposition 2. *Let L be a non-nilpotent Leibniz algebra of dimension 3 having the Leibniz kernel of dimension 1, $S = Ann_L(Leib(L))$. Suppose that $L \neq S$ and $L/Leib(L)$ is an abelian algebra. Then $L = Lei_{14}(3, F)$ is a Leibniz algebra of the following type*

$$\begin{gathered}Lei_{14}(3,F)=Fa_1\oplus Fa_2\oplus Fa_3 \text{ where}\\ [a_1,a_1]=a_3=[a_1,a_3],[a_1,a_2]=[a_2,a_1]=0,\\ [a_2,a_2]=[a_2,a_3]=[a_3,a_1]=[a_3,a_2]=[a_3,a_3]=0.\end{gathered}$$

In this case, $Leib(L) = Fa_3 = [L, L]$, $\zeta^{\text{left}}(L) = S = Fa_2 \oplus Fa_3$ is an abelian ideal. Moreover, $Fa_2 = \zeta(L) = \zeta^{\text{right}}(L)$, $Fa_1 \oplus Fa_3$ is a non-nilpotent subalgebra of L.

Proof. Let again $K = Leib(L)$, then L/K is a Lie algebra of dimension 2. As it was shown above $Ann_L^{\text{left}}(K)$ is an ideal of L such that $L/Ann_L^{\text{left}}(K)$ of dimension 1. The inclusion $Leib(L) \leq \zeta^{\text{left}}(L)$ implies that $Ann_L^{\text{left}}(K) = Ann_L(K) = S$. As it was shown above S/K is an ideal of L/K of dimension 1. In follows that S is nilpotent subalgebra of dimension 2. The factor-algebra L/K is a non-abelian Lie algebra of dimension 2. Let b be an element of L such that $b \notin S = Ann_L(K)$. Then the subalgebra $\langle b, K\rangle$ is not nilpotent. The equality $[K, b] = \langle 0\rangle$ implies that $\langle b, K\rangle$ is not a Lie algebra. Taking into account the information about the structure of a two-dimensional Leibniz algebra (see, for example, a survey [14]) we can suppose that $[b, b] = [b, c] = c$ where $K = Fc$.

Choose in S the element a such that $a \in K$. Then $S = Fa \oplus Fc$, and, moreover $[a, c] = [c, a] = 0$. Let b be an element of L such that $b \notin S$. The equality $K = [a + K, b + K]$ implies that $[a, b] = \kappa_1 c$ for some scalar $\kappa_1 \in F$. And similarly $[b, a] = \kappa_2 c$ for some scalar $\kappa_2 \in F$. We have

$$\begin{aligned}\kappa_1[b,c]&=[b,\kappa_1c]=[b,[a,b]]=[[b,a],b]+[a,[b,b]]\\&=[[b,a],b]=[\kappa_2c,b]=0.\end{aligned}$$

Since $b \notin Ann_L(K)$, $\kappa_1 = 0$.

Consider the mapping $L_b : S \to S$ defined by the rule: $L_b(y) = [b, y]$, $y \in S$. Clearly, L_b is a linear mapping, $Ker(L_b) = Ann_S^{\text{right}}(b)$, $Im(L_b) = [b, S] = K$. An isomorphism

$$K = [b, S] = Im(L_b) \cong S/Ker(L_b) = S/Ann_S^{\text{right}}(b),$$

and the fact that $dim_F(K) = 1$ implies that $Ann_S^{\text{right}}(b) = Y$ has dimension 1. Let $x = \alpha a + \beta b + \gamma c$ be an arbitrary element of L and let $y \in Y$. We have

$$[x, y] = [\alpha a + \beta b + \gamma c, y] = \alpha[a, y] + \beta[b, y] + \gamma[c, y] = 0,$$
$$[y, x] = [y, \alpha a + \beta b + \gamma c] = \alpha[y, a] + \beta[y, b] + \gamma[y, c] = \beta[y, b].$$

Furthermore, $[b, [y, b]] = [[b, y], b] + [y, [b, b]] = 0$, so that $[b, [y, x]] = [b, \beta[y, b]] = 0$. Thus, we can see that $[x, y], [y, x] \in Y$, and Y is an ideal of L. Since $b \notin Ann_L(K)$, $Y \cap K = \langle 0 \rangle$ and $S = K \oplus Y$. In particular, we obtain that S is an abelian ideal. In the coset $a + K$ we choose an element a_2 such that $a_2 \in Y$. By this choice, $[b, a_2] = 0$. From what has been established above, we obtain the following result: $[a_2, b] = 0$, so that, $Fa_2 = Y = Ann_S(b)$. Together with $[c, a_2] = 0 = [a_2, c]$ it follows that $Fa_2 = Y = \zeta(L)$.

Put now $a_1 = b$, $a_3 = c$. Then we come to the following Leibniz algebra

$$Lei_{14}(3, F) = Fa_1 \oplus Fa_2 \oplus Fa_3 \text{ where}$$
$$[a_1, a_1] = a_3 = [a_1, a_3], [a_1, a_2] = [a_2, a_1] = 0,$$
$$[a_2, a_2] = [a_2, a_3] = [a_3, a_1] = [a_3, a_2] = [a_3, a_3] = 0.$$

In this case, $Leib(L) = Fa_3 = [L, L]$, $\zeta^{\text{left}}(L) = S = Fa_2 \oplus Fa_3$ is an abelian ideal. Moreover, $Fa_2 = \zeta(L) = \zeta^{\text{right}}(L)$, $Fa_1 \oplus Fa_3$ is a non-nilpotent subalgebra of L.

Let us verify that an algebra with such defining relations is indeed a Leibniz algebra. Let x, y, z be arbitrary elements of L,

$$x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3,$$
$$y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3,$$
$$z = \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3,$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3, \sigma_1, \sigma_2, \sigma_3$ are arbitrary scalars. Then

$$[x, y] = [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3]$$

$$
\begin{aligned}
&= \xi_1\eta_1 a_3 + \xi_1\eta_3 a_3 = (\xi_1\eta_1 + \xi_1\eta_3)a_3,\\
[x,z] &= (\xi_1\sigma_1 + \xi_1\sigma_3)a_3,\\
[y,z] &= (\eta_1\sigma_1 + \eta_1\sigma_3)a_3.
\end{aligned}
$$

Thus

$$
\begin{aligned}
[x,[y,z]] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, (\eta_1\sigma_1 + \eta_1\sigma_3)a_3]\\
&= \xi_1(\eta_1\sigma_1 + \eta_1\sigma_3)[a_1, a_3]\\
&= \xi_1(\eta_1\sigma_1 + \eta_1\sigma_3)a_3 = (\xi_1\eta_1\sigma_1 + \xi_1\eta_1\sigma_3)a_3,\\
[[x,y],z] &= [(\xi_1\eta_1 + \xi_1\eta_3)a_3, \sigma_1 a_1 + \sigma_2 a_2 + \sigma_3 a_3] = 0,\\
[y,[x,z]] &= [\eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, (\xi_1\sigma_1 + \xi_1\sigma_3)a_3]\\
&= \eta_1(\xi_1\sigma_1 + \xi_1\sigma_3)[a_1, a_3]\\
&= \eta_1(\xi_1\sigma_1 + \xi_1\sigma_3)a_3 = (\eta_1\xi_1\sigma_1 + \eta_1\xi_1\sigma_3)a_3.
\end{aligned}
$$

Then

$$[[x,y],z] + [y,[x,z]] = 0 + (\eta_1\xi_1\sigma_1 + \eta_1\xi_1\sigma_3)a_3 = [x,[y,z]].$$

Thus, we have a Leibniz algebra. □

3. Main results

Theorem 1. *Let D be an algebra of derivations of the Leibniz algebra $Lei_{13}(3,F)$. Then D is isomorphic to a Lie subalgebra of $M_3(F)$ consisting of the matrices of the following form:*

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & 0 & \gamma \end{pmatrix},$$

$\beta, \gamma \in F$. Furthermore, D is abelian and D is a direct sum of two one-dimensional subalgebras.

Proof. Let $L = Lei_{13}(3,F)$ and $f \in Der(L)$. By Lemma 1, $Fa_3 = f(Leib(L)) = f(\zeta^{\text{left}}(L)) = \zeta^{\text{left}}(L)$, by Lemma 2, $f([L,L]) \leq [L,L]$. So that

$$
\begin{aligned}
f(a_1) &= \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3,\\
f(a_2) &= \beta_2 a_2 + \beta_3 a_3,\\
f(a_3) &= \gamma a_3,
\end{aligned}
$$

$\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3, \gamma \in F$. Then

$$\begin{aligned} f(a_3) &= f([a_1, a_1]) = [f(a_1), a_1] + [a_1, f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1[a_1, a_1] + \alpha_1[a_1, a_1] + \alpha_2[a_1, a_2] + \alpha_3[a_1, a_3] \\ &= 2\alpha_1 a_3 - \alpha_2 a_2 + \alpha_3 a_3 = -\alpha_2 a_2 + (2\alpha_1 + \alpha_3)a_3, \\ f(a_3) &= f([a_1, a_3]) = [f(a_1), a_3] + [a_1, f(a_3)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_3] + [a_1, \gamma a_3] \\ &= \alpha_1[a_1, a_3] + \gamma[a_1, a_3] = \alpha_1 a_3 + \gamma a_3 = (\alpha_1 + \gamma)a_3, \\ f(a_2) &= f([a_2, a_1]) = [f(a_2), a_1] + [a_2, f(a_1)] \\ &= [\beta_2 a_2 + \beta_3 a_3, a_1] + [a_2, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \beta_2[a_2, a_1] + \alpha_1[a_2, a_1] = \beta_2 a_2 + \alpha_1 a_2 = (\beta_2 + \alpha_1)a_2. \end{aligned}$$

Thus, we obtain

$$\begin{gathered} -\alpha_2 a_2 + (2\alpha_1 + \alpha_3)a_3 = (\alpha_1 + \gamma)a_3 = \gamma a_3, \\ (\beta_2 + \alpha_1)a_2 = \beta_2 a_2 + \beta_3 a_3. \end{gathered}$$

It follows that

$$\alpha_1 = \alpha_2 = 0, \alpha_3 = \gamma, \beta_3 = 0.$$

Hence, $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & 0 & \gamma \end{pmatrix},$$

$\beta, \gamma \in F$.

Conversely, let x, y be arbitrary elements of L,

$$\begin{aligned} x &= \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \\ y &= \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3, \end{aligned}$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ are arbitrary scalars. Then

$$\begin{aligned} [x, y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\ &= \xi_1\eta_1[a_1, a_1] + \xi_1\eta_2[a_1, a_2] + \xi_1\eta_3[a_1, a_3] + \xi_2\eta_1[a_2, a_1] \\ &= \xi_1\eta_1 a_3 - \xi_1\eta_2 a_2 + \xi_1\eta_3 a_3 + \xi_2\eta_1 a_2 \\ &= (\xi_2\eta_1 - \xi_1\eta_2)a_2 + (\xi_1\eta_1 + \xi_1\eta_3)a_3, \end{aligned}$$

$$\begin{aligned} f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\ &= \xi_1 \gamma a_3 + \xi_2 \beta a_2 + \xi_3 \gamma a_3 \\ &= \xi_2 \beta a_2 + (\xi_1 \gamma + \xi_3 \gamma) a_3, \\ f(y) &= \eta_2 \beta a_2 + (\eta_1 \gamma + \eta_3 \gamma) a_3, \\ f([x,y]) &= f((\xi_2 \eta_1 - \xi_1 \eta_2) a_2 + (\xi_1 \eta_1 + \xi_1 \eta_3) a_3) \\ &= (\xi_2 \eta_1 - \xi_1 \eta_2) f(a_2) + (\xi_1 \eta_1 + \xi_1 \eta_3) f(a_3) \\ &= (\xi_2 \eta_1 - \xi_1 \eta_2) \beta a_2 + (\xi_1 \eta_1 + \xi_1 \eta_3) \gamma a_3 \\ &= (\xi_2 \eta_1 \beta - \xi_1 \eta_2 \beta) a_2 + (\xi_1 \eta_1 \gamma + \xi_1 \eta_3 \gamma) a_3. \end{aligned}$$

Thus,

$$\begin{gathered} [f(x), y] + [x, f(y)] \\ = [\xi_2 \beta a_2 + (\xi_1 \gamma + \xi_3 \gamma) a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\ + [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_2 \beta a_2 + (\eta_1 \gamma + \eta_3 \gamma) a_3] \\ = \xi_2 \beta \eta_1 [a_2, a_1] + \xi_1 (\eta_1 \gamma + \eta_3 \gamma)[a_1, a_3] + \xi_1 \eta_2 \beta [a_1, a_2] \\ = \xi_2 \beta \eta_1 a_2 + (\xi_1 \eta_1 \gamma + \xi_1 \eta_3 \gamma) a_3 - \beta \xi_1 \eta_2 a_2 \\ = (\xi_2 \beta \eta_1 - \beta \xi_1 \eta_2) a_2 + (\xi_1 \eta_1 \gamma + \xi_1 \eta_3 \gamma) a_3, \end{gathered}$$

so that $f([x,y]) = [f(x), y] + [x, f(y)]$.

Denote by A the subset of $M_3(F)$ consisting of the matrices of the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \gamma & 0 & \gamma \end{pmatrix},$$

$\gamma \in F$. Denote by B the subset of $M_3(F)$ consisting of the matrices of the following form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\beta \in F$.

It is not hard to see that A, B are one-dimensional subalgebras of $\Xi(L)$, $[B, A] = \langle 0 \rangle$, $\Xi(L)$ is a direct sum $A \oplus B$. □

Theorem 2. *The algebra of derivations of the Leibniz algebra $Lei_{14}(3, F)$ is isomorphic to the derivation algebra of $Lei_{13}(3, F)$.*

Proof. Let $L = Lei_{14}(3, F)$ and $f \in Der(L)$. By Lemma 1, $\zeta^{\text{left}}(L) = f(\zeta^{\text{left}}(L)) = f(Fa_2 \oplus Fa_3)$, $Fa_2 = \zeta(L) = f(\zeta(L))$, by Lemma 2, $Fa_3 =$

$f(Leib(L)) = f([L,L]) \le [L,L]$. So that

$$f(a_1) = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3,$$
$$f(a_2) = \beta a_2,$$
$$f(a_3) = \gamma a_3,$$

$\alpha_1, \alpha_2, \alpha_3, \beta, \gamma \in F$. Then

$$\begin{aligned} f(a_3) &= f([a_1,a_1]) = [f(a_1),a_1] + [a_1, f(a_1)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_1] + [a_1, \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3] \\ &= \alpha_1 [a_1,a_1] + \alpha_1 [a_1,a_1] + \alpha_3 [a_1,a_3] \\ &= 2\alpha_1 a_3 + \alpha_3 a_3 = (2\alpha_1 + \alpha_3) a_3, \\ f(a_3) &= f([a_1,a_3]) = [f(a_1),a_3] + [a_1, f(a_3)] \\ &= [\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3, a_3] + [a_1, \gamma a_3] \\ &= \alpha_1 [a_1,a_3] + \gamma [a_1,a_3] = (\alpha_1 + \gamma) a_3. \end{aligned}$$

Thus, we obtain

$$(2\alpha_1 + \alpha_3) a_3 = (\alpha_1 + \gamma) a_3 = \gamma a_3.$$

It follows that $\alpha_1 = \alpha_2 = 0, \alpha_3 = \gamma$. Hence, $\Xi(f)$ is the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta & 0 \\ \gamma & 0 & \gamma \end{pmatrix},$$

$\beta, \gamma \in F$.

Conversely, let x, y be arbitrary elements of L,

$$x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3,$$
$$y = \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3,$$

where $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ are arbitrary scalars. Then

$$\begin{aligned} [x,y] &= [\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3, \eta_1 a_1 + \eta_2 a_2 + \eta_3 a_3] \\ &= \xi_1 \eta_1 [a_1,a_1] + \xi_1 \eta_3 [a_1,a_3] = \xi_1 \eta_1 a_3 + \xi_1 \eta_3 a_3 \\ &= (\xi_1 \eta_1 + \xi_1 \eta_3) a_3, \\ f(x) &= f(\xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3) = \xi_1 f(a_1) + \xi_2 f(a_2) + \xi_3 f(a_3) \\ &= \xi_1 \gamma a_3 + \xi_2 \beta a_2 + \xi_3 \gamma a_3 = \xi_2 \beta a_2 + (\xi_1 \gamma + \xi_3 \gamma) a_3, \end{aligned}$$

$$\begin{aligned} f(y) &= \eta_2\beta a_2 + (\eta_1\gamma + \eta_3\gamma)a_3, \\ f([x,y]) &= f((\xi_1\eta_1 + \xi_1\eta_3)a_3) = (\xi_1\eta_1 + \xi_1\eta_3)f(a_3) \\ &= (\xi_1\eta_1 + \xi_1\eta_3)\gamma a_3 = (\xi_1\eta_1\gamma + \xi_1\eta_3\gamma)a_3, \end{aligned}$$

so that $f([x,y]) = [f(x),y] + [x,f(y)]$.

As we can see that the algebra of derivations of Leibniz algebra $Lei_{14}(3,F)$ is isomorphic to the algebra of derivations of Leibniz algebra $Lei_{13}(3,F)$. □

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