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On block-supersymmetric polynomials on Banach spaces

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Abstract. We investigate algebras of block-supersymmetric polynomials on an infinite-dimensional Banach space $\ell_p(\mathbb{C}^s)$ of absolutely p-convergent sequences of vectors in $\mathbb{C}^s$, and ring structures on the set $\mathcal{M}_p$ of point evaluation homomorphisms of these algebras. In particular, we establish some necessary and sufficient conditions for a polynomial to be block-supersymmetric. Also, we describe complex ring homomorphisms of $\mathcal{M}_p$.

Introduction and preliminaries

Theory of symmetric functions is a natural subject in the classical theory of invariants, algebras, and combinatorics (see e.g. [19]). The main results and methods of this theory have nontrivial extensions to infinite-dimensional spaces. Symmetric (invariant) polynomials with respect to some groups of symmetries of Banach spaces ℓ_p were considered first in [11, 20]. Algebras of symmetric analytic functions on Banach spaces, their homomorphisms, and spectra have been studied in [1, 3, 6, 7, 23, 24, 25] and others (see [5] and the references therein).

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It is well-known that in the algebra of all symmetric polynomials of n scalar variables $x_1, \ldots, x_n$ with respect to permutations of the variables there is an algebraic basis of power symmetric polynomials $F_1, \ldots, F_n$,

$$F_k(x) = \sum_{i=1}^{n} x_i^k.$$

It means that any symmetric polynomial of n variables can be uniquely represented as an algebraic combination of $F_1, \ldots, F_n$. According to [11, 20], this result can be extended to some infinite-dimensional cases. It was shown that polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k, \quad k \geq \lceil p \rceil$$

form an algebraic basis in the algebra of all symmetric polynomials on ℓ_p, $1 \leq p < \infty$, where $\lceil p \rceil$ is the ceiling of p.

Let $\ell_p(\mathbb{C}^s)$, $1 \leq p < \infty$ be the Banach space of sequences

$$x = (x_1, x_2, \ldots, x_j, \ldots)$$

such that $x_j = (x_j^{(1)}, \ldots, x_j^{(s)})$ are vectors in $\mathbb{C}^s$ for every fixed $j \in \mathbb{N}$, and the series

$$\|x\|_p := \Big(\sum_{j=1}^{\infty} \sum_{r=1}^{s} \Big| x_j^{(r)} \Big|^p \Big)^{1/p}$$

converges. A polynomial P on $\ell_p(\mathbb{C}^s)$ is *block-symmetric (or vector-symmetric)* if

$$P(x_1, x_2, \ldots, x_m, \ldots) = P(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(m)}, \ldots)$$

for every permutation (injective mapping) $\sigma \colon \mathbb{N} \to \mathbb{N}$ and every $x_m \in \mathbb{C}^s$.

It is known [17] that polynomials

$$H^{\mathbf{k}}(x) = H^{k_1, k_2, \ldots, k_s}(x) = \sum_{j=1}^{\infty} \prod_{r=1}^{s} (x_j^{(r)})^{k_r}$$

for $|\mathbf{k}| \geq \lceil p \rceil$ form an algebraic basis in the algebra $\mathcal{P}_{vs}(\ell_p(\mathbb{C}^s))$ of block-symmetric polynomials on $\ell_p(\mathbb{C}^s)$, $1 \leq p < \infty$, where $x = (x_1, x_2, \ldots)$ are in $\ell_p(\mathbb{C}^s)$. Here we use the standard notations for multi-indexes $\mathbf{k} = (k_1, k_2, \ldots, k_s)$, and $|\mathbf{k}| = k_1 + k_2 + \cdots + k_s$. Block-symmetric (or

MacMahon) polynomials on finite-dimensional spaces where investigated in [21].

A supersymmetric polynomial P of $m+n$ complex variables

$$(y_1,\dots,y_m|x_1,\dots,x_n)\in\mathbb{C}^m\times\mathbb{C}^n$$

is defined as a polynomials that is invariant with respect to separate permutations $y_1,\dots,y_m$ and $x_1,\dots,x_n$ and satisfy the following cancelation law: $P(t,\dots,y_m|t,\dots,x_n)$ does not depend on t. In [22] Stembridge proved that polynomials $T_k=F_k(x)-F_k(y)$, $k\in\mathbb{N}$ form the set of generators of supersymmetric polynomials. Note that in [14] it was proved that this algebra is not finitely generated, that is, there does not exist a finite number of generators. Supersymmetric polynomials and supersymmetric analytic functions on sequences Banach spaces and some their generalizations were investigated in [4, 8, 9, 10, 13].

Let $\mathbb{Z}_0=\mathbb{Z}\setminus\{0\}$ and $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$ be the Banach space of two-sides sequences

$$\begin{aligned} z &= (\dots,z_{-n},\dots,z_{-1},z_1,\dots,z_n,\dots)=(y|x)\\ &= (\dots,y_n,\dots,y_1|x_1,\dots,x_n,\dots) \end{aligned}$$

such that

$$\|z\|_p:=\Big(\sum_{i=-\infty}^{\infty}\|z_i\|_p^p\Big)^{1/p}=\Big(\sum_{i=-\infty}^{\infty}\sum_{j=1}^{s}|z_i^{(j)}|^p\Big)^{1/p}<\infty,$$

where $1\le p<\infty$. Here the vectors $x=(x_1,x_2,\dots)$ and $y=(y_1,y_2,\dots)$ are in $\ell_p(\mathbb{C}^s)$, that is, the vector coordinates $x_i=(x_i^{(1)},\dots,x_i^{(s)})$ and $y_i=(y_i^{(1)},\dots,y_i^{(s)})$ are vectors in $\mathbb{C}^s$ with $z_n=x_n$, $z_{-n}=y_n$ for $n\in\mathbb{N}$.

Let us consider the following polynomials on $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$:

$$T^{\mathbf{k}}(z)=H^{\mathbf{k}}(x)-H^{\mathbf{k}}(y)=\sum_{j=1}^{\infty}\prod_{r=1}^{s}(x_j^{(r)})^{k_r}-\sum_{j=1}^{\infty}\prod_{r=1}^{s}(y_j^{(r)})^{k_r}, \tag{1}$$

$\mathbf{k}=(k_1,\dots,k_s)$, $|\mathbf{k}|\ge\lceil p\rceil$.

Definition 1. *A polynomial P on $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$ is called block-supersymmetric (or vector-supersymmetric) if it is an algebraic combination of polynomials $\{T^{\mathbf{k}}\}_{|\mathbf{k}|=\lceil p\rceil}^{\infty}$. That is, P can be written as a finite sum of finite products of polynomials in $\{T^{\mathbf{k}}\}_{|\mathbf{k}|=\lceil p\rceil}^{\infty}$ and constants. We denote by $\mathcal{P}_{vsup}(\ell_p)$ the algebra of all block-supersymmetric polynomials on $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$.*

Block-supersymmetric polynomials and their possible applications were studied in [8, 12, 15, 16]. In this paper, we continue to investigate algebras of block-supersymmetric polynomials on $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$ and ring structures on point evaluation homomorphisms of these algebras. In Section 1, we introduce a "supersymmetric semigroup" of symmetries $\mathfrak{S}$ and prove that a polynomial is supersymmetric if and only if it is $\mathfrak{S}$-invariant. In Section 2, we consider the case if $\mathbb{C}^s$ is an algebra and investigate a ring structure that appears on the set of point evaluation functionals on the algebra of block-supersymmetric polynomials.

1. Generators of block-supersymmetric polynomials

For a given $x = (x_1, x_2, \ldots) \in \ell_p(\mathbb{C}^s)$ we denote by $\operatorname{supp}(x)$ the support of x, that is,

$$\operatorname{supp}(x) = \{i \in \mathbb{N} \colon x_i \neq 0\}.$$

Let $x = (x_1, x_2, \ldots)$ and $v = (v_1, v_2, \ldots)$ are in $\ell_p(\mathbb{C}^s)$. We say that $x \sim v$ if there exists a bijection σ from $\operatorname{supp}(x)$ to $\operatorname{supp}(v)$ such that $x_i = v_{\sigma(i)}$ for every $i \in \operatorname{supp}(x)$.

Using Theorem 1 and Corollary 1 in [18], we can get the following proposition.

Proposition 1. *If there is an integer m such that $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(v)$ for every multi-index $\mathbf{k}$ such that $|\mathbf{k}| > m$, then $x \sim v$.*

Conversely, if $x \sim v$, then $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(v)$ for every $\mathbf{k}$, $|\mathbf{k}| \geq \lceil p \rceil$.

Proof. Note that if x has infinity many nonzero vector coordinates x_i then x is equivalent to a vector $\widetilde{x}$ such that all vector coordinates of $\widetilde{x}$ are nonzero and $H^{\mathbf{k}}(x) = H^{\mathbf{k}}(\widetilde{x})$ for every $\mathbf{k}$, $|\mathbf{k}| \geq \lceil p \rceil$. Indeed, we can obtain $\widetilde{x}$ from x removing zero coordinates. Thus, without the loss of generality, we can assume that either x or v has a finite number of nonzero vector coordinates or all vector coordinates x_i and v_i of both x and v respectively, are nonzero vectors in $\mathbb{C}^s$. But for this case, in [18] it was proved that x and v must be equivalent.

The converse statement is evident. $\square$

In [15] it was proved that the polynomials $T^{\mathbf{k}}$ are algebraically independent in $\mathcal{P}_{vsup}(\ell_1)$. From here we immediately have that $T^{\mathbf{k}}$ are algebraically independent in $\mathcal{P}_{vsup}(\ell_p)$, $|\mathbf{k}| \geq \lceil p \rceil$. Thus we have the following proposition.

Proposition 2. *The sequence of polynomials* $\left\{T^{\mathbf{k}}\right\}_{|\mathbf{k}|=\lceil p\rceil}^{\infty}$ *is an algebraic basis in* $\mathcal{P}_{vsup}(\ell_p)$.

We need the following operation "$\bullet$" of the "symmetric addition" on $\ell_p(\mathbb{C}^s)$ (see [6, 18]):

$$x \bullet u = (x_1, u_1, \ldots, x_n, u_n, \ldots), \quad x, u \in \ell_p(\mathbb{C}^s).$$

Clearly, $H^{\mathbf{k}}(x \bullet u) = H^{\mathbf{k}}(x) + H^{\mathbf{k}}(u)$ for every $\mathbf{k}$, $|\mathbf{k}| > \lceil p\rceil$.

The operation "$\bullet$" was extended to $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$ in [15] by

$$z \bullet w = (y \bullet v | x \bullet u) = (\ldots, v_n, y_n, \ldots, v_1, y_1 | x_1, u_1, \ldots, x_n, u_n, \ldots),$$

where $z = (y|x), w = (v|u)$. We denote by $z^- = (y|x)^- = (x, y)$ the "symmetric inverse" element to z. It is easy to see that $(z^-)^- = z$ and $z \bullet z^- \sim (0|0)$.

From the definitions and Proposition 1 we have the following properties of "$\bullet$" (cf. [15]).

Proposition 3. *The following statements hold:*

1. $T^{\mathbf{k}}(z \bullet w) = T^{\mathbf{k}}(z) + T^{\mathbf{k}}(w)$ *for every* $\mathbf{k} = (k_1, \ldots, k_s)$, $|\mathbf{k}| \geq \lceil p\rceil$.
2. $z \sim 0$ *if and only if we can write* $z = (d|b)$ *for some* $d, b \in \ell_p(\mathbb{C}^s)$ *and* $H^{\mathbf{k}}(d) = H^{\mathbf{k}}(b)$ *for all* $\mathbf{k}$, $|\mathbf{k}| \geq \lceil p\rceil$, *and if and only if* $d \sim b$.

Let us introduce the following equivalence relation on $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$. Given

$$z = (y|x) = (\ldots, y_2, y_1 | x_1, x_2, \ldots)$$

and

$$z' = (y'|x') = (\ldots, y'_2, y'_1 | x'_1, x'_2, \ldots)$$

in $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$ are equivalent $(z \sim z')$ if and only if there are $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ in $\ell_p(\mathbb{C}^s)$ such that $x \bullet a \sim x' \bullet b$ and $y \bullet a \sim y' \bullet b$ in $\ell_p(\mathbb{C}^s)$.

The quotient set $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})/\sim$ will be denoted by $\mathcal{M}_p$. Let $[z] \in \mathcal{M}_p$ be the equivalence class containing $z \in \ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$. Clearly, any block-supersymmetric polynomials P is well-defined on $\mathcal{M}$ by $P([z]) = P(z)$. The operations introduced above can be extended to $\mathcal{M}_p$ by

$$[z] + [w] = [z \bullet w] \quad \text{and} \quad -[z] = [z^-], \quad z, w \in \ell_p(\mathbb{C}^s_{\mathbb{Z}_0}). \tag{2}$$

It is easy to check that so defined operations on $\mathcal{M}_p$ do not depend on representatives and $(\mathcal{M}_p, +, -)$ is a commutative group, where $\mathbb{O} = (0|0)$ is the zero element.

Let $[z] = [(y|x)]$. We say that $(y|x)$ is an irreducible representative of $[z]$ if $x_i \neq y_j$ for every $i \in \operatorname{supp}(x)$ and $j \in \operatorname{supp}(y)$. It is easy to check (see e.g. [8]) that any $[z] \in \mathcal{M}_p$ has an irreducible representative.

Theorem 1. *Let $z = (y|x)$ and $z' = (y'|x')$ be in $\ell_p(\mathbb{C}^s)$. Then $z \sim z'$ if and only if $T^{\mathbf{k}}(z) = T^{\mathbf{k}}(z')$ for every $\mathbf{k}$, $|\mathbf{k}| \geq \lceil p \rceil$.*

Proof. Without loss of generality, we may assume that $(y|x)$ and $(y'|x')$ are irreducible representatives of $[z]$ and $[z']$ respectively. If $T^{\mathbf{k}}(z) = T_k(z')$, then $T^{\mathbf{k}}(z \bullet z'^{-}) = 0$, that is,

$$(x \bullet y'|y \bullet x') \sim 0.$$

Thus,

$$H^{\mathbf{k}}(x \bullet y') = H^{\mathbf{k}}(y \bullet x'), \quad |\mathbf{k}| \geq \lceil p \rceil.$$

Hence, $(x \bullet y') \sim (y \bullet x')$, that is, there exists a bijection from $\operatorname{supp}(x \bullet y')$ to $\operatorname{supp}(y \bullet x')$ such that, up to this bijection, the coordinates of $x \bullet y'$ are the same as the coordinates of $y \bullet x'$. If x is not equivalent to x', then there is a nonzero vector $a \in \mathbb{C}^s$ such that a is a coordinate of x with the multiplicity n and a is a coordinate of x' with the multiplicity n', and $n \neq n'$ (if a is not a coordinate of say x', then $n' = 0$). Suppose that $n > n'$. Then the relation $(x \bullet y') \sim (y \bullet x')$ is possible only if a is a coordinate of y with the multiplicity $n - n' > 0$. Thus, x and y have the same vector coordinate a of a nonzero multiplicity. But $(y|x)$ is irreducible. A contradiction.

Let $z \sim z'$. Then $x \bullet a \sim x' \bullet b$ and $y \bullet a \sim y' \bullet b$ for some a and b in $\ell_p(\mathbb{C}^s)$. Thus, for every $\mathbf{k}$, $|\mathbf{k}| \geq \lceil p \rceil$,

$$T^{\mathbf{k}}(z) = T^{\mathbf{k}}(x \bullet a|y \bullet a) = T^{\mathbf{k}}(x' \bullet b|y' \bullet b) = T^{\mathbf{k}}(z').$$

□

Let us denote by $\mathfrak{S}$ the following "semigroup of supersymmetry" consisting of affine operators on $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$ of the form:

$$(\ldots, y_2, y_1|x_1, x_2, \ldots) \mapsto (\ldots, y_2, y_1|x_{\sigma(1)}, x_{\sigma(2)}, \ldots),$$

$$(\ldots, y_2, y_1|x_1, x_2, \ldots) \mapsto (\ldots, y_{\mu(2)}, y_{\mu(1)}|x_1, x_2, \ldots),$$

and

$$(y|x) \mapsto (y \bullet a|x \bullet a),$$

where σ and μ are arbitrary permutations on $\mathbb{N}$, and $a \in \ell_p(\mathbb{C}^s)$. A polynomial P is said to be $\mathfrak{S}$-*symmetric* if $P(A(x)) = P(x)$ for every $A \in \mathfrak{S}$ and $x \in \ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$.

Theorem 2. *A polynomial P on $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$ is block-supersymmetric if and only if it is $\mathfrak{S}$-symmetric.*

Proof. Clearly, every block-supersymmetric polynomial is $\mathfrak{S}$-symmetric. Let P be $\mathfrak{S}$-symmetric. Consider polynomials $H^{\mathbf{k}}_+(y|x) = H^{\mathbf{k}}(x)$, and $H^{\mathbf{k}}_-(y|x) = H^{\mathbf{k}}(y)$, $|\mathbf{k}| \geq \lceil p \rceil$. From the definition of $\mathfrak{S}$ it follows that P is an algebraic combination of $H^{\mathbf{k}}_+$ and $H^{\mathbf{k}}_-$, $\lceil p \rceil \leq |\mathbf{k}| \leq \deg P$ because, P can be represented as a block-symmetric polynomial of x with coefficients that are block-symmetric polynomial of y. Thus, P is an algebraic combination of polynomials $T^{\mathbf{k}} = H^{\mathbf{k}}_+ - H^{\mathbf{k}}_-$ and $Q^{\mathbf{k}} = H^{\mathbf{k}}_+ + H^{\mathbf{k}}_-$. Note, that

$$Q^{\mathbf{k}}(y \bullet a|x \bullet a) = Q^{\mathbf{k}}(y|x) + 2H^{\mathbf{k}}(a)$$

Let $q = q(t_1, \ldots, t_m, r_1, \ldots, r_m)$ be a polynomial of several variables for some appropriate m such that

$$P(y|x) = q\big(T^{\mathbf{k}_1}(y|x), \ldots, T^{\mathbf{k}_m}(y|x), Q^{\mathbf{k}_1}(y|x), \ldots, Q^{\mathbf{k}_m}(y|x)\big),$$

$\lceil p \rceil \leq |\mathbf{k}| \leq \deg P$. It is known [2] that for an arbitrary finite sequence of complex numbers $(\lambda_1, \ldots, \lambda_m)$ there exists a vector $a \in \ell_p(\mathbb{C}^s)$ such that $H^{\mathbf{k}_1}(a) = \lambda_1$, ..., $H^{\mathbf{k}_m}(a) = \lambda_m$. Then

$$\begin{aligned}
P(y|x) &= P(y \bullet a|x \bullet a) \\
&= q\big(T^{\mathbf{k}_1}(y \bullet a|x \bullet a), \ldots, T^{\mathbf{k}_m}(y \bullet a|x \bullet a), Q^{\mathbf{k}_1}(y \bullet a|x \bullet a), \\
&\ldots, Q^{\mathbf{k}_m}(y \bullet a|x \bullet a)\big) \\
&= q\big(T^{\mathbf{k}_1}(y|x), \ldots, T^{\mathbf{k}_m}(y|x), Q^{\mathbf{k}_1}(y \bullet a|x \bullet a), \\
&\ldots, Q^{\mathbf{k}_m}(y \bullet a|x \bullet a)\big) \\
&= q\big(T^{\mathbf{k}_1}(y|x), \ldots, T^{\mathbf{k}_m}(y|x), Q^{\mathbf{k}_1}(y|x) + \lambda_1, \ldots, Q^{\mathbf{k}_m}(y|x) + \lambda_m\big).
\end{aligned}$$

But it is possible only if the polynomial q does not depend on variables $r_1, \ldots, r_m$ that completes the proof. □

Let us consider the finite-dimensional case. For any fixed positive integers n, m, and s we consider the space $\mathbb{C}^s_{n,m}$ consisting of the vectors

$$(y_m, \ldots, y_1|x_1, \ldots, x_n)$$

such that each x_i and y_j are in $\mathbb{C}^s$. The space $\mathbb{C}^s_{n,m}$ can be naturally included into $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$ by

$$(y_m, \ldots, y_1|x_1, \ldots, x_n) \mapsto (\ldots, 0, y_m, \ldots, y_1|x_1, \ldots, x_n, 0, \ldots).$$

The restrictions of block-supersymmetric polynomials on $\ell_1(\mathbb{C}^s_{\mathbb{Z}_0})$ to $\mathbb{C}^s_{n,m}$ define block-supersymmetric polynomials on $\mathbb{C}^s_{n,m}$. Hence, the restrictions of $H^{\mathbf{k}}$, $|\mathbf{k}| \geq 1$ are generators in the algebra of block-supersymmetric polynomials on $\mathbb{C}^s_{n,m}$. Of course, this family of generators is algebraically dependent because an infinity sequence of polynomials can not be algebraically independent in a finite dimensional space. On the other hand, as we mentioned above, even for $s = 1$ the algebra of block-supersymmetric polynomials on $\mathbb{C}^s_{n,m}$ is infinitely generated [14] and so, it is infinitely generated for any other s as well.

2. Ring structures associated with block-supersymmetric polynomials

The fact that the family $\{T^{\mathbf{k}}\}$, $|\mathbf{k}| \geq \lceil p \rceil$ forms an algebraic basis in $\mathcal{P}_{vsup}(\ell_p)$ guarantees us that every complex homomorphism φ of $\mathcal{P}_{vsup}(\ell_p)$ can be defined by a family of numbers $\{c_{\mathbf{k}}\}$, $|\mathbf{k}| \geq \lceil p \rceil$ so that $\varphi(T^{\mathbf{k}}) = c_{\mathbf{k}}$. However, for applications to algebras of block-supersymmetric analytic functions (see [15]), it is important to have a description of point evaluation functionals. For every $z \in \ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$ the point evaluation functional δ_z is defined as

$$\delta_z(P) = P(z), \quad P \in \mathcal{P}_{vsup}(\ell_p).$$

Clearly, $\delta_z = \delta_{z'}$ if and only if $T^{\mathbf{k}}(z) = T^{\mathbf{k}}(z')$, $|\mathbf{k}| \geq \lceil p \rceil$. From Theorem 1 it follows that $\delta_z = \delta_{z'}$ if and only if $z \sim z'$. It is known [8] that the set of point evaluation functionals on the algebra of block-supersymmetric polynomials admits a ring structure if there is a multiplication on $\mathbb{C}^s$. We suppose that an operation of multiplication "$\odot_{\mathcal{A}}$" is defined on $\mathbb{C}^s$ so that $\mathbb{C}^s$ with "$\odot_{\mathcal{A}}$" and the usual operation of addition is an algebra. We denote the algebra $(\mathbb{C}^s, +, \odot_{\mathcal{A}})$ by $\mathcal{A}$.

Let $x, y \in \ell_1(\mathbb{C}^s)$. Then the symmetric multiplication $x \diamond_{\mathcal{A}} y$ associated with the multiplication $\odot_{\mathcal{A}}$ is the set of elements $x_i \odot_{\mathcal{A}} y_j$, $i, j \in \mathbb{N}$ enumerated in some fixed order, where $x_i = (x_i^{(1)}, \ldots, x_i^{(s)}), y_i = (y_i^{(1)}, \ldots, y_i^{(s)}) \in \mathbb{C}^s$. Let now $u = (y|x)$ and $v = (d|b)$ be in $\mathbb{C}^s_{\mathbb{Z}_0}(\ell_p)$. As in [13], we define

$$u \diamond_{\mathcal{A}} v = [((y \diamond_{\mathcal{A}} b) \bullet (x \diamond_{\mathcal{A}} d)|(y \diamond_{\mathcal{A}} d) \bullet (x \diamond_{\mathcal{A}} b))].$$

The operations above can be lifted to $\mathcal{M}_p$ by

$$[u]+[v]=[u\bullet v],\quad -[u]=[u^-],\quad [u]\overset{\mathcal{A}}{\times}[v]=[u\diamond_{\mathcal{A}} v].$$

Proposition 4 ([8])**.** *The set* $(\mathcal{M},+,\overset{\mathcal{A}}{\times})$ *is a ring with zero* $\mathbb{O}=[(\mathbf{0}|\mathbf{0})]$. *If* e *is a unit of* $\mathcal{A}$, *then* $\mathbb{I}=[(\mathbf{0}|(e,0\ldots))]$ *is a unit of* $(\mathcal{M},+,\overset{\mathcal{A}}{\times})$. *If* $\mathcal{A}$ *is commutative, then* $(\mathcal{M},+,\overset{\mathcal{A}}{\times})$ *is a commutative ring.*

Let θ be a multiplicative map, $\theta\colon\mathcal{A}\to\mathbb{C}$ such that if $x\in\ell_p(\mathbb{C}^s)$, then the sequence $(\theta(x_n))$ is in ℓ_1. We consider the following function on $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$ associated with θ,

$$T_\theta(y|x)=\sum_{n=1}^{\infty}\theta(x_n)-\sum_{n=1}^{\infty}\theta(y_n).$$

Clearly, if θ is a polynomial on $\mathbb{C}^s$, then T_θ is a polynomial on $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$.

Theorem 3. *For every* θ *defined as above,*

(i) $T_\theta(u\bullet v)=T_\theta(u)+T_\theta(v)$ *and* $T_\theta(u\diamond_{\mathcal{A}} v)=T_\theta(u)T_\theta(v)$, $u,\ v\in\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$;

(ii) the function T_θ *can be lifted to a function* τ_θ *on* $\mathcal{M}_p$ *by*

$$\tau_\theta[(y|x)]=T_\theta(y|x),$$

and τ_θ *is a ring homomorphism.*

Proof. (i) Let $u=(y|x)$ and $v=(d|b)$. Then

$$T_\theta(u\bullet v)=\sum_{n=1}^{\infty}\theta(x_n)-\sum_{n=1}^{\infty}\theta(y_n)+\sum_{n=1}^{\infty}\theta(b_n)-\sum_{n=1}^{\infty}\theta(d_n)=T_\theta(u)+T_\theta(v).$$

Also, by the multiplicativity of θ,

$$\begin{aligned}
T_\theta(u\diamond_{\mathcal{A}} v)&=T_\theta((y\diamond_{\mathcal{A}} b)\bullet(x\diamond_{\mathcal{A}} d)|(y\diamond_{\mathcal{A}} d)\bullet(x\diamond_{\mathcal{A}} b))\\
&=\sum_{n,m=1}^{\infty}\theta(x_nb_m)+\sum_{n,m=1}^{\infty}\theta(y_nd_m)-\sum_{n,m=1}^{\infty}\theta(x_nd_m)-\sum_{n,m=1}^{\infty}\theta(y_nb_m)\\
&=\Big(\sum_{n=1}^{\infty}\theta(x_n)-\sum_{n=1}^{\infty}\theta(y_n)\Big)\Big(\sum_{n=1}^{\infty}\theta(b_n)-\sum_{n=1}^{\infty}\theta(d_n)\Big)\\
&=T_\theta(y|x)T_\theta(d|b)=T_\theta(u)T_\theta(v).
\end{aligned}$$

(ii) Let $u = (y|x) \sim u' = (y'|x')$ in $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$. Then there are a and b in $\ell_p(\mathbb{C}^s)$ such that, up to permutations, $(y \bullet a|x \bullet a) = (y' \bullet b|x' \bullet b)$. Thus,

$$T_\theta(y|x) = T_\theta(y \bullet a|x \bullet a) = T_\theta(y' \bullet b|x' \bullet b) = T_\theta(y'|x').$$

Hence, $\tau_\theta([u]) = T_\theta(u)$ does not depend on representatives. From (i) it follows that

$$\tau_\theta([u] + [v]) = \tau_\theta([u]) + \tau_\theta([v]), \quad \text{and} \quad \tau_\theta([u] \overset{\mathcal{A}}{\times} [v]) = \tau_\theta([u])\tau_\theta([v])$$

for every $[u]$, and $[v]$ in $\ell_p(\mathbb{C}^s_{\mathbb{Z}_0})$. □

Example 1. Let $x \odot_{\mathcal{A}} y = (x^{(1)}y^{(1)}, \dots, x^{(s)}y^{(s)})$ be the coordinate-wise multiplication, x, $y \in \mathbb{C}^s$, and

$$\theta\big((x^{(1)}, \dots, x^{(s)})\big) = \big(x^{(1)}\big)^{k_1} \cdots \big(x^{(s)}\big)^{k_s}$$

for some multi-index $\mathbf{k} = k_1, \dots, k_s, k_i \in \mathbb{Z}_0$, $|\mathbf{k}| \neq \lceil p \rceil$. Then θ satisfies conditions of Theorem 3, and $T_\theta = T^{\mathbf{k}}$.

We will use notations $u \diamond v$ instead of $u \diamond_{\mathcal{A}} v$ and $[u][v]$ instead of $[u] \overset{\mathcal{A}}{\times} [v]$ if $\mathcal{A}$ is $\mathbb{C}^s$ with the coordinate-wise multiplication as in Example 1. Thus, we have the following corollary.

Corollary 1. *For every* $\mathbf{k} = k_1, \dots, k_s, k_i \in \mathbb{Z}_0, |\mathbf{k}| \neq \lceil p \rceil$, $T^{\mathbf{k}}(u \diamond v) = T^{\mathbf{k}}(u)T^{\mathbf{k}}(v)$, u, $v \in \mathcal{M}_p$ *and* $\tau_{\mathbf{k}} \colon [u] \mapsto T^{\mathbf{k}}(u)$ *are ring homomorphisms of* $(\mathcal{M}_p, \cdot, +)$.

Remark 1. Note that functionals $\tau_{\mathbf{k}}$, $|\mathbf{k}| \neq \lceil p \rceil$ do not exhaust the set of complex valued ring homomorphisms of the ring $(\mathcal{M}_p, \cdot, +)$ as in Example 1. In particular, setting $\theta = \left|\big(x^{(1)}\big)^{k_1} \cdots \big(x^{(s)}\big)^{k_s}\right|$, we will get some others complex valued ring homomorphisms τ_θ.

Example 2. Let $s = n^2$ for some integer $n > 1$, and $\mathcal{A} = \mathbb{C}^s = M(n \times n)$ be the algebra of $n \times n$ matrices and $\odot_{\mathcal{A}}$ be the usual matrix multiplication. It is well-known that there is no nontrivial complex homomorphisms of $M(n \times n)$. But there are multiplicative mappings on $M(n \times n)$. For example, the mapping $a \mapsto \det(a)$ is a multiplicative n-homogeneous (continuous) polynomial. From the continuity and homogeneity of $\det(\cdot)$ it follows that there is a constant $C > 0$ such that

$$|\det(a)| \leq C\|a\|_p^n,$$

where $a = (a_{ij}) \in M(n \times n)$, and

$$\|a\|_p = \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^p \Big)^{1/p}.$$

Thus, if $x = (x_1, x_2, \ldots)$ is a sequence of matrices in $M(n \times n)$ such that $x \in \ell_p(M(n \times n))$, then the sequence $\big(\big(\det(x_1)\big)^k, \big(\det(x_2)\big)^k, \ldots\big)$ belongs to ℓ_1 for every k such that $kn \geq \lceil p \rceil$. Hence, $\theta_k := \big(\det(a)\big)^k$ satisfies conditions of Theorem 3 and so,

$$\tau_{\theta_k}[(y|x)] = \sum_{i=1}^{\infty} \big(\det(x_i)\big)^k - \sum_{i=1}^{\infty} \big(\det(y_i)\big)^k$$

is a complex ring homomorphism for every k such that $kn \geq \lceil p \rceil$.

Note that for any multiplicative unital subsemigroup with $\mathcal{A}_0 \subset \mathcal{A}$ the subset

$$\big\{[(y|x)] \in \mathcal{M}_p \colon x_i \in \mathcal{A}_0,\, y_i \in \mathcal{A}_0\big\}$$

is a subring of $\mathcal{M}_p$.

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