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RESEARCH ARTICLE

A study of homoderivations in 3-prime near-rings with centralizing constraints

Slimane El Amrani and Abderrahmane Raji

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Abstract. In this paper, we explore the commutativity of 3-prime near-rings that admit homoderivations satisfying specific differential identities. Furthermore, we present an example demonstrating the essential role of the 3-primeness assumption in several theorems, emphasizing that this condition cannot be disregarded.

Introduction

The study of derivations and their generalizations plays a central role in understanding the structural properties of algebraic systems, particularly in noncommutative ring theory. A key line of investigation concerns conditions under which the existence of certain derivation-like maps imposes commutativity or other rigid algebraic behaviors on the underlying structure. This theme has been extensively explored in the context of prime rings and has yielded a variety of structural theorems with wide applicability in algebra and its branches.

Let $\mathcal{N}$ be a left near-ring, i.e., a nonempty set equipped with two binary operations " $+$ " and " $\cdot$ " such that $(\mathcal{N}, +)$ is a group (not necessarily abelian), $(\mathcal{N}, \cdot)$ is a semigroup, and left distributivity holds: $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathcal{N}$. Right near-rings are defined analogously with right distributivity. Near-rings generalize the concept of rings by relaxing both distributivity and commutativity of addition, leading to a

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wider and more flexible class of algebraic structures, with connections to combinatorics, automata theory, and coding theory.

An important subclass is that of 3-prime near-rings, defined by the property that $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$. These structures provide a fertile ground for extending classical results from ring theory. Moreover, a near-ring is called zero-symmetric if $0 \cdot x = 0$ for all $x \in \mathcal{N}$, and 2-torsion free if $2x = 0$ implies $x = 0$. The symbol $\mathcal{Z}(\mathcal{N})$ will denote the multiplicative center of $\mathcal{N}$, that is, $\mathcal{Z}(\mathcal{N}) = \{x \in \mathcal{N} \mid xy = yx$ for all $y \in \mathcal{N}\}$.

Among the tools for probing the internal structure of such rings and near-rings are additive maps such as derivations. A derivation on a near-ring $\mathcal{N}$ is an additive function $d : \mathcal{N} \to \mathcal{N}$ satisfying the Leibniz rule: $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. In the literature, various additive constructions including derivations, semi-derivations, generalized derivations, and related operators satisfying centralizing or algebraic identities, have been successfully applied to establish commutativity results in privileged near-rings (see [4, 5, 7–13] and the references therein).

In a related context, let $\mathcal{R}$ be a ring. In a remarkable generalization, El Sofy [6] introduced the concept of a homoderivation, defined as an additive map $h : \mathcal{R} \to \mathcal{R}$ satisfying $h(xy) = h(x)h(y) + h(x)y + xh(y)$. Such maps encapsulate a broader class of additive functions, where the interaction between elements reflects not only linearity but also nonlinear multiplicative perturbations. Interestingly, in prime rings, any map that is simultaneously a derivation and a homoderivation is necessarily the zero map. These structures have since been studied further, notably by Alharfie et al. [1], who established commutativity conditions for prime rings admitting homoderivations satisfying various centralizing identities involving commutators, anticommutators, and central elements.

The present work aims to extend these developments to the setting of 3-prime zero-symmetric near-rings. We investigate the behavior of homoderivations $h : \mathcal{N} \to \mathcal{N}$ that satisfy identities of the form $\big[[x, h(y)] + x \circ y, t\big] \in \mathcal{Z}(\mathcal{N})$, $\big[[x, h(y)] + x \circ y, t\big] \circ r \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t, r \in \mathcal{N}$, and similar variants. Our goal is to determine sufficient conditions under which such identities enforce the commutativity of $\mathcal{N}$. The results presented here generalize known theorems in ring theory and illustrate how non-associative algebraic frameworks can be constrained via functional identities derived from homomorphisms and derivations.

1. Preliminary results

We begin this section with the following lemmas, which play a crucial role in establishing the proofs of our main results.

Lemma 1. *Let $\mathcal{N}$ be a 3-prime near-ring and $\mathcal{I}$ be a nonzero semigroup ideal of $\mathcal{N}$.*

(i) [2, Lemma 1.4(i)] *If $x, y \in \mathcal{N}$ and $x\mathcal{I}y = \{0\}$, then $x = 0$ or $y = 0$.*

(ii) [2, Lemma 1.3(i)] *If $x \in \mathcal{N}$ and $x\mathcal{I} = \{0\}$ or $\mathcal{I}x = \{0\}$, then $x = 0$.*

Lemma 2. *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) [2, Lemma 1.2(iii)] *If $z \in \mathcal{Z}(\mathcal{N})\setminus\{0\}$ and $xz \in \mathcal{Z}(\mathcal{N})$, then $x \in \mathcal{Z}(\mathcal{N})$.*

(ii) [2, Lemma 1.5] *If $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

(iii) [2, Lemma 1.2(ii)] *If $\mathcal{Z}(\mathcal{N})$ contains a nonzero element z for which $z + z \in \mathcal{Z}(\mathcal{N})$, then $(\mathcal{N}, +)$ is abelian.*

Lemma 3 ([3, Theorem 3.7(i)])**.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h, then the following assertions are equivalent:*

(i) $h(\mathcal{N}) \subseteq \mathcal{Z}(\mathcal{N})$;

(ii) *$\mathcal{N}$ is a commutative ring.*

2. Main results

A. Raji et al. [11] studied the commutativity of a prime near-ring $\mathcal{N}$ admitting a multiplicative semiderivation d that satisfies either of the properties $[d(\mathcal{I}), \mathcal{I}] \subseteq \mathcal{Z}(\mathcal{N})$, $d(\mathcal{I}) \circ \mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$ and $\mathcal{N} \circ d(\mathcal{I}) \subseteq \mathcal{Z}(\mathcal{N})$, where $\mathcal{I}$ is a nonzero semigroup ideal of $\mathcal{N}$. Motivated by these results in the multiplicative center, in the following, we investigate the commutativity of a near-ring $\mathcal{N}$ admitting a homoderivation h that is zero-power valued on $\mathcal{N}$.

Theorem 1. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$, then the following conditions are equivalent:*

(i) $\big[[x, h(y)] + x \circ y, t\big] \in \mathcal{Z}(\mathcal{N})$ *for all $x, y, t \in \mathcal{N}$;*

(*ii*) $\big[[x, h(y)] + x \circ y, t\big] \circ r \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t, r \in \mathcal{N}$;

(*iii*) $\big[\big([x, h(y)] + x \circ y\big) \circ t, r\big] \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t, r \in \mathcal{N}$;

(*iv*) $\mathcal{N}$ *is a commutative ring.*

Proof. It follows from condition (*iv*) that properties (*i*)–(*iii*) hold. Let us show that (*i*) $\Rightarrow$ (*iv*). By hypothesis, we have

$$\big[[x, h(y)] + x \circ y, t\big] \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{1}$$

Replace t in (1) with $([x, h(y)] + x \circ y)t$, we get $([x, h(y)] + x \circ y)\big[[x, h(y)] + x \circ y, t\big] \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t \in \mathcal{N}$. By applying Lemma 2(*i*) together with (1), the above relation proves that $\big[[x, h(y)] + x \circ y, t\big] = 0$ or $[x, h(y)] + x \circ y \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t \in \mathcal{N}$. Both conditions give the following

$$[x, h(y)] + x \circ y \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \tag{2}$$

Our next goal is to show that $\mathcal{Z}(\mathcal{N}) \neq \{0\}$. In fact, suppose that $\mathcal{Z}(\mathcal{N}) = \{0\}$, then (2) yields

$$[x, h(y)] + x \circ y = 0 \text{ for all } x, y \in \mathcal{N}. \tag{3}$$

Since h is zero-power valued on $\mathcal{N}$ and is a nonzero mapping, there exists an element $y_0 \in \mathcal{N}$ and a positive integer $k = k(y_0) > 1$ such that $h^k(y_0) = 0$ and $z = h^{k-1}(y_0) \neq 0$. According to (3), it follows that $x \circ z = 0$ for all $x \in \mathcal{N}$ which implies that $x(-z) = zx$ for all $x \in \mathcal{N}$. Replacing x by xt in the last relation and using it again, we obtain $\mathcal{N}[t, -z] = \{0\}$ for all $t \in \mathcal{N}$. In view of Lemma 1(*ii*), we infer that $-z \in \mathcal{Z}(\mathcal{N})$. Now, putting $x = -z$ in (3), we arrive at $\mathcal{N}(2(-z)) = \{0\}$. Based on Lemma 1(*ii*) and the fact that $\mathcal{N}$ is 2-torsion-free, we obtain $z = 0$; which leads to a contradiction. Accordingly, $\mathcal{Z}(\mathcal{N}) \neq \{0\}$. Let $0 \neq z_0 \in \mathcal{Z}(\mathcal{N})$, and taking $x = z_0$ in (2) we obtain $z_0 \circ y \in \mathcal{Z}(\mathcal{N})$ for all $y \in \mathcal{N}$, which implies that $z_0(y + y) \in \mathcal{Z}(\mathcal{N})$ for all $y \in \mathcal{N}$. Given that $z_0 \neq 0$ and in view of Lemma 2(*i*), it follows that $y + y \in \mathcal{Z}(\mathcal{N})$ for all $y \in \mathcal{N}$. Replacing y by y^2 in the above relation and using Lemma 2(*i*), we get either $2y = 0$ or $y \in \mathcal{Z}(\mathcal{N})$ for all $y \in \mathcal{N}$. Because $\mathcal{N}$ is 2-torsion free, both conditions force $y \in \mathcal{Z}(\mathcal{N})$. Consequently, $\mathcal{N}$ is a commutative ring by Lemma 2(*ii*).

(*ii*) $\Rightarrow$ (*iv*). Assume that the following condition holds

$$\big[[x, h(y)] + x \circ y, t\big] \circ r \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t, r \in \mathcal{N}. \tag{4}$$

As a result, $\big[[[x,h(y)]+x\circ y,t]\circ r,m\big]=0$ for all $x,y,t,r,m\in\mathcal{N}$. Substituting $\big[[x,h(y)]+x\circ y,t\big]r$ for r, we obtain $\Big[\big[[x,h(y)]+x\circ y,t\big]\big[[x,h(y)]+x\circ y,t\big]\circ r,m\Big]=0$ for all $x,y,t,r,m\in\mathcal{N}$, which implies that $\Big(\big[[x,h(y)]+x\circ y,t\big]\circ r\Big)\Big[\big[[x,h(y)]+x\circ y,t\big],m\Big]=0$ for all $x,y,t,r,m\in\mathcal{N}$. Taking into account (4), the preceding relation yields $\Big(\big[[x,h(y)]+x\circ y,t\big]\circ r\Big)\mathcal{N}\Big[\big[[x,h(y)]+x\circ y,t\big],m\Big]=\{0\}$ for all $x,y,t,r,m\in\mathcal{N}$. In the light of the 3-primeness of $\mathcal{N}$, we find that for all $x,y,t,r,m\in\mathcal{N}$ either

$$\big[[x,h(y)]+x\circ y,t\big]\circ r=0 \text{ or } \Big[\big[[x,h(y)]+x\circ y,t\big],m\Big]=0. \tag{5}$$

Let x,y be two arbitrary elements of $\mathcal{N}$.

- If $\big[[x,h(y)]+x\circ y,t\big]\circ r=0$ for all $t,r\in\mathcal{N}$, then $\big[[x,h(y)]+x\circ y,t\big]r=r\Big(-\big[[x,h(y)]+x\circ y,t\big]\Big)$ for all $t,r\in\mathcal{N}$. Taking $r=nr$, where $n\in\mathcal{N}$, in the last equation, we obtain $n(-\big[[x,h(y)]+x\circ y,t\big])r=nr(-\big[[x,h(y)]+x\circ y,t\big])$ for all $r,t,n\in\mathcal{N}$, which results in $\mathcal{N}\Big[-\big[[x,h(y)]+x\circ y,t\big],r\Big]=\{0\}$ for all $t,r\in\mathcal{N}$. Thus, by Lemma 1(ii), it follows that $-\big[[x,h(y)]+x\circ y,t\big]\in\mathcal{Z}(\mathcal{N})$ for all $t\in\mathcal{N}$. Substituting $\big([x,h(y)]+x\circ y\big)t$ for t in the last relation and applying Lemma 2(i), we obtain $\big[[x,h(y)]+x\circ y,t\big]=0$ or $[x,h(y)]+x\circ y\in\mathcal{Z}(\mathcal{N})$ which reduce to $[x,h(y)]+x\circ y\in\mathcal{Z}(\mathcal{N})$.

- Now, suppose that there exist $t_0,r_0\in\mathcal{N}$ such that $\big[[x,h(y)]+x\circ y,t_0\big]\circ r_0\neq 0$. In virtue of (5), $\big[[x,h(y)]+x\circ y,t_0\big]$ will be a nonzero element of $\mathcal{Z}(\mathcal{N})$. Replacing r by $\big[[x,h(y)]+x\circ y,t_0\big]$ in (4), we obtain $\big[[x,h(y)]+x\circ y,t_0\big]\Big(\big[[x,h(y)]+x\circ y,t\big]+\big[[x,h(y)]+x\circ y,t\big]\Big)\in\mathcal{Z}(\mathcal{N})$ for all $t\in\mathcal{N}$, and hence in view of Lemma 2(i), we conclude that $\Big(\big[[x,h(y)]+x\circ y,t\big]+\big[[x,h(y)]+x\circ y,t\big]\Big)\in\mathcal{Z}(\mathcal{N})$ for all $t\in\mathcal{N}$. Substituting $\big[[x,h(y)]+x\circ y,t\big]$ for r in (4), we find that $\big[[x,h(y)]+x\circ y,t\big]\big([[x,h(y)]+x\circ y,t]+[[x,h(y)]+x\circ y,t]\big)\in\mathcal{Z}(\mathcal{N})$ for all $t\in\mathcal{N}$. Once again, by Lemma 2(i) and the 2-torsion freeness of $\mathcal{N}$, we conclude that $\big[[x,h(y)]+x\circ y,t\big]\in\mathcal{Z}(\mathcal{N})$ for all $t\in\mathcal{N}$.

Consequently, from both cases, since x and y are arbitrary, it follows that $[[x,h(y)]+x\circ y,t]\in\mathcal{Z}(\mathcal{N})$ for all $x,y,t\in\mathcal{N}$. Hence, by the proof of $(i)\Rightarrow(iv)$, $\mathcal{N}$ must be a commutative ring.

$(iii)\Rightarrow(iv)$. Let us assume that

$$\Big[\big([x,h(y)]+x\circ y\big)\circ t,r\Big]\in\mathcal{Z}(\mathcal{N}) \text{ for all } x,y,t,r\in\mathcal{N}. \tag{6}$$

Putting $\Big(\big([x,h(y)]+x\circ y\big)\circ t\Big)r$ instead of r in (6) and invoking Lemma 2(i), we find that $\big([x,h(y)]+x\circ y\big)\circ t\in\mathcal{Z}(\mathcal{N})$ or $\Big[\big([x,h(y)]+x\circ y\big)\circ t,r\Big]=0$

for all $x, y, t, r \in \mathcal{N}$. It follows that

$$\big([x, h(y)] + x \circ y\big) \circ t \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{7}$$

Assume that $\mathcal{Z}(\mathcal{N}) = \{0\}$. Replacing t by $[x, h(y)] + x \circ y$ in (7), we get $2\big([x, h(y)] + x \circ y\big)^2 = 0$ for all $x, y \in \mathcal{N}$. Given the 2-torsion freeness, we infer that $\big([x, h(y)] + x \circ y\big)^2 = 0$ for all $x, y \in \mathcal{N}$. On the other hand, (7) yields $\big([x, h(y)] + x \circ y\big)t + t\big([x, h(y)] + x \circ y\big) = 0$ for all $x, y, t \in \mathcal{N}$. Left multiplying the previous equation by $\big([x, h(y)] + x \circ y\big)$, we get $\big([x, h(y)] + x \circ y\big)t\big([x, h(y)] + x \circ y\big) = 0$ for all $x, y, t \in \mathcal{N}$ which can be written as $\big([x, h(y)] + x \circ y\big)\mathcal{N}\big([x, h(y)] + x \circ y\big) = \{0\}$ for all $x, y \in \mathcal{N}$. In the light of the 3-primeness of $\mathcal{N}$, we obtain $[x, h(y)] + x \circ y = 0$ for all $x, y \in \mathcal{N}$. Since this relation matches expression (3), we apply the same technique used in the proof of $(i) \Rightarrow (iv)$, leading to a contradiction, and therefore $\mathcal{Z}(\mathcal{N}) \neq \{0\}$. Now, choosing $0 \neq z_0 \in \mathcal{Z}(\mathcal{N})$ and replacing t by z_0 in (7), we get $z_0\big([x, h(y)] + x \circ y + [x, h(y)] + x \circ y\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$ which, because of Lemma 2(i), implies that $2\big([x, h(y)] + x \circ y\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. Taking $[x, h(y)] + x \circ y$ instead of t in (7) gives $\big([x, h(y)] + x \circ y\big)\big(2\big([x, h(y)] + x \circ y\big)\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y, \in \mathcal{N}$. In view of Lemma 2(i), the preceding relation demonstrates that either $2\big([x, h(y)] + x \circ y\big) = 0$ or $[x, h(y)] + x \circ y \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. But, since $\mathcal{N}$ is 2-torsion free, the first condition assures that $[x, h(y)] + x \circ y = 0$. Hence, in all cases $[x, h(y)] + x \circ y \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$ which is identical to expression (2). Therefore, $\mathcal{N}$ is a commutative ring. □

Theorem 2. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h which is zero-power valued on $\mathcal{N}$, then the following conditions are equivalent:*

(i) $\big[[x, h(y)] + xy, t\big] \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t \in \mathcal{N}$*;*

(ii) $\big[\big([x, h(y)] + xy\big) \circ t, r\big] \in \mathcal{Z}(\mathcal{N})$ *for all* $x, y, t, r \in \mathcal{N}$*;*

(iii) $\mathcal{N}$ *is a commutative ring.*

Proof. It is straightforward to confirm that condition (iii) ensures properties (i) and (ii).

Let us prove that (i) implies (iii). Suppose that

$$\big[[x, h(y)] + xy, t\big] \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{8}$$

This leads us to $\big[\big[[x, h(y)] + xy, t\big], s\big] = 0$ for all $x, y, t, s \in \mathcal{N}$. By substituting t with $\big([x, h(y)] + xy\big)t$ in the previous relation, we obtain

$\big[\big([x,h(y)]+xy\big)\big[[x,h(y)]+xy,t\big],s\big]=0$ for all $x,y,t,s\in\mathcal{N}$ which, in view of (8), can be written as $\big[[x,h(y)]+xy,t\big]\big[[x,h(y)]+xy,s\big]=0$ for all $x,y,t,s\in\mathcal{N}$. On the other hand, left multiplying the preceding equation by an arbitrary element of $\mathcal{N}$ and in virtue of (8), we obtain $\big[[x,h(y)]+xy,t\big]\mathcal{N}\big[[x,h(y)]+xy,s\big]=\{0\}$ for all $x,y,t,s\in\mathcal{N}$. Considering the 3-primeness of $\mathcal{N}$, the previous relation implies that $\big[[x,h(y)]+xy,t\big]=0$ or $\big[[x,h(y)]+xy,s\big]=0$ for all $x,y,t,s\in\mathcal{N}$. Both conditions give

$$[x,h(y)]+xy\in\mathcal{Z}(\mathcal{N}) \text{ for all } x,y\in\mathcal{N}. \tag{9}$$

Next, we claim that $\mathcal{Z}(\mathcal{N})\neq\{0\}$. Indeed, suppose that $\mathcal{Z}(\mathcal{N})=\{0\}$. Then, (9) yields

$$[x,h(y)]+xy=0 \text{ for all } x,y\in\mathcal{N}. \tag{10}$$

Since h is a nonzero zero-power valued on $\mathcal{N}$, there exists an element $y_0\in\mathcal{N}$ and a minimal integer $k=k(y_0)>1$ such that $h^k(y_0)=0$ and $z=h^{k-1}(y_0)\neq 0$. Putting $y=z$ in (10) gives $xz=0$ for all $x\in\mathcal{N}$. Applying Lemma 1(ii), we find that $z=0$; a contradiction. Hence $\mathcal{Z}(\mathcal{N})\neq\{0\}$ as claimed. Let $z_0\in\mathcal{Z}(\mathcal{N})$ be such that $z_0\neq 0$ and replacing x by z_0 in (9), we find that $z_0y\in\mathcal{Z}(\mathcal{N})$ for all $y\in\mathcal{N}$. In virtue of Lemma 2(i), we conclude that $\mathcal{N}\subseteq\mathcal{Z}(\mathcal{N})$, and therefore $\mathcal{N}$ is a commutative ring by Lemma 2(ii).

To demonstrate that $(i)\Rightarrow(iii)$, assume that

$$\big[\big([x,h(y)]+xy\big)\circ t,r\big]\in\mathcal{Z}(\mathcal{N}) \text{ for all } x,y,t,r\in\mathcal{N}. \tag{11}$$

Replacing r by $\big(\big([x,h(y)]+xy\big)\circ t\big)r$ in (11) and using it again, we obtain $\big[\big([x,h(y)]+xy\big)\circ t\big[\big([x,h(y)]+xy\big)\circ t,r\big],s\big]=0$ for all $x,y,t,r,s\in\mathcal{N}$ which can be rewritten as $\big[\big([x,h(y)]+xy\big)\circ t,r\big]\big[\big([x,h(y)]+xy\big)\circ t,s\big]=0$ for all $x,y,t,r,s\in\mathcal{N}$. On the other hand, left multiplying the preceding equation by m, where $m\in\mathcal{N}$, and taking $r=s$, we obtain $\big[\big([x,h(y)]+xy\big)\circ t,r\big]\,m\,\big[\big([x,h(y)]+xy\big)\circ t,r\big]=0$ for all $x,y,t,r,m\in\mathcal{N}$. By 3-primeness of $\mathcal{N}$, the previous relation forces $\big[\big([x,h(y)]+xy\big)\circ t,r\big]=0$ for all $x,y,t,r\in\mathcal{N}$. That is,

$$\big([x,h(y)]+xy\big)\circ t\in\mathcal{Z}(\mathcal{N}) \text{ for all } x,y,t\in\mathcal{N}. \tag{12}$$

Suppose now that $\mathcal{Z}(\mathcal{N})=\{0\}$. Then (12) yields $\big([x,h(y)]+xy\big)t=t\big(-\big([x,h(y)]+xy\big)\big)$ for all $x,y,t\in\mathcal{N}$. For $t=mt$, where $m\in\mathcal{N}$, we infer that $m\big(-\big([x,h(y)]+xy\big)\big)t=mt\big(-\big([x,h(y)]+xy\big)\big)$ for all $x,y,m,t\in\mathcal{N}$. Accordingly, $\mathcal{N}\big[-\big([x,h(y)]+xy\big),t\big]=\{0\}$ for all $x,y,t\in\mathcal{N}$. Invoking

Lemma 1(ii) and using our assumption that $\mathcal{Z}(\mathcal{N}) = \{0\}$, we obtain $[x, h(y)] + xy = 0$ for all $x, y \in \mathcal{N}$. As this result is the same as (10), we conclude that $\mathcal{Z}(\mathcal{N}) \neq \{0\}$. Letting $0 \neq z_0 \in \mathcal{Z}(\mathcal{N})$ and replacing t by z_0 in (12), we get $z_0\big([x, h(y)]+xy+[x, h(y)]+xy\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. By Lemma 2(i) and the fact that $z_0 \neq 0$, it follows that $2([x, h(y)]+xy) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. Again, by replacing t with $[x, h(y)] + xy$ in (12), we get $\big([x, h(y)]+xy\big)\big(2\big([x, h(y)]+xy\big)\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. Using Lemma 2(i), the former equation yields

$$2\big([x, h(y)] + xy\big) = 0 \text{ or } [x, h(y)] + xy \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \quad (13)$$

Suppose that there exist $x_0, y_0 \in \mathcal{N}$ such that $2\big([x_0, h(y_0)] + x_0y_0\big) = 0$, then $[x_0, h(y_0)] + x_0y_0 = -\big([x_0, h(y_0)] + x_0y_0\big)$. Putting $x = x_0$, $y = y_0$ and $t = \big([x_0, h(y_0)] + x_0y_0\big)t$ in (12), we obtain $\big([x_0, h(y_0)] + x_0y_0\big)$ $\big([x_0, h(y_0)] + x_0y_0\big) \circ t \in \mathcal{Z}(\mathcal{N})$ for all $t \in \mathcal{N}$. Applying Lemma 2(i), we find that $\big([x_0, h(y_0)] + x_0y_0\big) \circ t = 0$ or $[x_0, h(y_0)] + x_0y_0 \in \mathcal{Z}(\mathcal{N})$ for all $t \in \mathcal{N}$. If $\big([x_0, h(y_0)] + x_0y_0\big) \circ t = 0$ holds for all $t \in \mathcal{N}$, then

$$\begin{aligned}\big([x_0, h(y_0)] + x_0y_0\big)t &= -t\big([x_0, h(y_0)] + x_0y_0\big)\\ &= t\big(-\big([x_0, h(y_0)] + x_0y_0\big)\big)\\ &= t\big([x_0, h(y_0)] + x_0y_0\big) \text{ for all } t \in \mathcal{N}\end{aligned}$$

which shows that $[x_0, h(y_0)] + x_0y_0 \in \mathcal{Z}(\mathcal{N})$. Consequently, (13) reduces to $[x, h(y)] + xy \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$ which is identical to (9), and therefore $\mathcal{N}$ is a commutative ring. □

Theorem 3. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero homoderivation h, then the following conditions are equivalent:*

(i) $\big[[h(x), y] + y \circ h(x), t\big] \circ r \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t, r \in \mathcal{N}$;

(ii) $\big[\big([h(x), y] + y \circ h(x)\big) \circ t, r\big] \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t, r \in \mathcal{N}$;

(iii) $\mathcal{N}$ is a commutative ring.

Proof. One can readily check that property (iii) implies both properties (i) and (ii).

Let us demonstrate that $(i) \Rightarrow (iii)$. We are assuming that

$$\big[[h(x), y] + y \circ h(x), t\big] \circ r \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t, r \in \mathcal{N}. \quad (14)$$

Setting $r = \big[[h(x), y] + y \circ h(x), t\big]r$ in (14) and applying Lemma 2(i), we obtain

$$\big[[h(x), y] + y \circ h(x), t\big] \circ r = 0 \text{ or } \big[[h(x), y] + y \circ h(x), t\big] \in \mathcal{Z}(\mathcal{N}) \quad (15)$$

for all $x, y, t \in \mathcal{N}$. Let $x, y, t \in \mathcal{N}$, and suppose that $\big[[h(x), y] + y \circ h(x), t\big] \circ r = 0$ for all $r \in \mathcal{N}$. So that, $\big[[h(x), y] + y \circ h(x), t\big] r = r\big(-\big[[h(x), y] + y \circ h(x), t\big]\big)$ for all $r \in \mathcal{N}$. Now, putting $r = nr$, where $n \in \mathcal{N}$, in the last equation, thereby obtaining $n\big(-\big[[h(x), y] + y \circ h(x), t\big]\big) r = nr\big(-\big[[h(x), y] + y \circ h(x), t\big]\big)$ for all $n \in \mathcal{N}$, which leads to $\mathcal{N}\big[-\big[[h(x), y] + y \circ h(x), t\big], r\big] = \{0\}$ for all $r \in \mathcal{N}$, and hence $-\big[[h(x), y] + y \circ h(x), t\big] \in \mathcal{Z}(\mathcal{N})$. Consequently, (15) yields

$$\big[[h(x), y] + y \circ h(x), t\big] \in \mathcal{Z}(\mathcal{N}) \text{ or } -\big[[h(x), y] + y \circ h(x), t\big] \in \mathcal{Z}(\mathcal{N}) \quad (16)$$

for all $x, y, t \in \mathcal{N}$. Assume that there are $x_0, y_0, t_0 \in \mathcal{N}$ such that $k = \big[[h(x_0), y_0] + y_0 \circ h(x_0), t_0\big] \notin \mathcal{Z}(\mathcal{N})$. From (16), we find that $-k \in \mathcal{Z}(\mathcal{N}) \setminus \{0\}$. Substituting $-k$ for r in (14), we obtain $(-k)\big(2\big[[h(x), y] + y \circ h(x), t\big]\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t \in \mathcal{N}$. In virtue of Lemma 2(i) and the fact that $-k \neq 0$, it follows $2\big[[h(x), y] + y \circ h(x), t\big] \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t \in \mathcal{N}$. Replacing r by $\big[[h(x), y] + y \circ h(x), t\big]$ in (14), we find that $\big[[h(x), y] + y \circ h(x), t\big]\big(2\big[[h(x), y] + y \circ h(x), t\big]\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t \in \mathcal{N}$. Invoking Lemma 2(i) and taking into account that the 2-torsion freeness of $\mathcal{N}$, we deduce that $\big[[h(x), y] + y \circ h(x), t\big] \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t \in \mathcal{N}$. Specifically, for $x = x_0, y = y_0$ and $t = t_0$, we obtain $k \in \mathcal{Z}(\mathcal{N})$ which contradicts our assumption that $k \notin \mathcal{Z}(\mathcal{N})$. Therefore, (16) shows that $\big[[h(x), y] + y \circ h(x), t\big] \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t \in \mathcal{N}$. In particular, for $t = \big([h(x), y] + y \circ h(x)\big)t$, we get $\big([h(x), y] + y \circ h(x)\big)\big[[h(x), y] + y \circ h(x), t\big] \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t \in \mathcal{N}$. Using Lemma 2(i), we arrive at

$$[h(x), y] + y \circ h(x) \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y \in \mathcal{N}. \quad (17)$$

Putting $y = h(x)y$, where $x \in \mathcal{N}$, in (17), we find that $h(x)\big([h(x), y] + y \circ h(x)\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. By Lemma 2(i), this leads to

$$h(x) \in \mathcal{Z}(\mathcal{N}) \text{ or } [h(x), y] + y \circ h(x) = 0 \text{ for all } x, y \in \mathcal{N}. \quad (18)$$

Suppose that $[h(x), y] + y \circ h(x) = 0$ for all $x, y \in \mathcal{N}$. Given the 2-torsion freeness of $\mathcal{N}$, we deduce that $h(x)\mathcal{N} = \{0\}$ and hence $h = 0$ by Lemma 1(ii); a contradiction to our hypothesis. Consequently, there exist $x_0, y_0 \in \mathcal{N}$ such that $[h(x_0), y_0] + y_0 \circ h(x_0) \neq 0$, which implies that $h(x_0) \neq 0$ and from (18) its follows that $h(x_0) \in \mathcal{Z}(\mathcal{N})$. Replacing x by x_0 in (17), we get $h(x_0)(y + y) \in \mathcal{Z}(\mathcal{N})$ for all $y \in \mathcal{N}$. In view of Lemma 2(i), it follows that $y + y \in \mathcal{Z}(\mathcal{N})$ for all $y \in \mathcal{N}$. Replacing y by ty, where $t \in \mathcal{N}$ in the last result and using it again, we conclude that either $t \in \mathcal{Z}(\mathcal{N})$ or $2y = 0$ for all $y, t \in \mathcal{N}$. Taking into account that $\mathcal{N}$

is 2-torsion free, the above condition imply that $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$. Thus, $\mathcal{N}$ is a commutative ring by Lemma 2(ii).

$(ii) \Rightarrow (iii)$. Suppose that

$$\big[\big([h(x), y] + y \circ h(x)\big) \circ t, r\big] \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t, r \in \mathcal{N}. \tag{19}$$

Putting $\big(\big([h(x), y] + y \circ h(x)\big) \circ t\big)r$ instead of r in (19), we obtain $\big([h(x), y] + y \circ h(x)\big) \circ t\big[\big([h(x), y] + y \circ h(x)\big) \circ t, r\big] \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t, r \in \mathcal{N}$. By application of Lemma 2(i), the latter relation shows that $\big[\big([h(x), y] + y \circ h(x)\big) \circ t, r\big] = 0$ or $\big([h(x), y] + y \circ h(x)\big) \circ t \in \mathcal{Z}(\mathcal{N})$ for all $x, y, t, r \in \mathcal{N}$. Which leads us to the following conclusion

$$\big([h(x), y] + y \circ h(x)\big) \circ t \in \mathcal{Z}(\mathcal{N}) \text{ for all } x, y, t \in \mathcal{N}. \tag{20}$$

Suppose that $\mathcal{Z}(\mathcal{N}) = \{0\}$. It follows that $\big([h(x), y] + y \circ h(x)\big) \circ t = 0$ for all $x, y, t \in \mathcal{N}$ and thus $\big([h(x), y] + y \circ h(x)\big)t = t\big(-\big([h(x), y] + y \circ h(x)\big)\big)$ for all $x, y, t \in \mathcal{N}$. Taking $t = ts$, where $s \in \mathcal{N}$, in the previous equation and using it again, we find that $t\big(-\big([h(x), y] + y \circ h(x)\big)\big)s = ts\big(-\big([h(x), y] + y \circ h(x)\big)\big)$ and hence by Lemma 1(ii) that $-\big([h(x), y] + y \circ h(x)\big) \in \mathcal{Z}(\mathcal{N}) = \{0\}$ which means that $[h(x), y] + y \circ h(x) = 0$ for all $x, y \in \mathcal{N}$. However, this result, developed after (18), leads to a contradiction. Therefore $\mathcal{Z}(\mathcal{N}) \neq \{0\}$. Now, choosing $0 \neq z_0 \in \mathcal{Z}(\mathcal{N})$ and replacing t by z_0 in (20), we obtain $z_0\big(2\big([h(x), y] + y \circ h(x)\big)\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. In the light of Lemma 2(i), we infer that $2\big([h(x), y] + y \circ h(x)\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. Once again, replacing t by $[h(x), y] + y \circ h(x)$ in (20), we get $\big([h(x), y] + y \circ h(x)\big)\big([h(x), y] + y \circ h(x) + [h(x), y] + y \circ h(x)\big) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. By Lemma 2(i) and 2-torsion freeness of $\mathcal{N}$, we conclude that $[h(x), y] + y \circ h(x) \in \mathcal{Z}(\mathcal{N})$ for all $x, y \in \mathcal{N}$. Finally, by applying the same techniques as those used after relation (17), we deduce that $\mathcal{N}$ is a commutative ring. □

We show in the following results that no nonzero homoderivation h, which is zero-power valued on $\mathcal{N}$, can satisfy the given conditions.

Theorem 4. *Let $\mathcal{N}$ be a 2-torsion-free 3-prime near-ring. There exists no nonzero homoderivation h, which is zero-power valued on $\mathcal{N}$, that satisfies any of the following conditions:*

(i) *$h(x) \circ h(y) = x \circ y$ for all $x, y \in \mathcal{N}$;*

(ii) *$h(x) \circ h(y) = h(x) \circ y$ for all $x, y \in \mathcal{N}$;*

(iii) *$h(x) \circ y = x \circ y$ for all $x, y \in \mathcal{N}$.*

Proof. (i) Suppose that $\mathcal{N}$ has a nonzero homoderivation h such that $h(x) \circ h(y) = x \circ y$ for all $x, y \in \mathcal{N}$. By applying recurrence, we have $h^n(x) \circ h^n(y) = x \circ y$ for all $x, y \in \mathcal{N}$ and $n \in \mathbb{N}$. Since h is zero-power valued on $\mathcal{N}$, then there exist positive integers $k(x) > 1$ and $k(y) > 1$ such that $h^{k(x)}(x) = h^{k(y)}(y) = 0$. Let us define $p = max(k(x), k(y))$, then $0 = h^p(x) \circ h^p(y) = x \circ y$ for all $x, y \in \mathcal{N}$. From the proof of [14, Theorem 3.9], we deduce that $\mathcal{N} = \{0\}$, a contradiction.

(ii) Given that $\mathcal{N}$ has a nonzero homoderivation h satisfies $h(x) \circ h(y) = h(x) \circ y$ for all $x, y \in \mathcal{N}$. By induction, it follows that $h(x) \circ h^n(y) = h(x) \circ y$ for all $x, y \in \mathcal{N}$ and $n \in \mathbb{N}$. Since h is zero-power valued on $\mathcal{N}$, there exists a positive integer $k = k(y) > 1$ such that $h^k(y) = 0$. Hence, we deduce that

$$0 = h(x) \circ h^k(y) = h(x) \circ y \text{ for all } x, y \in \mathcal{N}. \tag{21}$$

Thus, we obtain $h(x)y = y(-h(x)) = yh(-x)$ for all $x, y \in \mathcal{N}$. Taking $y = ty$ and replacing x by $-x$ in the last equation, we infer that $th(-x)y = tyh(-x)$ for all $x, y, t \in \mathcal{N}$. That is, $\mathcal{N}[h(x), y] = \{0\}$ for all $x, y \in \mathcal{N}$. By Lemma 1(ii), this implies that $h(\mathcal{N}) \subseteq \mathcal{Z}(\mathcal{N})$, and hence $\mathcal{N}$ is a commutative ring by Lemma 3. So, from (21), it follows that $\mathcal{N}(h(x) + h(x)) = \{0\}$ for all $x \in \mathcal{N}$, which yields $h = 0$.

(iii) Assume that $\mathcal{N}$ admits a nonzero homoderivation h such that $h(x) \circ y = x \circ y$ for all $x, y \in \mathcal{N}$. By recurrence, we obtain $h^n(x) \circ y = x \circ y$ for all $x, y \in \mathcal{N}$ and $n \in \mathbb{N}$. Since h is zero-power valued on $\mathcal{N}$, for each $x \in \mathcal{N}$, there exists a positive integer $k = k(x) > 1$ such that $h^k(x) = 0$. Thus, $0 = h^k(x) \circ y = x \circ y$ for all $x, y \in \mathcal{N}$ which is the same result obtained in the proof of (i), leading to a contradiction. □

The following example illustrates the necessity of assuming that $\mathcal{N}$ is 3-prime in the hypotheses of our theorems.

Example 1. Let $\mathcal{S}$ be a 2-torsion free left near-ring. Let us define $\mathcal{N}$, h_1 and $h_2 : \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} \mid 0, x, y \in \mathcal{S} \right\},$$

$$h_1 \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & x & 0 \end{pmatrix} \text{ and } h_2 \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & y & 0 \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, $\mathcal{N}$ is a 2-torsion-free near-ring that is not 3-prime. Clearly, h_1 is a nonzero zero-power-valued homoderivation on $\mathcal{N}$, and h_2 is a nonzero homoderivation on $\mathcal{N}$, satisfying the following properties:

1) $[[A, h_1(B)] + A \circ B, C] \in \mathcal{Z}(\mathcal{N})$;
2) $[[A, h_1(B)] + A \circ B, C] \circ D \in \mathcal{Z}(\mathcal{N})$;
3) $[([A, h_1(B)] + A \circ B) \circ C, D] \in \mathcal{Z}(\mathcal{N})$;
4) $[[A, h_1(B)] + AB, C] \in \mathcal{Z}(\mathcal{N})$;
5) $[([A, h_1(B)] + AB) \circ C, D] \in \mathcal{Z}(\mathcal{N})$;
6) $[[h_2(A), B] + A \circ h_2(B), C] \circ D \in \mathcal{Z}(\mathcal{N})$;
7) $[\big([h_2(A), B] + A \circ h_2(B)\big) \circ C, D] \in \mathcal{Z}(\mathcal{N})$;
8) $h_1(A) \circ h_1(B) = A \circ B$;
9) $h_1(A) \circ h_1(B) = h_1(A) \circ B$;
10) $h_1(A) \circ B = A \circ B$

for all $A, B, C, D \in \mathcal{N}$. However, $\mathcal{N}$ is a noncommutative near-ring due to the noncommutativity of its addition operation.

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Contact information

S. El Amrani, A. Raji

LMACS Laboratory, Faculty of Sciences and Technology, Sultan Moulay Slimane University, Beni Mellal, Morocco
E-Mail: slimaneelamrani1994@gmail.com, rajiabd2@gmail.com

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