© Algebra and Discrete Mathematics
Volume **40** (2025). Number 2, pp. 158–182
DOI:10.12958/adm2408

RESEARCH ARTICLE

Coxeter spectral classification of non-negative posets of Dynkin type $\mathbb{E}_m$

Marcin Gąsiorek

Communicated by Yu. Drozd

Abstract. We give a complete description of connected non-negative Dynkin type $\mathrm{Dyn}_I = \mathbb{E}_m$ posets and prove that the number of such posets is finite. Moreover, by means of computer assisted analysis, we give a complete Coxeter classification of this class and prove that the pair $(\mathrm{Dyn}_I = \mathbb{E}_m, \mathbf{specc}_I)$, where $\mathbf{specc}_I \subseteq \mathbb{C}$ denotes the Coxeter spectrum of I, determines I uniquely, up to the strong Gram $\mathbb{Z}$-congruence.

Introduction

This work is devoted to the Coxeter spectral study of finite posets, inspired by their applications in the representation theory of posets, finite groups, classical orders, finite-dimensional algebras over a field K, and cluster K-algebras; see [1, 3, 18, 21, 25, 26, 28, 29] and Section 1.

By a finite poset I of *size* n we mean a pair $I = (V, \leq_I)$, where $V := \{1, \ldots, n\}$ and $\leq_I$ is a reflexive, antisymmetric, and transitive binary relation. Every poset I is uniquely determined by its *incidence matrix*

$$C_I = [c_{ij}] \in \mathbb{M}_n(\mathbb{Z}),\ \text{where } c_{ij} = 1 \text{ if } i \leq_I j \text{ and } c_{ij} = 0 \text{ otherwise,} \quad (1)$$

i.e., a square binary matrix that encodes the relation $\leq_I$. A poset I is defined to be *non-negative* of *rank* m if its *symmetric Gram matrix* $G_I := \frac{1}{2}(C_I + C_I^{tr}) \in \mathbb{M}_n(\mathbb{R})$ is positive semi-definite of rank m; see [29].

2020 Mathematics Subject Classification: *06A07, 06A11, 11E04, 68W30.*
Key words and phrases: *non-negative poset, unit quadratic form, Coxeter-Dynkin type, Coxeter spectrum.*

Our overall aim is to describe all connected non-negative posets (up to poset isomorphism) and study the interplay between their combinatorial and algebraic properties. For a discussion of the origins and applications of this study, the reader is referred to [10, Section 6.1], [14, 21, 29] and Section 1.

Following [29], we associate with a poset I the *Coxeter matrix* $\mathrm{Cox}_I := -C_I \cdot (C_I^{-1})^{tr} \in \mathbb{M}_n(\mathbb{Z})$ and its complex spectrum $\mathbf{specc}_I \subseteq \mathbb{C}$, called the *Coxeter spectrum* of I. We call two posets I_1 and I_2 *strongly Gram $\mathbb{Z}$-congruent* (denoted by $I_1 \approx_{\mathbb{Z}} I_2$) if their incidence matrices C_{I_1} and C_{I_2} are *$\mathbb{Z}$-congruent*, i.e.,

$$C_{I_2} = B^{tr} C_{I_1} B \text{ for some } B \in \mathrm{Gl}_n(\mathbb{Z}) := \{A \in \mathbb{M}_n(\mathbb{Z});\, \det A = \pm 1\},$$

see [26]. It is easy to check that this relation preserves definiteness, rank, and the Coxeter spectrum. The main results of this manuscript give a partial solution to the following variants of the problems formulated in [15, 23, 26, 28].

Problem 1. Give a complete description (up to poset isomorphism) of connected non-negative posets I.

Problem 2. When does the Coxeter spectrum $\mathbf{specc}_I \subseteq \mathbb{C}$ of a finite poset I determines the incidence matrix $C_I \in \mathbb{M}_n(\mathbb{Z})$ uniquely, up to the strong Gram $\mathbb{Z}$-congruence?

Every non-negative connected poset I, up to $\mathbb{Z}$-congruence of the symmetric Gram matrix G_I, is determined by a unique simply-laced Dynkin diagram $\mathrm{Dyn}_I \in \{\mathbb{A}_m, \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$, called the *Dynkin type* of I (see Definition 3 for details). The present work is a significant step toward the complete classification of all connected non-negative posets. It complements the results of [13], where posets of Dynkin type $\mathrm{Dyn}_I = \mathbb{A}_m$ are described (see Theorem 3 for more details). In this paper, we focus on the Dynkin type $\mathrm{Dyn}_I \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$.

One of our main results is the following theorem, which provides a partial solution to Problem 1 and [13, Conjecture 6.1].

Theorem 1. *There is only a finite number of connected non-negative posets I of Dynkin types $\mathrm{Dyn}_I \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$, up to poset isomorphism.*

Furthermore, by employing computational tools, we give a full description of these posets in Theorem 4 and perform Coxeter spectral analysis of them, see Theorem 5. In particular, we give a partial solution to Problem 2.

Theorem 2. *If I and J are non-negative connected posets of Dynkin type* $\mathrm{Dyn}_I = \mathrm{Dyn}_J \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$*, then* $I \approx_{\mathbb{Z}} J \Leftrightarrow \mathbf{specc}_I = \mathbf{specc}_J$.

In particular, our results can be viewed as a generalization of those presented in the recent work of Bondarenko and Styopochkina [6], as we explain in the next section.

1. Motivation and known results

Integer quadratic forms have proved to be a powerful tool in the representation theory of algebras. In certain situations, there is a strong relationship between the properties of representations of various objects and the properties of quadratic forms associated with them. In particular, P. Gabriel introduced in 1972 [8] an integer quadratic form $q_Q\colon \mathbb{Z}^{|Q_0|} \to \mathbb{Z}$ of a finite directed graph (quiver) $Q = (Q_0, Q_1)$, called the *quadratic Tits form* of Q. This form is given by

$$q_Q(x) \coloneqq \sum_{i \in Q_0} x_i^2 - \sum_{i \to j \in Q_1} x_i x_j, \tag{2}$$

where i runs through the set of vertices and $i \to j$ runs through the set of arrows. It is well known (see [1, Corollary 4.6]) that for a connected Q, the form q_Q is positive definite (i.e., $q_Q(v) > 0$ for all $0 \neq v \in \mathbb{Z}^{|Q_0|}$) if and only if $\overline{Q}$ (i.e., Q viewed as a non-oriented graph) is one of the Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$ (see Table 1.1). On the other hand, Gabriel asserts in [8] that this is the case if and only if KQ is a representation–finite connected hereditary algebra.

In the study of finite posets $T = (T, \preceq)$, combinatorial structures similar to quivers, Drozd introduced in 1974 [7] the *quadratic Tits form* $q_T^*\colon \mathbb{Z}^{T \cup \{*\}} \to \mathbb{Z}$ given by the formula:

$$q_T^*(x) \coloneqq x_*^2 + \sum_{i \in T} x_i^2 + \sum_{i \prec j} x_i x_j - x_* \sum_{i \in T} x_i, \tag{3}$$

where $* \notin T$. Drozd showed that a poset T is representation-finite (in the sense of Nazarova-Roiter [20]) if and only if its quadratic Tits form is weakly positive, i.e., $q_T^*(v) > 0$ for all $0 \neq v \in \mathbb{N}^{T \cup \{*\}}$.

Other quadratic forms associated with posets are also considered in the literature; see [23]. In particular, in 1993 Simson [22] introduced the following quadratic form $\widehat{q}_I\colon \mathbb{Z}^I \to \mathbb{Z}$:

$$\widehat{q}_I(x) \coloneqq \sum_{i \in I} x_i^2 + \sum_{\substack{i \prec j \\ j \in I \setminus \max I}} x_i x_j - \sum_{p \in \max I} \left(x_p \sum_{i \prec p} x_i \right). \tag{4}$$

It is shown in [22] that the category $prin(KI)$ of finitely generated prinjective right KI-modules is of finite representation type if and only if the quadratic form (4) is weakly positive.

We note that, from the combinatorial point of view, the case of posets is more complex than that of quivers. In contrast to quivers, the classes of posets with *weakly positive* and with *positive* Tits form $q_T^*(x)$ (3) do not coincide. The former case has been described by Kleiner as follows: the form $q_T^*(x)$ is weakly positive if and only if T does not contain any of the five *critical* subposets, see [19, Theorem 1]. The latter class, i.e., posets with Tits form positive (analogs of the Dynkin diagrams), is described up to the *minimax equivalence* by Bondarenko-Stepochkina in [4] (see also [2, Sections 1–3]) and by Gąsiorek-Simson in [14] (up to isomorphism). Since the quadratic form $q_T^*(x)$ coincides with $\widehat{q}_{T\sqcup\{*\}}(x)$, where $T \sqcup \{*\}$ is a poset with exactly one maximal element $*$, the one-peak posets $I = T \sqcup \{*\}$ described in [14] are in direct correspondence with the posets T given in [2,4].

The *non-negative* case is also strikingly different for quivers and posets (we recall that a quadratic form $q\colon \mathbb{Z}^n \to \mathbb{Z}$ is called *non-negative* iff $q(v) \geq 0$ for all $v \in \mathbb{Z}^n$). It is well known that the form $q_Q(x)$ is non-negative if and only if $\overline{Q}$ is a Dynkin or Euclidean diagram (see [1, Proposition 4.5]). It follows that the kernel $\operatorname{Ker} q_Q := \{v \in \mathbb{Z}^n;\, q_Q(v) = 0\}$ of a non-negative connected quiver that is not positive is an infinite cyclic subgroup of $\mathbb{Z}^n$ (such a quadratic form is called *principal*); see [1, Lemma 4.2]. This is not the case for posets, as the rank of the free group $\operatorname{Ker} q_T^*$ of a poset with a non-negative quadratic Tits form $q_T^*(x)$ can be arbitrarily large; see Proposition 5.

It is a natural problem to search for a description of posets T with the form $q_T^*(x)$ non-negative, that are not positive. Such a description is given in [9,15] and, independently in [5,6], for *principal* posets (these are direct analogs of Euclidean diagrams).

For the general case, only partial results are known. In particular, in [17,18] non-negative posets of size $n \leq 16$ and rank $n-2$ are computed and analysed. For posets of arbitrary size, the following theorem is given in [13] (we recall from [21] that the *Hasse quiver* $\mathcal{H}(I)$ of I is the transitive reduction of the acyclic quiver $\mathcal{D}(I) = (V, A_I)$, with $i \to j \in A_I$ iff $i <_I j$; see Definition 1).

Theorem 3. *Assume that I is a connected poset of size n.*

(a) *I is non-negative of Dynkin type* $\mathrm{Dyn}_I = \mathbb{A}_n$ *if and only if $\overline{\mathcal{H}}(I)$ is a path graph.*

(b) *I is non-negative with* $\mathrm{Dyn}_I = \mathbb{A}_{n-1}$ *if and only if* $\overline{\mathcal{H}}(I)$ *is a cycle graph and* $\mathcal{H}(I)$ *has at least two sinks (i.e., maximal elements).*

(c) *If I is non-negative of Dynkin type* $\mathrm{Dyn}_I = \mathbb{A}_m$, *then* $m \in \{n, n-1\}$.

This class is further analysed in [12], where the following is shown.

Corollary 1. *If I and J are non-negative connected posets of Dynkin type $\mathbb{A}_m$, then $I \approx_{\mathbb{Z}} J$ if and only if* $\mathbf{specc}_I = \mathbf{specc}_J$.

The main aim of this work is to prove similar results for Dynkin type $\mathrm{Dyn}_I \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ non-negative connected posets.

Remark 1. The reader is referred to [10, Section 6] for a discussion of the origins and applications of the Coxeter-type study of finite posets.

2. Preliminaries

Throughout, we mainly use the terminology and notation introduced in [14, 18, 21, 26, 29]. In particular, by $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$ we denote the set of non-negative integers, the ring of integers, and the real number field, respectively. We use a row notation for vectors $v = [v_1, \dots, v_n]$ and write v^{tr} to denote column vectors. By $e_1, \dots, e_n$ we denote the standard $\mathbb{Z}$-basis of $\mathbb{Z}^n$. We say that two square integer matrices $X \in \mathbb{M}_n(\mathbb{Z})$ and $Y \in \mathbb{M}_n(\mathbb{Z})$ are $\mathbb{Z}$-*congruent* (denoted by $X \sim_{\mathbb{Z}} Y$) if there exists such a matrix $B \in \mathrm{Gl}_n(\mathbb{Z}) := \{A \in \mathbb{M}_n(\mathbb{Z});\ \det A = \pm 1\}$, that $B^{tr} \cdot X \cdot B = Y$.

All graphs considered in the paper are finite and simple. In particular, by a *bigraph* we mean a *signed graph* $G = (V_G, E_G, \mathrm{sgn}_G)$ consisting of a finite set of *vertices* V_G, a finite set of *edges* E_G and a *sign* function $\mathrm{sgn}_G \colon E_G \to \{-1, 1\}$. By an *edge*, we mean a pair of vertices and we denote them graphically as follows:

- *positive* edges $E_G^+ := \{e \in E_G;\ \mathrm{sgn}(e) = +1\}$ by dotted lines $u \cdots\cdots v$;
- *negative* edges $E_G^- := \{e \in E_G;\ \mathrm{sgn}(e) = -1\}$ by full lines u — v.

By a *graph* we mean a bigraph with negative edges only, and by a quiver (*digraph*, *directed graph*) we mean a graph $\mathcal{D}$, whose edges $e \in E_{\mathcal{D}}$ have a designated source $s(e) \in V_{\mathcal{D}}$ and a target $t(e) \in V_{\mathcal{D}}$ (such oriented edges are called *arrows*). By the *underlying graph* $\overline{\mathcal{D}}$ of a quiver $\mathcal{D}$ we mean a graph obtained from $\mathcal{D}$ by forgetting the orientation of its arrows.

We note that every bigraph $\Delta = (V, E, \sigma)$ is uniquely determined by the upper-triangular non-symmetric Gram matrix

$$\check{G}_\Delta = [d_{ij}^\Delta], \text{ where } d_{ij}^\Delta := \begin{cases} |E^+(i,j)| - |E^-(i,j)| & \text{if } i < j, \\ 1 & \text{if } i = j, \\ 0 & \text{if } i > j, \end{cases} \tag{5}$$

and the symmetric Gram matrix $G_\Delta := \frac{1}{2}(\check{G}_\Delta + \check{G}_\Delta^{tr}) \in \mathbb{M}_n(\mathbb{Q})$. A bigraph Δ is defined to be *non-negative* of *rank* m if its *symmetric Gram matrix* $G_\Delta \in \mathbb{M}_n(\mathbb{Q})$ is positive semi-definite of rank m. Two bigraphs Δ_1 and Δ_2 are said to be *strongly* (*weakly*) Gram $\mathbb{Z}$-congruent $\Delta_1 \approx_\mathbb{Z} \Delta_2$ ($\Delta_1 \sim_\mathbb{Z} \Delta_2$) if and only if $\check{G}_{\Delta_1} \sim_\mathbb{Z} \check{G}_{\Delta_2}$ ($G_{\Delta_1} \sim_\mathbb{Z} G_{\Delta_2}$).

Definition 1. The *Hasse quiver* of a poset $I = (V, \leq_I)$ is a simple directed graph $\mathcal{H}(I) = (V, A)$ with the set of arrows defined as follows: $x \to y \in A$ iff $x <_I y$ and there is no such $z \in V$, that $x <_I z <_I y$.

We call a poset I *connected* if the graph $\overline{\mathcal{H}}(I) := \overline{\mathcal{H}(I)}$ is connected. Given a set of vertices $\{s_1, \dots, s_t\} \subseteq I$, we denote by $I^{(s_1,\dots,s_t)}$ the induced subposet $I^{(s_1,\dots,s_t)} := I \setminus \{s_1, \dots, s_t\}$.

Following [14, 26, 29], we associate with a poset $I = (\{1, \dots, n\}, \preceq)$:

- the symmetric Gram matrix $G_I := \frac{1}{2}(C_I + C_I^{tr}) \in \mathbb{M}_n(\frac{1}{2}\mathbb{Z})$; (6)
- the (incidence) *bilinear form* $b_I \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$,
$$b_I(x, y) := \sum_{i \preceq j} x_i y_j = x \cdot C_I \cdot y^{tr}; \tag{7}$$
- the (incidence) unit quadratic form $q_I \colon \mathbb{Z}^n \to \mathbb{Z}$ defined by the formula
$$q_I(x) := b_I(x, x) = \sum_{i \in \{1,\dots,n\}} x_i^2 + \sum_{i \prec j} x_i x_j = x \cdot G_I \cdot x^{tr}; \tag{8}$$
- the kernel $\operatorname{Ker} q_I := \{v \in \mathbb{Z}^n;\ q_I(v) = 0\} \subseteq \mathbb{Z}^n$; (9)
- the set of roots $\mathcal{R}_I := \{v \in \mathbb{Z}^n;\ q_I(v) = 1\} \subseteq \mathbb{Z}^n$; (10)
- the Coxeter matrix $\mathrm{Cox}_I := -C_I \cdot (C_I^{-1})^{tr} \in \mathbb{M}_n(\mathbb{Z})$; (11)
- the Coxeter polynomial $\mathrm{cox}_I(t) := \det(tE - \mathrm{Cox}_I) \in \mathbb{Z}[t]$; (12)
- the Coxeter spectrum $\mathbf{specc}_I := \{\lambda \in \mathbb{C};\ \mathrm{cox}_I(\lambda) = 0\} \subseteq \mathbb{C}$. (13)

It is known that a poset I is non-negative of rank m if and only if the quadratic form $q_I(x)$ is positive semi-definite (i.e., $q_I(v) \geq 0$ for every $v \in \mathbb{Z}^n$) and its kernel $\operatorname{Ker} q_I \subseteq \mathbb{Z}^n$ is a free abelian subgroup of rank $\mathrm{crk}_I := n - m$, see [26]. By setting $G_{\Delta_I} := G_I$ we associate the bigraph Δ_I with an arbitrary finite poset I (we note that Δ_I has positive edges only).

Definition 2. Two posets I and J are said to be strongly (weakly) Gram $\mathbb{Z}$-congruent and denoted by $I \approx_{\mathbb{Z}} J$ ($I \sim_{\mathbb{Z}} J$) if their incidence matrices (symmetric Gram matrices) are $\mathbb{Z}$-congruent.

We call a non-negative poset I *positive* if $\mathrm{crk}_I = 0$, *principal* if $\mathrm{crk}_I = 1$, and *indefinite* if its symmetric Gram matrix G_I is not positive/negative semidefinite.

Proposition 1. *Let $I = (\{1, \ldots, n\}, \preceq)$ be a finite poset.*

(a) *The incidence quadratic form $q_I\colon \mathbb{Z}^{|I|} \to \mathbb{Z}$ (8) is non-negative of rank m if and only if the quadratic form $\widehat{q}_I\colon \mathbb{Z}^{|I|} \to \mathbb{Z}$ (4) is non-negative of rank m.*

(b) *The Tits quadratic form $q^*_T\colon \mathbb{Z}^{|T|+1} \to \mathbb{Z}$ (3) of a poset T is non-negative if and only if the incidence quadratic form $q_{T^*}\colon \mathbb{Z}^{|T^*|} \to \mathbb{Z}$ (8) of a one-peak poset T^* is non-negative, where $T^* := T \sqcup \{*\}$ is the enlargement of T by a unique maximal element $* \notin T$.*

Proof. (a) Without loss of generality, we may assume that the set of maximal elements of I has the form $\max I = \{k, k+1, \ldots, n\}$. Since $\widehat{q}_I = q_I \circ \widehat{t}$, where the homomorphism $\widehat{t}\colon \mathbb{Z}^n \to \mathbb{Z}^n$ is defined as

$$[x_1, \ldots, x_n] \overset{\widehat{t}}{\longmapsto} [x_1, \ldots, x_{k-1}, -x_k, \ldots, -x_n],$$

the thesis follows. Similarly in the case (b), since $q^*_T = q_{T^*} \circ t^-_*$ with

$$\mathbb{Z}^{|T|+1} \ni [x_1, \ldots, x_{|T|}, x_*] \overset{t^-_*}{\longmapsto} [x_1, \ldots, x_{|T|}, -x_*] \in \mathbb{Z}^{|T|+1}. \qquad \square$$

The kernel $\operatorname{Ker} q_I \subseteq \mathbb{Z}^n$ (9) of a non-negative poset I is a free abelian subgroup of rank crk_I, that admits a $(k_1, \ldots, k_{\mathrm{crk}_I})$-special $\mathbb{Z}$-basis in the following sense.

Proposition 2. *Assume that $I = (\{1, \ldots, n\}, \preceq)$ is a connected non-negative poset of corank $\mathrm{crk}_I = r \geq 1$.*

(a) *There exist integers* $1 \leq k_1 < \ldots < k_r \leq n$ *such that the rank* $r \geq 1$ *group* $\operatorname{Ker} q_I \subseteq \mathbb{Z}^n$ *admits a* $(k_1, \ldots, k_r)$*-special* $\mathbb{Z}$*-basis* $h^{(k_1)}, \ldots, h^{(k_r)}$*, i.e., a basis satisfying* $h^{(k_i)}_{k_i} = 1$ *and* $h^{(k_i)}_{k_j} = 0$ *for* $1 \leq i, j \leq r$ *and* $i \neq j$.

(b) *For every* $\{s_1, \ldots, s_t\} \subseteq \{k_1, \ldots, k_r\}$ *the subposet* $I^{(s_1,\ldots,s_t)}$ *is connected and non-negative of corank* $r - t \geq 0$.

(c) *The poset* $I^{(k_1,\ldots,k_r)}$ *is positive and connected.*

Proof. Since, without loss of generality, one may assume that $C_I = \check{G}_{\Delta_I}$ (i.e., assume that the vertices of the digraph $\mathcal{H}(I)$ are topologically ordered), apply [27, Proposition 5.1] and [30, Theorem 2.1]. □

Following [10,29], we associate with a connected non-negative poset I its *Dynkin type* $\operatorname{Dyn}_I \in \{\mathbb{A}_m, \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$.

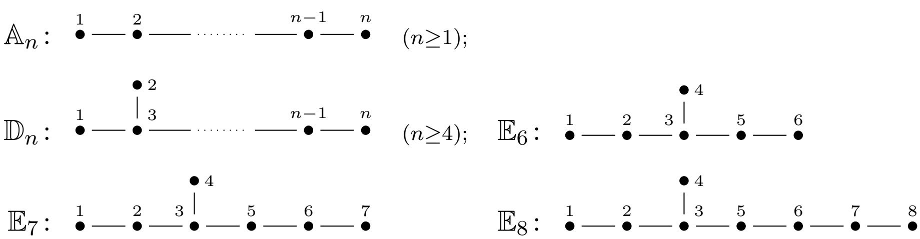


Table 1.1: Simply laced Dynkin diagrams

Definition 3. Assume that I is a connected non-negative poset of corank $r \geq 0$. The Dynkin type Dyn_I is defined to be the unique simply laced Dynkin diagram of Table 1.1 viewed as a bigraph

$$\operatorname{Dyn}_I \in \{\mathbb{A}_m, \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$$

such that $m = n - r$ and $\check{\Delta}_I \sim_{\mathbb{Z}} \operatorname{Dyn}_I$, where

- $\check{\Delta}_I := \Delta_I$ if $r = 0$ (i.e., I is positive);
- $\check{\Delta}_I := \Delta_{I^{(k_1,\ldots,k_r)}} \subseteq \Delta_I$ if $r > 0$ (see Proposition 2(c)).

We note that the bigraph Dyn_I can be efficiently calculated algorithmically, by means of an *inflation algorithm* (see [31, Algorithm 4.2]).

Proposition 3. *Assume that* I *and* J *are finite partially ordered sets, and* Δ *is a bigraph.*

(a) $I \simeq J \Rightarrow J \approx_{\mathbb{Z}} I$ (*isomorphism implies strong Gram* $\mathbb{Z}$*-congruence*);

(b) $I \approx_{\mathbb{Z}} J \Rightarrow \mathbf{specc}_I = \mathbf{specc}_J, \ \operatorname{cox}_I(t) = \operatorname{cox}_J(t)$;

(c) $I \approx_{\mathbb{Z}} J \Rightarrow I \sim_{\mathbb{Z}} J$;

(d) *If a poset* I *is non-negative of corank* r *and* $I \sim_{\mathbb{Z}} J$ $(I \sim_{\mathbb{Z}} \Delta)$, *then the poset* J (*bigraph* Δ) *is non-negative of corank* r.

Proof. The thesis follows from the definitions and direct calculations. In particular, it is straightforward to verify that $B_\sigma \cdot C_J \cdot B_\sigma^{tr} = C_I$, where σ is a permutation defining the $I \simeq J$ isomorphism and $B_\sigma \in \mathrm{Gl}_n(\mathbb{Z})$ is a matrix obtained from the identity matrix E by σ permutation of its rows. Further details can be found in [12, Fact 2.14], [26, Lemma 2.1], and [29, Lemma 3]. □

3. Main results

We start by showing that under certain circumstances (i.e., when $q_I(x)$ determines I uniquely), every connected non-negative corank $\mathrm{crk}_I > 0$ poset I can be constructed from a positive poset $J \subseteq I$. The following lemma is a generalization of [17, Theorem 3.4] (see also [30, Theorem 2.1]).

Lemma 1. *Let* I *be a connected non-negative poset of size* n *and corank* $\mathrm{crk}_I = r > 0$ *with a* $(k_1, \dots, k_r)$*-special* $\mathbb{Z}$*-basis* $h^{(k_1)}, \dots, h^{(k_r)}$ *of* $\operatorname{Ker} q_I$, *and let* $I' := I^{(k_1,\dots,k_r)} = I \setminus \{k_1, \dots, k_r\}$.

(a) *The vectors*

$$u^{(k_1)} := h^{(k_1)} - e_{k_1}, \ \dots, \ u^{(k_r)} := h^{(k_r)} - e_{k_r} \tag{14}$$

are roots of I *and the vectors* $\check{u}^{(k_i)} := u^{(k_i)}|_{\{1,\dots,n\}\setminus\{k_1,\dots,k_r\}} \in \mathbb{Z}^{n-r}$ *are roots of the positive connected poset* I'.

(b) *The incidence quadratic form* (8) *of* I *is given by*

$$\begin{aligned} q_I(x) = q_{I'}(\check{x}) &+ \sum_{i=1}^{r} x_{k_i}^2 - \sum_{i=1}^{r} 2b_{I'}(\check{x}, \check{u}^{(k_i)}) x_{k_i} \\ &+ \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} 2b_{I'}(\check{u}^{(k_j)}, \check{u}^{(k_i)}) x_{k_i} x_{k_j}, \end{aligned} \tag{15}$$

where $\check{x} := x|_{\{1,\dots,n\}\setminus\{k_1,\dots,k_r\}}$.

Proof. Throughout, by $\check{v} := v|_{\{1,\dots,n\}\setminus\{k_1,\dots,k_r\}} \in \mathbb{Z}^{n-r}$ we denote the restriction of a vector $v = [v_1, \dots, v_n] \in \mathbb{Z}^n$ to coordinates $\{1, \dots, n\} \setminus \{k_1, \dots, k_r\}$ and by $\overline{v} \in \mathbb{Z}^n$ the vector $\overline{v} := v - \sum_{i=1}^{r} v_{k_i} e_{k_i}$.

(a) It follows from [24, Theorem 3.2(a)] that for every $1 \le i \le r$ we have

$$q_I(u^{(k_i)}) = q_I(h^{(k_i)} - e_{k_i}) = q_I(-e_{k_i}) = 1.$$

Moreover, by the definition of the vector $u^{(k_i)}$ and properties of a $(k_1, \dots, k_r)$-special $\mathbb{Z}$-basis we have $q_{I'}(\check{u}^{(k_i)}) = q_I(u^{(k_i)}) = 1$, where the poset $I' := I^{(k_1,\dots,k_r)}$ is positive and connected, see Proposition 2.

(b) By Definition (8) we know that $q_I(v + w) = q_I(v) + q_I(w) + 2b_I(v, w)$ for any $v, w \in \mathbb{Z}^n$ and it follows that

$$\begin{aligned} q_I(v) &= q_I(\overline{v} + \sum_{i=1}^{r} v_{k_i} e_{k_i}) = q_I(\overline{v}) + q_I(\sum_{i=1}^{r} v_{k_i} e_{k_i}) + 2b_I(\overline{v}, \sum_{i=1}^{r} v_{k_i} e_{k_i}) \\ &= q_{I'}(\check{v}) + \sum_{i=1}^{r} v_{k_i}^2 + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} 2b_I(e_{k_i}, e_{k_j}) v_{k_i} v_{k_j} \qquad (16) \\ &\quad + \sum_{i=1}^{r} v_{k_i} 2b_I(\overline{v}, e_{k_i}). \end{aligned}$$

We recall from [24, Proposition 2.8(b)] that $b_I(-, h) = 0$ for any $h \in \operatorname{Ker} q_I$. Hence

$$0 = b_I(v, h) = b_I(v, \overline{h}) + \sum_{i=1}^{r} h_{k_i} b_I(v, e_{k_i})$$

and it follows that $\sum_{i=1}^{r} h_{k_i} b_I(v, e_{k_i}) = -b_I(v, \overline{h})$. Moreover, for a fixed $s \in \{k_1, \dots, k_r\}$ and $w = [w_1, \dots, w_n]$ with $w_{k_1} = \dots = w_{k_r} = 0$ we have

$$b_I(w, e_s) = -b_I(w, \overline{h}^{(s)}) = -b_I(w, u^{(s)}) = -b_{I'}(\check{w}, \check{u}^{(s)}),$$

and for every $s, t \in \{k_1, \dots, k_r\}$, $s \neq t$, we have

$$b_I(e_t, e_s) = -b_I(h^{(t)} - u^{(t)}, \overline{h}^{(s)}) = b_{q_I}(u^{(t)}, u^{(s)}) = b_{I'}(\check{u}^{(s)}, \check{u}^{(t)}).$$

Applying those equalities to (16) we conclude that

$$q_I(v) = q_{I'}(\check{v}) + \sum_{i=1}^{r} v_{k_i}^2 - \sum_{i=1}^{r} 2b_{I'}(\check{v}, \check{u}^{(k_i)}) v_{k_i}$$

$$+\sum_{i=1}^{r-1}\sum_{j=i+1}^{r} 2b_{I'}(\check{u}^{(k_j)}, \check{u}^{(k_i)})v_{k_i}v_{k_j},$$

and the proof is finished. □

Now we prove one of the main results of the paper.

Proof of Theorem 1. Let $I = (\{1,\ldots,n\},\preceq)$ be a Dynkin type $\mathrm{Dyn}_I \in \{\mathbb{E}_6,\mathbb{E}_7,\mathbb{E}_8\}$ connected non-negative poset I of corank $\mathrm{crk}_I = r > 0$ and let $h^{(k_1)},\ldots,h^{(k_r)}$ be a $(k_1,\ldots,k_r)$-special $\mathbb{Z}$-basis of $\operatorname{Ker} q_I$. By Definition 3 and Lemma 1 we have

$$\begin{aligned} q_I(x) = q_{I'}(\check{x}) + \sum_{i=1}^{r} x_{k_i}^2 - \sum_{i=1}^{r} 2b_{I'}(\check{x}, u^{(k_i)})x_{k_i} \\ + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} 2b_{I'}(u^{(k_i)}, u^{(k_j)})x_{k_i}x_{k_j}, \end{aligned} \tag{17}$$

where

- $\check{v} := v|_{\{1,\ldots,n\}\setminus\{k_1,\ldots,k_r\}} \in \mathbb{Z}^{n-r}$;
- I' is a connected positive poset of Dynkin type $\mathrm{Dyn}_I = \mathbb{E}_{n-r}$;
- for every $1 \leq i \leq r$ we have $\check{u}^{(k_i)} \in \mathcal{R}_{I'}$,where $u^{(k_i)} = h^{(k_i)} - e_{k_i}$.

Without loss of generality, we may assume that the elements of I are topologically sorted (i.e., $i \preceq j \Rightarrow i < j$), that is, the incidence matrix $C_I \in \mathbb{M}_n(\mathbb{Z})$ (1) is upper triangular. It follows that I is uniquely determined by its incidence quadratic form $q_I\colon \mathbb{Z}^n \to \mathbb{Z}$ (8). Therefore, up to isomorphism, the poset I is determined by

- positive connected poset $I' \subseteq I$ of Dynkin type $\mathrm{Dyn}_I = \mathbb{E}_{n-r}$;
- list of pairs (k_i, u_i), where $1 \leq k_1 < k_2 < \cdots < k_r \leq n$ are r different integer indices and $u_i \in \mathcal{R}_{I'}$ are roots of I', i.e., $q_{I'}(u_i) = 1$;

that is, $I = I'(k_1,\ldots,k_r; u_1,\ldots,u_r)$.

The assumption that $q_I(x)$ (17) is an incidence quadratic form of a poset implies that the roots $u_1,\ldots,u_r$ are pairwise different. Otherwise, we would have

$$\{0,1\} \ni 2b_{I'}(u_i,u_j) = 2b_{I'}(u_i,u_i) = 2q_{I'}(u_i) = 2.$$

Since the set $\mathcal{R}_{I'}$ is finite (see [25, Proposition 4.1]), we conclude that there is a finite number of posets of the form $I'(k_1, \ldots, k_r; u_1, \ldots, u_r)$ for any fixed positive connected poset I'. To finish the proof we note that the assumption $\mathrm{Dyn}_{I'} \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ yields $6 \leq |I'| = |\mathrm{Dyn}_{I'}| \leq 8$ and the number of posets of fixed size is finite. □

Although the proof of Theorem 1 yields a bound on the number of non-negative connected posets I of Dynkin type $\mathrm{Dyn}_I \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$, this estimate is far from optimal. We know that there exist, up to isomorphism, exactly $43, 197$, and 548 positive connected posets I' of Dynkin type $\mathbb{E}_6, \mathbb{E}_7$ and $\mathbb{E}_8$, respectively (see [10, Table 2.15]) and every such poset has exactly 72, 126, and 240 roots (since $|\mathcal{R}_{I'}| = |\mathcal{R}_{\mathrm{Dyn}_{I'}}|$, it follows from [1]). Therefore:

- $\mathrm{crk}_I \leq 240$;
- the total number of all, up to isomorphism, aforementioned posets I is less than

$$\#I \leq 43 \sum_{r=1}^{72} \binom{72}{r} P(6+r, r) + 197 \sum_{r=1}^{126} \binom{126}{r} P(7+r, r) + 548 \sum_{r=1}^{240} \binom{240}{r} P(8+r, r) < 1.858 \times 10^{486},$$

where $P(n, k) \coloneqq \frac{n!}{(n-k)!}$.

One can obtain a better estimation by a more detailed examination of the formula (15). By $\mathrm{Pos}_D^{\geq 0} \coloneqq \{I;\ \mathrm{Dyn}_I = D \text{ and } \mathrm{crk}_I > 0\}$ we denote a set of all, up to isomorphism, connected non-negative posets of Dynkin type D, that are not positive.

Corollary 2. *For the sets* $\mathrm{Pos}_{\mathbb{E}_n}^{\geq 0}$, *where* $n \in \{6, 7, 8\}$, *the maximum corank and cardinality are bounded as follows.*

- $\max(\{\mathrm{crk}_I;\ I \in \mathrm{Pos}_{\mathbb{E}_6}^{\geq 0}\}) \leq 10$ *and* $|\mathrm{Pos}_{\mathbb{E}_6}^{\geq 0}| < 4.93 \times 10^{12}$;
- $\max(\{\mathrm{crk}_I;\ I \in \mathrm{Pos}_{\mathbb{E}_7}^{\geq 0}\}) \leq 20$ *and* $|\mathrm{Pos}_{\mathbb{E}_7}^{\geq 0}| < 1.926 \times 10^{27}$;
- $\max(\{\mathrm{crk}_I;\ I \in \mathrm{Pos}_{\mathbb{E}_8}^{\geq 0}\}) \leq 28$ *and* $|\mathrm{Pos}_{\mathbb{E}_8}^{\geq 0}| < 8.472 \times 10^{40}$.

Proof. The proof is computational. By Lemma 1 we know that, up to isomorphism, every connected non-negative poset I of corank $\mathrm{crk}_I = r$ and Dynkin type $\mathrm{Dyn}_I = \mathbb{E}_{n-r}$ has a form

$$I \coloneqq I'(u_1, \ldots, u_r; i_1, \ldots, i_r), \tag{18}$$

where I' is a positive connected poset with $\mathrm{Dyn}_{I'} = \mathbb{E}_{n-r} \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$, $u_1, \ldots, u_r \in \mathcal{R}_{I'}$, and $1 \leq i_1 < \cdots < i_r \leq n$. To estimate the maximal corank and the number of such posets I, it suffices to estimate the number of posets of the form (18) and the maximal possible value of r. Since

$$q_I(x) = q_{I'}(\check{x}) + \sum_{i=1}^{r} x_{k_i}^2 - \sum_{i=1}^{r} 2b_{I'}(\check{x}, u_i)x_{k_i} + \sum_{i=1}^{r-1}\sum_{j=i+1}^{r} 2b_{I'}(u_j, u_i)x_{k_i}x_{k_j}$$

by (15), and, on the other hand, $q_I(x)$ is an incidence quadratic form (8), it follows that

$$\widehat{G}_{I'} \cdot u_i^{tr} \in \{0, -1\}^{n-r} \text{ and } u_j \cdot \widehat{G}_{I'} \cdot u_i^{tr} \in \{0, 1\}, \tag{19}$$

where $\widehat{G}_{I'} \coloneqq C_{I'} + C_{I'}^{tr}$ (see Definition (7)). Now, our aim is to estimate the maximal size and the number of sets $\{u_1, \ldots, u_r\} \in 2^{\mathcal{R}_{I'}}$ that agree with (19).

We proceed as follows: given a topologically sorted poset I' we compute its set of roots $\mathcal{R}_{I'} \subseteq \mathbb{Z}^{|I'|}$ and we construct the graph $G_{I'}^{\mathcal{R}} = (V, E)$, where $V = \{u \in \mathcal{R}_{I'};\ \widehat{G}_{I'} \cdot u^{tr} \in \{0, -1\}^{|I'|}\}$ and there is an edge between vertices u and v if $u \cdot \widehat{G}_{I'} \cdot v^{tr} \in \{0, 1\}$. It follows that every set of vectors $\{u_1, \ldots, u_r\} \subseteq \mathcal{R}_{I'}$ that satisfies the conditions (19) corresponds to a clique in the graph $G_{I'}^{\mathcal{R}}$. It follows that there are at most

$$\#\{I;\ I = I'(u_1, \ldots, u_r; i_1, \ldots, i_r)\} \leq \sum_{r=1}^{|V|} |\mathrm{clique}(G_{I'}^{\mathcal{R}}, r)| \cdot P(|I'| + r, r)$$

posets of the shape (18), where we denote by $\mathrm{clique}(G, k)$ the set of size k cliques in the graph G. By considering all, up to isomorphism, positive connected posets of Dynkin type $\mathbb{E}_6$, $\mathbb{E}_7$, and $\mathbb{E}_8$ we obtain the estimates. Moreover, by computing $\max(\{\omega(G_{I'}^{\mathcal{R}});\ I' \in \mathrm{Pos}_{\mathbb{E}_m}^0\})$ for $m \in \{6, 7, 8\}$, where $\mathrm{Pos}_D^0 \coloneqq \{I;\ \mathrm{Dyn}_I = D \text{ and } \mathrm{crk}_I = 0\}$ and $\omega(G)$ denotes the *clique number* of a graph G (i.e., the number of vertices in a maximum clique in G), we get the bound on the corank. □

Although the estimates of Corollary 2 are much better than the ones discussed earlier, they show that it is problematic to use the construction $I \mapsto I(u_1, \ldots, u_r; i_1, \ldots, i_r)$ to compute all connected non-negative posets of Dynkin type $\mathbb{E}_n$. This follows from the fact that for any positive connected I with $\mathrm{Dyn}_I = \mathbb{E}_n$ there exists many admissible sets $\{u_1, \ldots, u_r\} \in 2^{\mathcal{R}_I}$ and every such a set yields $P(n+r, r)$ possible quadratic forms. Therefore, we use a different strategy and, in particular, show that the estimates of Corollary 2 are off by orders of magnitude from the real values.

$\mathrm{crk}_I = r\backslash n$	6	7	8	9	10	11	# all
$\mathrm{crk}_I = 0$	43	197	548	–	–	–	788
$\mathrm{crk}_I = 1$	–	84	470	2102	–	–	2656
$\mathrm{crk}_I = 2$	–	–	40	244	1566	–	1850
$\mathrm{crk}_I = 3$	–	–	–	–	2	31	33
# all	43	281	1058	2346	1568	31	5327

Table 1.2: Number of connected non-negative posets I of Dynkin type $\mathbb{E}_{n-r}$ and corank $\mathrm{crk}_I = r$

Theorem 4. *If I is a finite connected non-negative poset of Dynkin type $\mathbb{E}_6$, $\mathbb{E}_7$ or $\mathbb{E}_8$, then $\mathrm{crk}_I \leq 3$. Moreover, the precise number of non-negative connected posets (up to isomorphism) with $\mathrm{crk}_I = r$ and $\mathrm{Dyn}_I = \mathbb{E}_{n-r}$ is detailed in Table 1.2. In particular, we have the following:*

- $\max(\{\mathrm{crk}_I;\ I \in \mathrm{Pos}^{\geq 0}_{\mathbb{E}_6}\}) = 2$ *and* $|\mathrm{Pos}^{\geq 0}_{\mathbb{E}_6}| = 124$;
- $\max(\{\mathrm{crk}_I;\ I \in \mathrm{Pos}^{\geq 0}_{\mathbb{E}_7}\}) = 3$ *and* $|\mathrm{Pos}^{\geq 0}_{\mathbb{E}_7}| = 716$;
- $\max(\{\mathrm{crk}_I;\ I \in \mathrm{Pos}^{\geq 0}_{\mathbb{E}_8}\}) = 3$ *and* $|\mathrm{Pos}^{\geq 0}_{\mathbb{E}_8}| = 3699$.

Proof. The first part of the proof is a computational one. By employing a slightly modified version of [15, Algorithm 3.1], where we do not limit calculations to principal posets only, we compute all (up to isomorphism) connected non-negative posets I with $|I| \leq 12$, encoded in the form of upper-triangular incidence matrices. There are exactly 40047 such posets. For every I we calculate its

- corank (equal $|I| - \mathbf{rank}\, G_I$);
- $(k_1^I, \ldots, k_r^I)$-special $\mathbb{Z}$-basis $h_I^{(k_1^I)}, \ldots, h_I^{(k_r^I)}$ of $\mathrm{Ker}\, q_I$ (by implementing procedure described in the proof of [27, Proposition 5.1]), and
- Dynkin type $\mathrm{Dyn}_I \in \{\mathbb{A}_m, \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ (by applying [31, Algorithm 4.2] to the bigraph $\Delta_{I^{(k_1,\ldots,k_r)}}$).

Next, we select Dynkin type $\mathrm{Dyn}_I \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ posets only. It follows that there are exactly 167, 913, and 4247 such posets, respectively. In particular, there is no such poset I with $|I| = 12$ (nor with $\mathrm{crk}_I = 4$). Comprehensive summary of computations is given in Table 1.2.

To finish the proof we need to show that there are no other Dynkin type $\mathrm{Dyn}_I = \mathbb{E}_{|I|-\mathrm{crk}_I}$ connected non-negative posets. Assume, to the contrary, that J is such a connected non-negative poset and $|J| = n > 12$. By Proposition 2(a) there exists an $(s_1, \ldots, s_r)$-special $\mathbb{Z}$-basis $h_J^{(s_1)}, \ldots, h_J^{(s_r)}$ of $\mathrm{Ker}\, q_J$ and $\mathrm{Dyn}_{J^{(s_1,\ldots,s_r)}} = \mathbb{E}_{n-r} = \mathbb{E}_m$, where $m \in \{6, 7, 8\}$ (see Definition 3). Consider the subposet $J' := J^{(s_1,\ldots,s_t)}$, where $t := 12-m$. It is easy to check that vectors $\check{h}_J^{(s_1)}, \ldots, \check{h}_J^{(s_t)}$, where $\check{x} := x|_{\{1,\ldots,n\}\setminus\{s_{t+1},\ldots,s_r\}}$, constitute a $(s_1, \ldots, s_t)$-special $\mathbb{Z}$-basis of $\mathrm{Ker}\, q_{J'}$. By construction, we have $|J'| = 12$ and $\Delta_{J^{(s_1,\ldots,s_r)}} = \Delta_{J'^{(s_1,\ldots,s_t)}}$. It follows that $\mathrm{Dyn}_{J'} = \mathrm{Dyn}_J = \mathbb{E}_m$ and we get the contradiction that finishes the proof. □

We recall that there is a strong connection between non-negative posets in the sense of the Tits quadratic form (3) and the incidence quadratic form (8). In particular, one associates with an arbitrary (not necessarily connected) finite poset T its unique *one-peak enlargement* $T^* := T \sqcup \{*\}$ by adding to T a single maximal element $* \notin T$. It is straightforward to verify that the form $q_T^*(x)$ (3) is non-negative of corank $\mathrm{crk}_T^* = r$ if and only if the form $q_{T^*}(x)$ (8) is non-negative of corank $\mathrm{crk}_{T^*} = r$, see Proposition 1(b). Here, by corank $\mathrm{crk}_T^* \geq 0$, we mean the rank of the free abelian group $\mathrm{Ker}\, q_T^* := \{v \in \mathbb{Z}^{n+1};\ q_T^*(v) = 0\} \subseteq \mathbb{Z}^{n+1}$. Consequently, Theorem 4 has the following interpretation in terms of the Tits quadratic form.

Corollary 3. *Let T be a finite poset with $q_T^*(x)$ (3) form non-negative. If $\mathrm{Dyn}_{T^*} \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$, then $\mathrm{crk}_T^* \leq 2$. Moreover, the precise number of non-negative connected posets (up to isomorphism) with $\mathrm{crk}_T^* = r$ and $\mathrm{Dyn}_{T^*} = \mathbb{E}_{n-r}$ is given in Table 1.3.*

$\mathrm{crk}_I = r \backslash n$	6	7	8	9	10	11	# all
$\mathrm{crk}_I = 0$	16	56	121	–	–	–	193
$\mathrm{crk}_I = 1$	–	31	132	422	–	–	585
$\mathrm{crk}_I = 2$	–	–	18	79	329	–	426
# all	16	87	271	501	329	–	1204

Table 1.3: Number of posets T with $q_T^*(x)$ form non-negative with $\mathrm{Dyn}_{T^*} = \mathbb{E}_{n-r}$ and corank $\mathrm{crk}_T^* = r$

Proof. The corollary follows by the results of Theorem 4, where one has to consider one-peak posets $I = T^*$ only. As an example, we give in Table 1.4 a full list of all, up to isomorphism, posets T with $\mathrm{crk}_T^* = 2$ and $\mathrm{Dyn}_{T^*} = \mathbb{E}_6$.

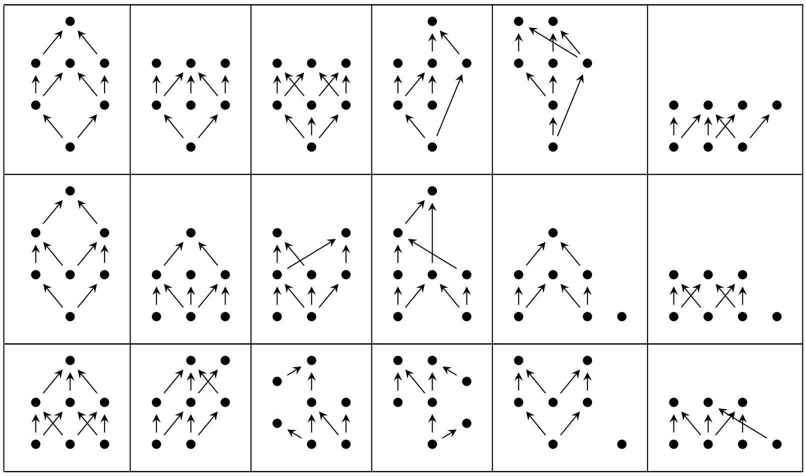


Table 1.4: Posets T with $\mathrm{crk}_T^* = 2$ and $\mathrm{Dyn}_{T^*} = \mathbb{E}_6$.

Details of the proof are left to the reader. In particular, we note that every poset I with the incidence quadratic form $q_I(x)$ (8) non-negative, $\mathrm{Dyn}_I \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ and $\mathrm{crk}_I = 3$ has at least two maximal elements, hence is not of the form T^*. □

Now we give a detailed Coxeter spectral analysis of connected non-negative posets I of Dynkin type $\mathrm{Dyn}_I \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$.

Theorem 5. *Assume that I is a finite connected non-negative poset of Dynkin type $\mathrm{Dyn}_I \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ and $n := |I|$.*

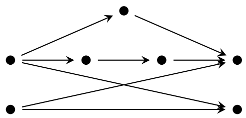
(a) *If $n = 6$, then I is positive (i.e. $\mathrm{crk}_I = 0$), $\mathrm{cox}_I(t) = t^6 + t^5 - t^3 + t + 1$ and, up to isomorphism, I is one of 43 posets presented in [16].*

(b) *If $n = 7$, then one of three possibilities hold:*

- *I is positive (i.e. $\mathrm{crk}_I = 0$), $\mathrm{cox}_I(t) = t^7 + t^6 - t^4 - t^3 + t + 1$ and, up to isomorphism, I is one of 197 posets described in [11];*
- *I is principal (i.e. $\mathrm{crk}_I = 1$), $\mathrm{cox}_I(t) = t^7 + t^6 - 2t^4 - 2t^3 + t + 1$ and, up to isomorphism, I is one of 83 posets described in [11];*

- *I is principal,* $\mathrm{cox}_I(t) = t^7 - t^5 - t^2 + 1$ *and, up to isomorphism, I has the following shape.*

$\mathcal{H}(I)$:

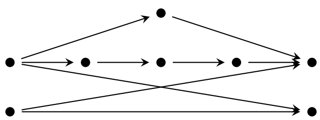


(c) *If $n = 8$, then one of seven possibilities hold:*

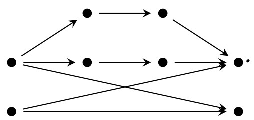
crk_I	cox_I	# I
0	$t^8 + t^7 - t^5 - t^4 - t^3 + t + 1$	548
1	$t^8 + t^7 - t^5 - 2t^4 - t^3 + t + 1$	463
	$t^8 - t^6 - t^2 + 1$	1
	$t^8 - 2t^6 + 2t^4 - 2t^2 + 1$	4
	$t^8 - t^5 - t^3 + 1$	2
2	$t^8 + t^7 + t^6 - 2t^5 - 2t^4 - 2t^3 + t^2 + t + 1$	39
	$t^8 - t^6 - t^2 + 1$	1

In particular, if I is a poset with $\mathrm{cox}_I(t) = t^8 - t^6 - t^2 + 1$, *then I is isomorphic to J or J', where* $\mathrm{crk}_J = 1$, $\mathrm{crk}_{J'} = 2$ *and*

$\mathcal{H}(J)$:



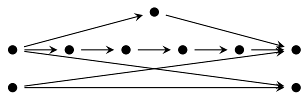
$\mathcal{H}(J')$:



(d) *If $n = 9$, then one of nine possibilities hold:*

crk_I	cox_I	# I
1	$t^9 + t^8 - t^6 - t^5 - t^4 - t^3 + t + 1$	2078
	$t^9 - 2t^7 - t^6 + 2t^5 + 2t^4 - t^3 - 2t^2 + 1$	11
	$t^9 - t^7 - t^2 + 1$	1
	$t^9 - t^7 - t^6 + t^5 + t^4 - t^3 - t^2 + 1$	9
	$t^9 - t^5 - t^4 + 1$	3
2	$t^9 + t^8 - 2t^5 - 2t^4 + t + 1$	236
	$t^9 - 2t^7 + t^5 + t^4 - 2t^2 + 1$	5
	$t^9 - 3t^7 - t^6 + 3t^5 + 3t^4 - t^3 - 3t^2 + 1$	1
	$t^9 - t^6 - t^3 + 1$	2

In particular, if I is a poset with $\mathrm{cox}_I(t) = t^9 - t^7 - t^2 + 1$, *then I has the following shape.*

$\mathcal{H}(I)$:


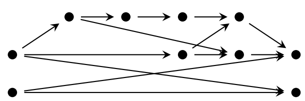
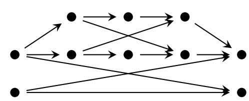
(e) *If $n = 10$, then one of seven possibilities hold:*

crk_I	cox_I	$\# I$
2	$t^{10} + t^9 - t^7 - t^6 - t^4 - t^3 + t + 1$	1485
	$t^{10} - 2t^8 - t^7 + t^6 + 2t^5 + t^4 - t^3 - 2t^2 + 1$	27
	$t^{10} - 2t^8 - 2t^7 + t^6 + 4t^5 + t^4 - 2t^3 - 2t^2 + 1$	7
	$t^{10} - t^8 - t^2 + 1$	16
	$t^{10} - t^8 - 2t^7 + t^6 + 2t^5 + t^4 - 2t^3 - t^2 + 1$	27
	$t^{10} - 2t^5 + 1$	4
3	$t^{10} - 3t^8 + 2t^6 + 2t^4 - 3t^2 + 1$	2

In particular, if I is a poset with

$$\mathrm{cox}_I(t) = t^{10} - 3t^8 + 2t^6 + 2t^4 - 3t^2 + 1,$$

then $\mathrm{crk}_I = 3$ *and I is isomorphic to one of the following posets.*

$\mathcal{H}(J)$:

$\mathcal{H}(J')$:


(f) *If $n = 11$, then* $\mathrm{crk}_I = 3$,

- $\mathrm{cox}_I(t) = t^{11} - 2t^9 - t^8 + 2t^6 + 2t^5 - t^3 - 2t^2 + 1$ *and I is, up to isomorphism, one of* 20 *posets described in [11];*
- $\mathrm{cox}_I(t) = t^{11} - t^9 - 3t^8 + 3t^6 + 3t^5 - 3t^3 - t^2 + 1$ *and I is, up to isomorphism, one of* 11 *posets described in [11].*

Proof. The proof is a computational one. Using standard Computer Algebra System (e.g. SageMath or Maple) tools we divide the list of connected non-negative posets I of Dynkin type $\mathrm{Dyn}_I \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ (obtained in the proof of Theorem 4) up to the Coxeter polynomial. In particular, there are exactly 19 671 such posets and they split into 29 classes, as described by the theorem (the list is available in [11]). □

As discussed earlier, posets with the Tits quadratic form non-negative are closely related to the ones with the incidence quadratic form (8) non-negative, see Proposition 1(b). On the other hand, these two classes are strikingly different on the level of Coxeter spectral analysis. In particular, when $\mathrm{Dyn}_{T^*} \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$, there is only one Coxeter class for a given corank crk^*_T.

Corollary 4. *Let T be a finite poset with $q^*_T(x)$ (3) form non-negative. If $\mathrm{Dyn}_{T^*} \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$, then the corank $\mathrm{crk}^*_T \geq 0$ determines the Coxeter polynomial $\mathrm{cox}_{T^*}(t)$ uniquely. Moreover, there exist exactly 9 Coxeter classes of such posets, as detailed in Table 1.8.*

Proof. The proof is a computational one: it suffices to use the arguments given in the proof of Theorem 5 and Corollary 3. Details are left to the reader. □

n	crk_{T^*}	$\mathrm{cox}_{T^*}(t)$	# I
5	0	$t^6 + t^5 - t^3 + t + 1$	16
6	0	$t^7 + t^6 - t^4 - t^3 + t + 1$	56
	1	$t^7 + t^6 - 2t^4 - 2t^3 + t + 1$	31
7	0	$t^8 + t^7 - t^5 - t^4 - t^3 + t + 1$	121
	1	$t^8 + t^7 - t^5 - 2t^4 - t^3 + t + 1$	132
	2	$t^8 + t^7 + t^6 - 2t^5 - 2t^4 - 2t^3 + t^2 + t + 1$	18
8	1	$t^9 + t^8 - t^6 - t^5 - t^4 - t^3 + t + 1$	422
	2	$t^9 + t^8 - 2t^5 - 2t^4 + t + 1$	79
9	2	$t^{10} + t^9 - t^7 - t^6 - t^4 - t^3 + t + 1$	329

Table 1.8: Number of all non-negative posets T, up to the corank crk^*_T and Coxeter polynomial $\mathrm{cox}_{T^*}(t)$

Proof of Theorem 2. Our aim is to show that

$$I \approx_{\mathbb{Z}} J \Leftrightarrow \mathbf{specc}_I(t) = \mathbf{specc}_J(t)$$

given non-negative connected posets I and J of Dynkin type $\mathrm{Dyn}_I = \mathrm{Dyn}_J \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$. Since the equivalence

$$(\mathrm{crk}_I, \mathbf{specc}_I) = (\mathrm{crk}_J, \mathbf{specc}_J) \;\Leftrightarrow\; (\mathrm{crk}_I, \mathrm{cox}_I) = (\mathrm{crk}_J, \mathrm{cox}_J)$$

is obvious, it suffices to show that $I \approx_{\mathbb{Z}} J \Leftrightarrow (\mathrm{crk}_I, \mathrm{cox}_I) = (\mathrm{crk}_J, \mathrm{cox}_J)$.

The implication "$\Rightarrow$" is a consequence of the fact that the strong Gram $\mathbb{Z}$-congruence of matrices implies the equality of ranks and Coxeter polynomials, see Proposition 3.

The proof of the implication "$\Leftarrow$" is a computational one. First, using procedures described in the proofs of Lemma 5 and Theorem 4, we generate the list **nnegE** of all, up to isomorphism, non-negative connected posets I of Dynkin type $\mathrm{Dyn}_I \in \{\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$, encoded in the form of incidence matrices. This list is finite (see Theorem 1) and consists of exactly 5327 elements (see Theorem 4). Next, we divide **nnegE** into sublists, up to the corank and Coxeter polynomial, i.e., we calculate the sublist

$$\mathbf{nnegE}^{m,r}_{\mathrm{cpol}} \coloneqq \{C_I \in \mathbf{nnegE};\ |I| = m,\ \mathrm{crk}_I = r \text{ and } \mathrm{cox}_I = \mathrm{cpol}\}.$$

There are exactly 29 such sublists and 24 of them contain more than one poset. Now, for each of the 24 lists $\mathbf{nnegE}^{m,r}_{\mathrm{cpol}}$ with $|\mathbf{nnegE}^{m,r}_{\mathrm{cpol}}| > 1$:

- we select a single matrix $C_I \in \mathbf{nnegE}^{m,r}_{\mathrm{cpol}} \subseteq \mathbb{M}_m(\mathbb{Z})$;
- for every remaining matrices $C_J \in \mathbf{nnegE}^{m,r}_{\mathrm{cpol}} \setminus \{C_I\}$, using the procedure described in [10, Section 5], we calculate such a $\mathbb{Z}$-invertible matrix $B_{JI} \in \mathbb{M}_m(\mathbb{Z})$ that $B^{tr}_{JI} \cdot C_J \cdot B_{JI} = C_I$.

The list of all calculated matrices is given in [11]. □

Conclusion and future work

In the present work, we give a complete description of connected non-negative posets I of Dynkin type $\mathrm{Dyn}_I = \mathbb{E}_m$, and we show that there is only a finite number of such posets. In particular, we prove [13, Conjecture 6.1] and show that $\mathrm{crk}_I \leq 3$, i.e., all such posets have a corank bounded by 3. This makes the case of $\mathrm{Dyn}_I = \mathbb{E}_m$ posets similar to the $\mathrm{Dyn}_I = \mathbb{A}_m$ case, where the corank is less than or equal to one, see [13]. This result has a direct application to the Dynkin type recognition problem.

Proposition 4. *The Dynkin type* $\mathrm{Dyn}_I \in \{\mathbb{A}_m, \mathbb{D}_m, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8\}$ *of a connected non-negative poset* I *of size* $n \coloneqq |I|$ *and corank* $\mathrm{crk}_I = r$ *can be computed with the following time complexity, depending on the data structure used to encode the poset.*

(a) *$O(n^3)$, using the incidence matrix $C_I \in \mathbb{M}_{|I|}(\mathbb{Z})$ (1).*

(b) *$O(n^2)$, using the adjacency matrix $Ad_{\mathcal{H}(I)} \in \mathbb{M}_{|I|}(\mathbb{Z})$ of its Hasse quiver $\mathcal{H}(I)$ (see Definition 1).*

(c) *$O(n)$, using the adjacency list of its Hasse digraph $\mathcal{H}(I)$.*

(d) *$O(1)$, using the adjacency list of its Hasse digraph $\mathcal{H}(I)$, sorted by degrees of vertices.*

Proof. Since we are interested in asymptotic complexity, without loss of generality we may assume that $|I| \geq 12$. Hence, in view of Theorem 4, we get that $\mathrm{Dyn}_I \in \{\mathbb{A}_m, \mathbb{D}_m\}$. Moreover, it follows from [13, Theorem 4.4] (see also Theorem 3) that exactly one of the following two situations holds:

- $\overline{\mathcal{H}}(I)$ is a path or a cycle graph, and then $\mathrm{Dyn}_I = \mathbb{A}_m$;
- $\mathrm{Dyn}_I = \mathbb{D}_m$,

where $m \coloneqq n - \mathrm{crk}_I$. Summing up, in order to distinguish between Dynkin types $\mathbb{A}_m$ and $\mathbb{D}_m$ it suffices to check if all vertices of the Hasse quiver $\mathcal{H}(I)$ have degree at most 2. The time complexity of this operation depends on the data structure used for poset encoding and is as follows.

(b) In the adjacency matrix case, computing the degree of a vertex requires $O(n)$ operations and, in the pessimistic case, we have to check all n vertices, thus we have $n \cdot O(n) = O(n^2)$ complexity.

(c) In the pessimistic case one has to check degrees of all n vertices, each of which has at most two arrows, thus we have $n \cdot O(2) = O(n)$.

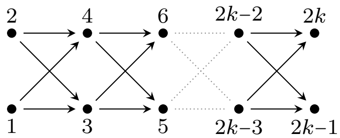
(d) Since the vertices are sorted by their degrees, one only has to check whether the vertex $v \in I$ of the largest degree satisfies $deg(v) < 3$, which is an $O(1)$ operation.

To finish the proof, we note that in case (a) it suffices to transform the incidence matrix C_I into the adjacency matrix $Ad_{\mathcal{H}(I)}$, which can be done in $O(n^3)$, and proceed as in (b). □

The case of non-negative connected posets I of the Dynkin type $\mathbb{D}_m$ is more complex, as there is no restriction on the poset corank.

Proposition 5. *Let $k \geq 3$ be a natural number. The garland $\mathcal{G}_k$, defined by the Hasse digraph*

$$\mathcal{H}(\mathcal{G}_k) =$$



,

is non-negative of Dynkin type $\mathrm{Dyn}_{\mathcal{G}_k} = \mathbb{D}_{k+1}$ *and corank* $\mathrm{crk}_{\mathcal{G}_k} = k - 1$*.*

Proof. It is straightforward to verify that

$$q_{\mathcal{G}_k}(x) = \left(x_1 + \frac{1}{2}\sum_{i=3}^{2k} x_i\right)^2 + \left(x_2 + \frac{1}{2}\sum_{i=3}^{2k} x_i\right)^2 + \frac{1}{2}\sum_{i=2}^{k}(x_{2i-1} - x_{2i})^2,$$

hence the garland poset $\mathcal{G}_k$ is non-negative. Moreover, integer vectors $h^3, h^5, \ldots, h^{2k-1} \in \mathbb{Z}^{2k}$, where $h^i := e_i + e_{i+1} - e_1 - e_2$ form such a basis of the free abelian group $\mathrm{Ker}\, q_{\mathcal{G}_k} \subseteq \mathbb{Z}^{2k}$, that $h^i_i = 1$ and $h^i_j = 0$ for $i, j \in \{3, 5, \ldots, 2k-1\}$ and $i \neq j$. Since $\mathcal{G}_k^{(3,5,\ldots,2k-1)} = {}_0\mathbb{D}^*_{k+1} \diamond \mathbb{A}_0$ is positive of Dynkin type $\mathbb{D}_{k+1}$ (see [14, Theorem 5.2]), the thesis follows from Definition 3. □

We do not know a general description of type $\mathrm{Dyn}_I = \mathbb{D}_m$ non-negative connected posets, but the experimental results (see [10, 18]) suggest that their number grows exponentially.

Conjecture 1. *Let $n \geq 4$ be an integer. Up to poset isomorphism, there exist exactly:*

$$Nneg(n, \mathbb{D}_n) = \begin{cases} 5, & n = 4, \\ (n+5)2^{n-4} - 1, & n > 4, \end{cases}$$

positive posets I of the Dynkin type $\mathrm{Dyn}_I = \mathbb{D}_n$*.*

Moreover, the experimental results suggest that the number of Coxeter types depends on the corank in the following way.

Conjecture 2. *The number of Coxeter types of* $\mathrm{Dyn}_I = \mathbb{D}_m$ *non-negative connected posets of size $n = |I|$ equals*

$$|\mathbf{CTypes}_I^{\mathrm{crk}_I}| = \begin{cases} 1, & \textit{if } \mathrm{crk}_I = 0, \\ \max(0, n - 2\,\mathrm{crk}_I - 2), & \textit{if } \mathrm{crk}_I = 2k+1, \\ \max(0, n - 2\,\mathrm{crk}_I - 1 - (\mathrm{crk}_I \bmod 2)), & \textit{if } \mathrm{crk}_I = 4k \\ \max(0, n - 2\,\mathrm{crk}_I - 1 - (n \bmod 2)), & \textit{otherwise.} \end{cases}$$

Nevertheless, this description does not give any insights into the structure of $\mathbb{D}_m$ type non-negative connected posets.

Open problem 1. Give a structural description of non-negative connected posets of Dynkin type $\mathrm{Dyn}_I = \mathbb{D}_m$.

References

[1] Assem, I., Skowroński, A., Simson, D.: Elements of the representation theory of associative algebras: Techniques of representation theory, vol. **65**. London Math. Soc. Student Texts. Cambridge University Press, Cambridge (2006). https://doi.org/10.1017/CBO9780511614309

[2] Bondarenko, V.M. Minimax equivalence method: initial ideas, first applications and new concepts. Algebra Discrete Math. **38**(1), 1–22 (2024). https://doi.org/10.12958/adm2332

[3] Bondarenko, V.M., Futorny, V., Klimchuk, T., Sergeichuk, V.V., Yusenko, K.: Systems of subspaces of a unitary space. Linear Algebra Appl. **438**(5), 2561–2573 (2013). https://doi.org/10.1016/j.laa.2012.10.038

[4] Bondarenko V.M., Stepochkina, M.V.: (Min, max)-ekvivalentnost' chastichno uporyadochennyh mnozhestv i kvadratichnaya forma Titsa ((Min, max)-equivalence of partially ordered sets and quadratic Tits form). Zb. Pr. Inst. Mat. NAN Ukr. **2**(3), 18–58 (2005)

[5] Bondarenko, V.M., Styopochkina, M.V.: The classification of serial posets with the non-negative quadratic Tits form being principal. Algebra Discrete Math. **27**(2), 202–211 (2019)

[6] Bondarenko, V.M., Styopochkina, M.V.: Classification of the almost positive posets. Algebra Discrete Math. **39**(1), 65–88 (2025). https://doi.org/10.12958/adm2391

[7] Drozd, Y.A.: Coxeter transformations and representations of partially ordered sets. Funct. Anal. Its. Appl. **8**, 219–225 (1974). https://dx.doi.org/10.1007/BF01075695

[8] Gabriel, P.: Unzerlegbare Darstellungen I. Manuscripta Math. **6**(1), 71–103 (1972). https://doi.org/10.1007/BF01298413

[9] Gąsiorek, M.: A Coxeter type classification of one-peak principal posets. Linear Algebra Appl. **582**, 197–217 (2019). https://doi.org/10.1016/j.laa.2019.08.001

[10] Gąsiorek, M.: On algorithmic Coxeter spectral analysis of positive posets. Appl. Math. Comput. **386**, 125507 (2020). https://doi.org/10.1016/j.amc.2020.125507

[11] Gąsiorek, M.: Complete list of connected non-negative posets of Dynkin type $\mathbb{E}$. `http://www.umk.pl/~mgasiorek/nneg_typ_e.zip`

[12] Gąsiorek, M.: A Coxeter type classification of Dynkin type $\mathbb{A}_\mathrm{n}$ non-negative posets. Preprint at https://arxiv.org/abs/2205.15813 (2023)

[13] Gąsiorek, M.: Structure of non-negative posets of Dynkin type $\mathbb{A}_\mathrm{n}$. Preprint at https://doi.org/10.48550/arXiv.2205.15032 (2023)

[14] Gąsiorek, M., Simson, D.: One-peak posets with positive quadratic Tits form, their mesh translation quivers of roots, and programming in Maple and Python. Linear Algebra Appl. **436**(7), 2240–2272 (2012). https://doi.org/10.1016/j.laa.2011.10.045

[15] Gąsiorek, M., Simson, D., Zając, K.: Algorithmic computation of principal posets using Maple and Python. Algebra Discrete Math. **17**(1), 33–69 (2014)

[16] Gąsiorek, M., Simson, D., Zając, K.: On Coxeter type study of non-negative posets using matrix morsifications and isotropy groups of Dynkin and Euclidean diagrams. European J. Combin. **48**, 127–142 (2015). https://doi.org/10.1016/j.ejc.2015.02.015

[17] Gąsiorek, M., Simson, D., Zając, K.: Structure and a Coxeter–Dynkin type classification of corank two non-negative posets. Linear Algebra Appl. **469**, 76–113 (2015). https://doi.org/10.1016/j.laa.2014.11.003

[18] Gąsiorek, M., Zając, K.: On algorithmic study of non-negative posets of corank at most two and their Coxeter-Dynkin types. Fundam. Inform. **139**(4), 347–367 (2015). https://doi.org/10.3233/FI-2015-1238

[19] Kleiner, M.M.: Partially ordered sets of finite type. J. Math. Sci. **3**, 607–615 (1975). https://doi.org/10.1007/BF01084663

[20] Nazarova, L.A., Roiter, A.V.: Representations of partially ordered sets. J. Math. Sci. **3**, 585–606 (1975).https://doi.org/10.1007/BF01084662

[21] Simson, D.: Linear representations of partially ordered sets and vector space categories. In: Göbel, R., Macintyre, A. (eds.) Algebra, logic and applications, vol. **4**. Gordon and Breach Science Publishers, Montreux (1992)

[22] Simson, D.: Posets of finite prinjective type and a class of orders. J. Pure Appl. Algebra **90**(1), 77–103 (1993). https://doi.org/10.1016/0022-4049(93)90138-J

[23] Simson, D.: Integral bilinear forms, Coxeter transformations and Coxeter polynomials of finite posets. Linear Algebra Appl. **433**(4), 699–717 (2010). https://doi.org/10.1016/j.laa.2010.03.041

[24] Simson, D.: Mesh algorithms for solving principal Diophantine equations, sand-glass tubes and tori of roots. Fundam. Inform. **109**(4), 425–462 (2011). https://doi.org/10.3233/FI-2011-520

[25] Simson, D.: Mesh geometries of root orbits of integral quadratic forms. J. Pure Appl. Algebra **215**(1), 13–34 (2011). https://doi.org/10.1016/j.jpaa.2010.02.029

[26] Simson, D.: A Coxeter-Gram classification of positive simply laced edge-bipartite graphs. SIAM J. Discrete Math. **27**(2), 827–854 (2013). https://doi.org/10.1137/110843721

[27] Simson, D.: Symbolic algorithms computing Gram congruences in the Coxeter spectral classification of edge-bipartite graphs, II. Isotropy mini-groups. Fundam. Inform. **145**(1), 49–80 (2016). https://doi.org/10.3233/FI-2016-1346

[28] Simson, D.: Weyl orbits of matrix morsifications and a Coxeter spectral classification of positive signed graphs and quasi-Cartan matrices of Dynkin type $\mathbb{A}_n$. Adv. Math. **404**, 108389 (2022). https://doi.org/10.1016/j.aim.2022.108389

[29] Simson, D., Zając, K.: A framework for Coxeter spectral classification of finite posets and their mesh geometries of roots. Int. J. Math. Sci. 743734 (2013). https://doi.org/10.1155/2013/743734

[30] Zając, K.: On the structure of loop-free non-negative edge-bipartite graphs. Linear Algebra Appl. **579**, 262–283 (2019). https://doi.org/10.1016/j.laa.2019.06.002

[31] Zając, K.: On polynomial time inflation algorithm for loop-free non-negative edge-bipartite graphs. Discrete Appl. Math. **283**, 28–43 (2020). https://doi.org/10.1016/j.dam.2019.12.002

Contact information

M. Gąsiorek Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, 87-100 Toruń, Poland
E-Mail: mgasiorek@mat.umk.pl

Received by the editors: 11.06.2025
and in final form 15.09.2025.