Integer quadratic forms and extensions of subsets of linearly independent roots

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ABSTRACT. We consider subsets of linearly independent roots in a certain root system Φ. Let S' be such a subset, and let S' be associated with any Carter diagram Γ'. The main question of the paper: what root $\gamma \in \Phi$ can be added to S' so that $S' \cup \gamma$ is also a subset of linearly independent roots? This extra root γ is called the *linkage root*. The vector γ^∇ of inner products $\{(\gamma, \tau'_i) \mid \tau'_i \in S'\}$ is called the *linkage label vector*. Let $B_{\Gamma'}$ be the Cartan matrix associated with Γ'. It is shown that γ is a linkage root if and only if $\mathscr{B}^\vee_{\Gamma'}(\gamma^\nabla) < 2$, where $\mathscr{B}^\vee_{\Gamma'}$ is a quadratic form with the matrix inverse to $B_{\Gamma'}$. The set of all linkage roots for Γ' is called a *linkage system* and is denoted by $\mathscr{L}(\Gamma')$. The sizes of $\mathscr{L}(\Gamma')$ and $\mathscr{L}(\Gamma)$ are the same for diagrams Γ and Γ' that have the same rank and ADE type. Let $W^\vee$ be the Weyl group of the quadratic form $\mathscr{B}^\vee_{\Gamma'}$. The sizes and structure of orbits for linkage systems $\mathscr{L}(D_l)$ and $\mathscr{L}(D_l(a_k))$ are presented.

1. Introduction

1.1. Brief summary

The Carter diagram is a generalization of the Dynkin diagram that allows cycles of even length. Let Γ be an arbitrary Carter diagram. A set of linearly independent roots $S = \{\tau_1, \ldots, \tau_n\}$ is called Γ-set, if the vertices

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of the diagram Γ are in one-to-one correspondence with the roots of S and the inner products (τ_i, τ_j) correspond to the edges of Γ. Precise definitions of what a Carter diagram and Γ-set are will be given a little later, see § 1.2 and § 1.3.

Let $\widetilde{\Gamma}$ be the diagram obtained from a Carter diagram Γ by adding one extra root γ, with its bonds, so that the roots corresponding to $\widetilde{\Gamma}$ form a linearly independent root subset. We would like to describe such roots in the general case. The extra root γ is called a *linkage root*, the diagram $\widetilde{\Gamma}$ is called a *linkage diagram*. More precise definitions will be given shortly.

The simple criterion for a given vector γ to be a linkage root given in the terms of the inverse quadratic form $\mathscr{B}^\vee$ to the Cartan matrix associated with Γ is as follows:

$$\mathscr{B}_\Gamma^\vee(\gamma^\nabla) < 2, \tag{1.1}$$

where γ^∇ is so-called *linkage label vector*, the vector of the inner products of γ with roots from a given Γ-set (Theorem 3.3). There exists one-to-one correspondence between linkage diagrams and linkage label vectors. The Weyl group $W^\vee$ preserving the quadratic form $\mathscr{B}^\vee$ acts on the space of the linkage diagrams, or, equivalently, on the space of linkage label vectors, see § 2.4.

The set $\mathscr{L}_{\widetilde{\Gamma}}(\Gamma)$ of all linkage systems for the given vertex extension $\Gamma \prec \widetilde{\Gamma}$ is said to be *partial linkage system*. (For the definition of vertex extension, see § 4.1.) The union of all partial systems using all possible extensions $\Gamma \prec \widetilde{\Gamma}$ is called the *full linkage system* of Γ, or, for short, *linkage system*:

$$\mathscr{L}(\Gamma') = \bigcup_{\Gamma' \prec \widetilde{\Gamma}} \mathscr{L}_{\widetilde{\Gamma}}(\Gamma'). \tag{1.2}$$

The size and structure of linkage systems $\mathscr{L}(D_l)$ and $\mathscr{L}(D_l(a_k))$ are described in Theorem 5.3, and Figs. 9–12, Figs. 4–5, see also Table 2.1. The group $W^\vee$ acts on the linkage system and forms several orbits that are not connected to each other.

The Carter diagrams having the same rank and type form *homogeneous classes*. For example, E_6, $E_6(a_1)$ and $E_6(a_2)$ form the homogeneous class $C(E_6)$. Any homogeneous class contains only one Dynkin diagram, see § 1.5. For each pair of diagrams $\{\widetilde{\Gamma}, \Gamma\}$ out of one homogeneous class, there exists *transition* matrix M mapping each $\widetilde{\Gamma}$-set $\widetilde{S}$ to some Γ-set S (Theorem 1.1). The matrix M acts only on one element in $\widetilde{S}$, the remaining elements remain fixed. The transition matrix is an

involution. Transition matrices make it possible to connect objects associated with different Carter diagrams from the same homogeneous class, see § 1.4.

The transition mappings are close in properties to the transformations introduced by A. Gabrielov for systematic study of quadratic forms associated with singularities, and to the transformations introduced by S. Ovsienko for the study weakly positive unit quadratic forms, see § 1.6.1.

1.2.   Root system, Cartan matrix, Carter diagram

Let Φ be a finite root system, W be the corresponding finite Weyl group. We consider only root systems with simply-laced Dynkin diagrams. The Cartan matrix associated with Φ is denoted by B. Let $\mathscr{B}$ be the quadratic Tits form associated with B and $(\cdot, \cdot)$ be the inner product induced by $\mathscr{B}$. For the Cartan matrix B, the following well-known property[1] holds:

$$\mathscr{B}(\alpha) = 2 \iff \alpha \in \Phi. \tag{1.3}$$

In [4], Carter introduced admissible diagrams to describe conjugacy classes in W. These diagrams can be used to describe other objects, such as root subsets[2] and quadratic forms. Let $S \subset \Phi$ be a root subset, roots of S are not necessarily simple. To the subset S we associate some diagram Γ that provides one-to-one correspondence between roots of S and nodes of Γ. The diagram Γ is said to be *admissible* if the following two conditions hold:

    (a) Nodes Γ correspond to a subset of linearly independent

       roots in Φ. (1.4)

    (b) Any cycle in Γ contains an even number of nodes.

In admissible diagrams all edges are solid, whereas Carter diagrams have solid and dotted diagrams that distinguish negative and positive inner products, see § 1.7. In [26], it was observed that the cycles in the Carter diagrams contain at least one solid edge $\{\alpha_1, \beta_1\}$ (on which $(\alpha_1, \beta_1) < 0$) and at least one dotted edge $\{\alpha_2, \beta_2\}$ (on which $(\alpha_2, \beta_2) > 0$), see § 1.9. In this paper we introduce new objects-linkage roots, linkage diagrams and others, which are also closely related to Carter diagrams.

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[1]For the matrix $B/2$, (1.3) looks like this: $\mathscr{B}(\alpha) = 1 \Leftrightarrow \alpha \in \Phi$, see [15, Prop. 1.6].
[2]In what follows, the phrase "root subset" means a subset of linearly independent roots.

1.3. The partial Cartan matrix

Let Γ be a Carter diagram. A set of linearly independent roots $S = \{\tau_1, \ldots, \tau_n\}$ is called Γ-set, if the vertices of the diagram Γ are on one-to-one correspondence with the roots of S and the inner products (τ_i, τ_j) correspond to the edges of Γ as follows:

$$(\tau_i, \tau_j) = \begin{cases} 0, & \text{if } \tau_i \text{ and } \tau_j \text{ are not connected,} \\ -1, & \text{if edge } \{\tau_i, \tau_j\} \text{ is solid,} \\ 1, & \text{if edge } \{\tau_i, \tau_j\} \text{ is dotted.} \end{cases}$$

Besides this, $(\tau_i, \tau_i) = 2$ for any i. Similarly to the Cartan matrix associated with Dynkin diagrams, for each pair $\{\Gamma, S\}$, we determine the following matrix:

$$B_\Gamma := \begin{pmatrix} (\tau_1, \tau_1) & \ldots & (\tau_1, \tau_n) \\ \ldots & \ldots & \ldots \\ (\tau_n, \tau_1) & \ldots & (\tau_n, \tau_n) \end{pmatrix}. \tag{1.5}$$

We call the matrix B_Γ the *partial Cartan matrix* corresponding to Γ. Why partial? Since in this case some off-diagonal elements are 1, they correspond to the dotted edges. The partial Cartan matrix B_Γ is well-defined since products (τ_i, τ_j) in (1.5) do not depend on the choice of the Γ-set S. The elements of the partial Cartan matrix are uniquely determined by the diagram Γ.

1.4. Some properties of Dynkin and Carter diagrams

Any simply-laced Dynkin diagram is the particular case of the Carter diagram. We will use the phrase "Carter diagram" to refer to a Carter diagram which is not a Dynkin diagram unless otherwise stated. For a list of all Carter diagrams, see Fig. 1. This list is divided into homogeneous classes, see § 1.5.

(i) Consider the orbit of any root subset associated with a Carter diagram (resp. Dynkin diagram) under the action of the Weyl group. In the case of a Carter diagram, each root subset in this orbit contains non-simple roots. In the case of a Dynkin diagram, some root subset on this orbit consists only of simple roots.

(ii) For both Carter and Dynkin diagrams, the associated quadratic forms are positive definite, see Propositions 3.1. The partial Cartan

matrix $B_{\Gamma'}$ for the Carter diagram Γ' is related to the Cartan matrix B_Γ for some Dynkin diagram Γ by a certain transition matrix M:

$$^tM \cdot B_{\Gamma'} \cdot M = B_\Gamma, \qquad (1.6)$$

see § 1.6.

(iii) The transition matrix M modifies the root subset, the quadratic form as (1.6) and even the underlying diagram. The transition matrix combines the Carter diagrams and the corresponding Dynkin diagram together into one homogeneous class, see § 1.5. Each homogeneous class contains only one Dynkin diagram.

(iv) The Carter diagrams contain 4-cycles $D_4(a_1)$ as subdiagrams. In principle, Carter diagrams can contain the cycles of length > 4. It is shown in [26, Theorem 3.1] that any Carter diagram containing l-cycles, where $l > 4$, is equivalent to another Carter diagram containing only 4-cycles. To realize this equivalence, an explicit transformation was constructed that maps each Carter diagram with long cycles into some Carter diagram containing only 4-cycles.

(v) The correspondence between root subsets and Carter diagrams is not one-to-one. For example, the set of four linear independent roots $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ connected as shown in Fig. 5, can be associated with exactly two conjugacy classes having the following representatives: $w_o = s_{\alpha_1} s_{\beta_1} s_{\alpha_2} s_{\beta_2}$ (associated with D_4) and $w_1 = s_{\alpha_1} s_{\alpha_2} s_{\beta_1} s_{\beta_2}$ (associated with $D_4(a_1)$), [26, §§ 1.2.3–1.2.4]. The element w_o corresponds to the Coxeter transformation class, and w_1 corresponds to the non-Coxeter class. As Vavilov and Migrin noted in [28, § 1.2], this single phenomenon explains all non-Coxeter classes for all Carter diagrams with cycles.

1.5.   Homogeneous Carter diagrams

The Dynkin diagrams A_l, where $l \geq 1$ (resp. D_l, $l \geq 4$; resp. E_l, $l = 6, 7, 8$) are said to be the *Dynkin diagrams of A-type* (resp. *D-type*, resp. *E-type*). The Carter diagrams D_l, $D_l(a_k)$, $l \geq 4$, $1 \leq k \leq \left[\frac{l-2}{2}\right]$ (resp. E_l, $E_l(a_k)$, $l = 6, 7, 8$, k are given in (1.7)) are said to be the *Carter diagrams of D-type*, resp. *E-type*).

The Carter diagrams of the same type and the same index constitute a *homogeneous class* of Carter diagrams. Denote by $C(\Gamma)$ the homogeneous class containing the Carter diagram Γ, see (1.7) and Fig. 1.
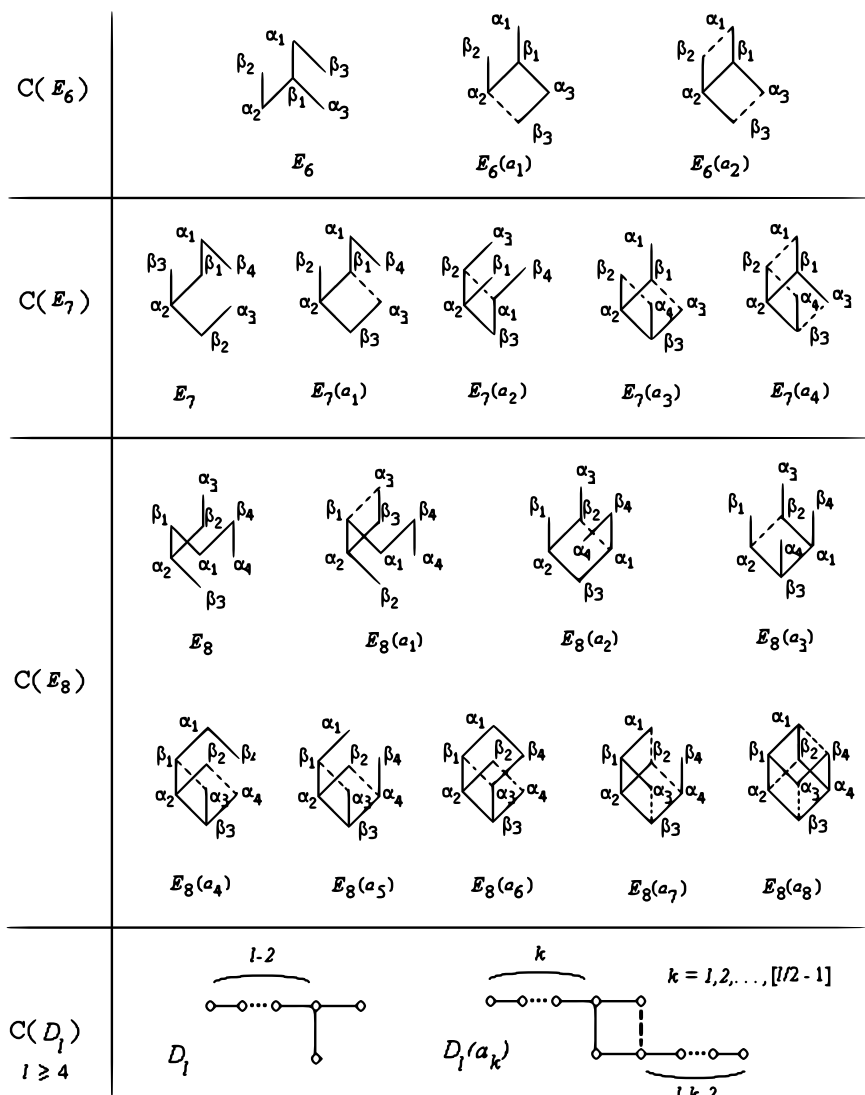
Figure 1: Homogeneous classes of Carter diagrams $C(D_l)$ and $C(E_l)$.

$$
\begin{aligned}
&C(E_6) = \{E_6, E_6(a_k), k = 1, 2\},\\
&C(E_7) = \{E_7, E_7(a_k), 1 \le k \le 4\},\\
&C(E_8) = \{E_8, E_8(a_k), 1 \le k \le 8\},\\
&C(D_l) = \{D_l, D_l(a_k), 1 \le k \le \left[\tfrac{l-2}{2}\right]\}, \text{ where } l \ge 4.
\end{aligned}
\tag{1.7}
$$

Let $\widetilde{S}$ (resp. S) be a $\widetilde{\Gamma}$-set (resp. Γ-set). In [27, Tables 2–5], the *transition matrices* $M : \widetilde{S} \longmapsto S$ are constructed for the following homogeneous pairs $\{\widetilde{\Gamma}, \Gamma\}$:

$$
\begin{aligned}
&(1)\ \{D_4(a_1), D_4\}; &\quad &(9)\ \{E_8(a_1), E_8\}; \\
&(2)\ \{D_l(a_k), D_l\}; &\quad &(10)\ \{E_8(a_2), E_8\}; \\
&(3)\ \{E_6(a_1), E_6\}; &\quad &(11)\ \{E_8(a_3), E_8(a_2)\}; \\
&(4)\ \{E_6(a_2), E_6(a_1)\}; &\quad &(12)\ \{E_8(a_4), E_8(a_1)\}; \\
&(5)\ \{E_7(a_1), E_7\}; &\quad &(13)\ \{E_8(a_5), E_8(a_4)\}; \\
&(6)\ \{E_7(a_2), E_7\}; &\quad &(14)\ \{E_8(a_6), E_8(a_4)\}; \\
&(7)\ \{E_7(a_3), E_7(a_1)\}; &\quad &(15)\ \{E_8(a_7), E_8(a_5)\}; \\
&(8)\ \{E_7(a_4), E_7(a_3)\}; &\quad &(16)\ \{E_8(a_8), E_8(a_7)\}.
\end{aligned}
\tag{1.8}
$$

To our considerations, let us add *diagram similarity mapping* L_{τ_i} reflecting the root $\tau_i \in \widetilde{S}$ and fixing the remaining roots in $\widetilde{\Gamma}$-set:

$$
L_{\tau_i} : \tau_i \longmapsto -\tau_i. \tag{1.9}
$$

Equation (1.9) means that any solid (resp. dotted) edge with vertex τ_i is mapped to the dotted (resp. solid) edge. Two Carter diagrams obtained from each other by a sequence of reflections (1.9), are called *similar Carter diagrams*. The diagrams obtained as images of the transition matrices M in (1.6) are considered up to similarity of Carter diagrams, see [26, § 1.3]. The list (1.8) is called the *adjacency list*.

1.6. Transitions

The transition matrices proposed in [27] transform root subsets associated with the corresponding Carter diagrams. Given a Carter diagram Γ, each transition changes only one root in the subset associated with Γ.

Theorem 1.1 (Transition theorem [27, Theorem 4.1]). *For each pair of diagrams $\{\widetilde{\Gamma}, \Gamma\}$ out of the list (1.8), there exists the <u>transition matrix</u> M mapping each $\widetilde{\Gamma}$-set $\widetilde{S}$ to some Γ-set S. The matrix M acts only on one element $\widetilde{\alpha} \in \widetilde{S}$ and does not change remaining elements in $\widetilde{S}$:*

$$
\begin{cases}
M\tau_i = \tau_i \text{ for all } \tau_i \in \widetilde{S}, \tau_i \neq \widetilde{\alpha}, \\
M\widetilde{\alpha} = \alpha = -\widetilde{\alpha} + \sum t_i \tau_i, \text{ the sum is taken over all } \tau_i \in \widetilde{S} \\
\qquad \text{except } \widetilde{\alpha};\ t_i \in \mathbb{Z}.
\end{cases}
\tag{1.10}
$$

The image $S = M\widetilde{S}$ the Γ-set. The transition matrix $M : \widetilde{S} \longmapsto S$ is an <u>*involution*</u>*:*

$$M\widetilde{\alpha} = \alpha \text{ and } M\alpha = \widetilde{\alpha}.$$

The transition matrix change the underlying diagram for given subset of roots. For more details about the transitions matrices M corresponding to the adjacency list (1.8), see Tables 2–5 in [27, § 4.6]. In § 3 we use transitions M to relate partial Cartan matrices associated with homogeneous Carter diagrams, as well as to relate linkage systems associated with them, see § 1.8.

1.6.1.   Transition mappings in other contexts

I would like to highlight two areas that use transformations close to the transition matrices from Theorem 1.1. In 1973, A. Gabrielov introduced transformations for the systematic study of quadratic forms associated with singularities, [10], see Gabrielov's examples in § A.1. In 1978, S. Ovsienko introduced so-called inflation technique for study the weakly positive unit quadratic forms, [22]. For the Ovsienko theorem, see § A.2.

1.7.   Admissible diagrams and Carter diagrams

Each element $w \in W$ can be expressed in the form

$$w = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_k}, \text{ where } \alpha_i \in \Phi \text{ for all } i. \tag{1.11}$$

Carter proved that k in the decomposition (1.11) is the smallest if and only if the subset of roots $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is linearly independent; such a decomposition is said to be *reduced*. The admissible diagram corresponding to the given element w is not unique, since the reduced decomposition of the element w is not unique.

Denote by $l_C(w)$ the smallest value k corresponding to any reduced decomposition (1.11). The corresponding set of roots $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ consists of linearly independent and not necessarily simple roots, see Lemma 1.2. If $l(w)$ is the smallest value k in (1.11) such that all roots α_i are *simple*, then $l_C(w) \leq l(w)$.

Lemma 1.2 ([4]). *Let $\alpha_1, \alpha_2, \ldots, \alpha_k \in \Phi$. Then, $s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_k}$ is reduced if and only if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are linearly independent.*

The admissible diagram may contain cycles, since the roots of S are non necessarily simple, [26]. Let us fix some basis of roots corresponding to the given admissible diagram Γ:

$$S = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_h\}. \tag{1.12}$$

According to $(1.4(a))$, there exists a certain set (1.12) of linearly independent roots. Thanks to $(1.4(b))$, there exists a partition $S = S_\alpha \coprod S_\beta$ which is said to be the *bicolored partition*. The admissible diagram is bicolored, i.e., the set of nodes can be partitioned into two disjoint subsets $S_\alpha = \{\alpha_1, \ldots, \alpha_k\}$ and $S_\beta = \{\beta_1, \ldots, \beta_h\}$, where roots of S_α (resp. S_β) are mutually orthogonal.

Let $w = w_1 w_2$ be the decomposition of w into the product of two involutions. By [4, Lemma 5] each of w_1 and w_2 can be expressed as a product of reflections as follows:

$$w = w_1 w_2, \ \text{ where } \ w_1 = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_k}, \quad w_2 = s_{\beta_1} s_{\beta_2} \ldots s_{\beta_h}, \tag{1.13}$$

subset $S_\alpha = \{\alpha_1, \ldots, \alpha_k\}$ (resp. $S_\beta = \{\beta_1, \ldots, \beta_h\}$) consists of mutually orthogonal roots. The decomposition (1.13) is said to be a *bicolored decomposition*. The subset of roots corresponding to w_1 (resp. w_2) is said to be α-*set* (resp. β-*set*):

$$\alpha\text{-set} = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}, \quad \beta\text{-set} = \{\beta_1, \beta_2, \ldots, \beta_h\}. \tag{1.14}$$

The element w in (1.13) is said to be *semi-Coxeter element*; it represents the conjugacy class associated with the admissible diagram Γ and root subset S.

Any admissible diagram Γ is said to be a *Carter diagram* if any edge connecting a pair of roots $\{\alpha, \beta\}$ with inner product $(\alpha, \beta) > 0$ (resp. $(\alpha, \beta) < 0$) is drawn as dotted (resp. solid) edge. There is no edge for the inner product $(\alpha, \beta) = 0$.

Up to solid/dotted edges, the classification of Carter diagrams coincides with the classification of admissible diagrams.

1.8. Linkage root and linkage diagram

We consider the *extension* of the root subset S from the bicolored partition (1.12) by means of the root $\gamma \in \Phi$, so that the set of roots

$$S' = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_h, \gamma\}$$

is linearly independent. The new diagram Γ' is obtained by addition new edges, these edges are *solid* (resp. *dotted*) for $(\gamma, \tau) = -1$ (resp. $(\gamma, \tau) = 1$), where $\tau \in S$.

The diagram Γ' is said to be a *linkage diagram* and the root γ is said to be a *linkage root*. The roots τ corresponding to the new edges $((\gamma, \tau) \neq 0)$ are said to be *endpoints* of the linkage diagram. Consider vector γ^∇ defined by (1.15).

$$\gamma^\nabla := \begin{pmatrix} (\gamma, \alpha_1) \\ \dots, \\ (\gamma, \alpha_k) \\ (\gamma, \beta_1) \\ \dots, \\ (\gamma, \beta_h) \end{pmatrix}. \tag{1.15}$$

This vector is said to be the *linkage label vector*. There is a one-to-one correspondence between linkage label vectors γ^∇ (with labels $\gamma_i^\nabla \in \{-1, 0, 1\}$) and linkage diagrams such that $(\gamma, \tau) \in \{-1, 0, 1\}$.

1.9. Solid and dotted edges of linkage diagrams

Lemma 1.3 ([26, Lemma A.1]). *Every cycle in the Carter diagram contains at least one solid edge and at least one dotted edge.*
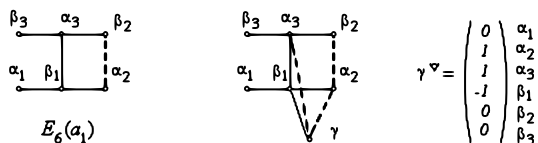

Figure 2: Example of linkage diagram for $E_6(a_1)$.

This lemma is also valid for cycles of any linkage diagram. In Fig. 2 if $\{\beta_1, \gamma\}$ is solid then edge $\{\alpha_3, \gamma\}$ is necessarily dotted, see triangle $\{\alpha_3, \gamma, \beta_1\}$. Similarly, from triangle $\{\alpha_2, \gamma, \beta_1\}$ the edge $\{\alpha_2, \gamma\}$ is necessarily dotted.

1.10. Getting rid of triangles

By adding the linkage root to a Carter diagram, we get a new diagram which is not necessarily a Carter diagram. After this, we act by the Weyl

group of the inverse quadratic form and obtain the orbit of the linkage diagrams. In the most of cases, this orbit contains a Carter diagram, see Figs. 5, 9, 10. Sometimes the orbit does not contain a Carter diagram because every linkage diagram on the orbit contains a triangle or other odd cycle. By changing the basis we can eliminate a triangle or other odd cycle. The procedure for eliminating the triangle is shown in Fig. 3.
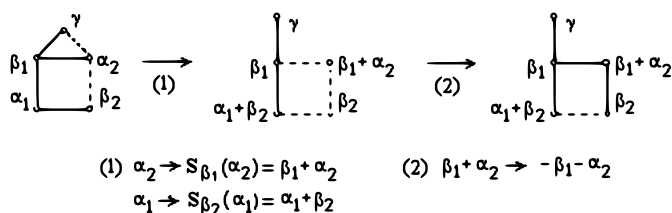

(1) $\alpha_2 \to S_{\beta_1}(\alpha_2) = \beta_1 + \alpha_2$ (2) $\beta_1 + \alpha_2 \to -\beta_1 - \alpha_2$

$\alpha_1 \to S_{\beta_2}(\alpha_1) = \alpha_1 + \beta_2$

Figure 3: Eliminating the triangle

1.11. Nilpotent orbits in the simple Lie algebras

There is a connection (which was pointed out to me by A. Elashivili) between the representatives of nilpotent orbits in simple Lie algebras and root subsets of linearly independent roots represented by Carter diagrams. By theorem of Jacobson-Morozov, this connection is also a connection between simple three-dimensional S-subalgebras (or $sl2$-triples) listed by Dynkin, and root subsets of linearly independent roots, see [6, 8, 12, 13, 17, 21].

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2. The main results

2.1. When a vector is a linkage root?

According with § 1.8, the linkage diagram is obtained from a Carter diagram Γ by adding some linkage root γ, with its bonds, so that the roots corresponding to vertices of Γ together with γ form a linearly independent root subset. We would like to describe such roots in the general case,

namely:

> *Let Γ be a Carter diagram, S be some Γ-set. What root γ can be added to S so that $S \cup \gamma$ is a set of linearly independent roots in some finite root system?*   (2.1)

With every linkage diagram we associate the linkage label vector. There is one-to-one correspondence between linkage label vectors and linkage diagrams, see § 1.8. The linkage diagram and their linkage label vector γ^∇ for the Carter diagram $E_6(a_1)$ is depicted in Fig. 2. For more examples of linkage diagrams and linkage label vectors see Fig. 5 (case $D_4(a_1)$), Figs. 10 (case $D_k(a_l)$).

It turns out that the answer to question (2.1) is very simple in terms of the quadratic form associated with the partial Cartan matrix. Let γ be a linkage root for Γ, γ^∇ be the linkage label vector, see (1.15). We denote by L the space spanned by roots of S, and by $L(\gamma)$ the space spanned by L and γ. Let us write

$$L = [\tau_1, \ldots, \tau_l], \qquad L(\gamma) = [\tau_1, \ldots, \tau_l, \gamma].   \tag{2.2}$$

Note that $L(\gamma)$ corresponds to a certain linkage diagram. Let $\mathscr{B}_\Gamma$ be the quadratic form associated with the partial Cartan matrix B_Γ. Then,

$$\mathscr{B}_\Gamma(\gamma) = \langle B_\Gamma \gamma, \gamma \rangle = \langle \gamma^\nabla, B_\Gamma^{-1} \gamma^\nabla \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual dot product. Since B_Γ is positive definite, the eigenvalues of B_Γ are positive. Therefore, the matrix B_Γ^{-1} is also positive definite. We call the quadratic form $\mathscr{B}_\Gamma^\vee$ corresponding to the matrix B_Γ^{-1} the *inverse quadratic form*. The form $\mathscr{B}_\Gamma^\vee$ is positive definite.

2.2. Criterion that a vector is a linkage root

Theorem (see Theorem 3.3, Linkage root criterion)**.** *(i) Let θ^∇ be the linkage label vector corresponding to a certain root $\theta \in L(\gamma)$. The root θ is a linkage root if and only if*

$$\mathscr{B}_\Gamma^\vee(\theta^\nabla) < 2.   \tag{2.3}$$

(ii) Let $\theta \in L(\gamma)$ be a root with only one endpoint $\tau_i \in S$. Then θ is a linkage root if and only if

$$b_{i,i}^\vee < 2,$$

where $b_{i,i}^{\vee}$ is the ith diagonal element of B_{Γ}^{-1}.

Note that criterion (2.3) does not depend on the choice of the Γ-set S, but only on the quadratic form $\mathscr{B}$ or, what the same, on the diagram Γ.

Let $\{\Gamma', \Gamma\}$ be a pair of Carter diagrams from the list (1.8), M be the transition matrix transforming Γ'-set S' to Γ-set S, see § 1.6, and let $B_{\Gamma'}$ and B_{Γ} be the partial Cartan matrices for Γ' and Γ. Then, M relates the partial Cartan matrices $B_{\Gamma'}$ and B_{Γ} as follows:

$$^{t}M \cdot B_{\Gamma'} \cdot M = B_{\Gamma}.$$

It is shown in **Proposition 3.4** that vector γ^{∇} is the linkage label vector for Γ if and only if $\,^{t}M\gamma^{\nabla}$ is the linkage label vector for Γ'.

2.3. Full linkage systems $\mathscr{L}(D_l)$ and $\mathscr{L}(D_l(a_k))$

Denote by $\mathscr{L}_{\widetilde{\Gamma}}(\Gamma', S')$ the set of linkage diagrams

$$\mathscr{L}_{\widetilde{\Gamma}}(\Gamma', S') = \{\gamma^{\nabla} \mid \gamma \in \Phi(\widetilde{\Gamma}), \gamma \notin L(S')\}, \qquad (2.4)$$

where $\Phi(\widetilde{\Gamma})$ is the root system associated with $\widetilde{\Gamma}$, S' some Γ'-set and $L(S')$ is the linear span $\text{Span}(S')$. It was shown in **Proposition 4.2** that $\mathscr{L}_{\widetilde{\Gamma}}(\Gamma', S')$ does not depend on choosing Γ'-set S'. Then, the set (2.4) can be denoted by $\mathscr{L}_{\widetilde{\Gamma}}(\Gamma')$. The set $\mathscr{L}_{\widetilde{\Gamma}}(\Gamma')$ is said to be *partial linkage system* $\widetilde{\Gamma}$ *over* Γ'. The pair $\{\Gamma', \widetilde{\Gamma}\}$, where $\Gamma' \in C(\Gamma)$, is said to be the *vertex extension* and is denoted by $\Gamma' \prec \widetilde{\Gamma}$. The union of all partial systems by all possible vertex extensions is said to be the *linkage system* or *full linkage system* of Γ:

$$\mathscr{L}(\Gamma') = \bigcup_{\Gamma' \prec \widetilde{\Gamma}} \mathscr{L}_{\widetilde{\Gamma}}(\Gamma'). \qquad (2.5)$$

For instance, for $\Gamma = A_7$, we have:

$$\mathscr{L}(A_7) = \mathscr{L}_{A_8}(A_7) \cup \mathscr{L}_{D_8}(A_7) \cup \mathscr{L}_{E_8}(A_7).$$

If the diagram $\widetilde{\Gamma}$ in (2.4) is the A-type (resp. D-type, resp. E-type) Dynkin diagram, the component $\mathscr{L}_{\widetilde{\Gamma}}(\Gamma')$ is called the *A-component* (resp. *D-component*, resp. *E-component*) in the full linkage system (2.5). In **Theorem 5.3**, the D- and E-components, the size of each component for the full linkage systems $\mathscr{L}(D_l)$ and $\mathscr{L}(D_l(a_k))$ are found. The results are shown in Table 2.1.

Γ	Number of components	Number of linkage diagrams, and $p = \mathscr{B}_\Gamma^\vee(\gamma^\nabla)$		In all
		D-components $p = 1$	E-components $p = \frac{l}{4}$	
$D_4, \ D_4(a_1)$	3	$3 \times 8 = 24$	-	24
$D_5, \ D_5(a_1)$	3	10	$2 \times 16 = 32$	42
$D_6, D_6(a_1), D_6(a_2)$	3	12	$2 \times 32 = 64$	76
$D_7, D_7(a_1), D_7(a_2)$	3	14	$2 \times 64 = 128$	142
$D_l, \ D_l(a_k), l > 7$	1	$2l$	-	$2l$

Table 2.1: Number of linkage diagrams in extensions for diagrams D_l and $D_l(a_k)$

For any pair of homogeneous Carter diagrams Γ and Γ' the sizes of linkage systems $\mathscr{L}(\Gamma)$ and $\mathscr{L}(\Gamma')$ are the same (**Proposition 4.3**).

The linkage systems $\mathscr{L}(D_4)$ (resp. $\mathscr{L}(D_4(a_1))$, resp. $\mathscr{L}(D_l)$ for $l > 4$, resp. $\mathscr{L}(D_l(a_k))$ for $l > 4$) are depicted in Fig. 4 (resp. Fig. 5, resp. Fig. 9, resp. Fig. 10).

One of two orbits of the E-component of the linkage system $\mathscr{L}(D_5)$ are depicted in Fig. 11. Two E-components of the linkage system $\mathscr{L}(D_6)$ are depicted in Fig. 12.

2.4. The Weyl group of quadratic form $\mathscr{B}_\Gamma^\vee$

The Weyl group of $\mathscr{B}_\Gamma^\vee$ is generated by dual reflections $\{s_{\tau_i}^* \mid \tau_i \in S\}$, where

$$s_{\tau_i}^* \gamma^\nabla := \gamma^\nabla - \langle \gamma^\nabla, \tau_i \rangle \tau_i^\nabla.$$

If γ^∇ is a linkage label vector then values $(s_{\tau_i}^* \gamma^\nabla)_{\tau_k}$ in (3.18) belong to $\{-1, 0, 1\}$, i.e., we get new linkage label vector $s_{\tau_i}^* \gamma^\nabla$ (**Proposition 3.6**). The action of reflections s_{τ_i} and dual reflections $s_{\tau_i}^*$ related as follows:

$$(s_{\tau_i} \gamma)^\nabla = (s_{\tau_i}^* \gamma^\nabla).$$

Reflections $s_{\tau_i}^*$ preserve the quadratic form $\mathscr{B}_\Gamma^\vee$ (**Proposition 3.8**):

$$\mathscr{B}_\Gamma^\vee(s_{\tau_i}^* \gamma^\nabla) = \mathscr{B}_\Gamma^\vee(\gamma^\nabla) \text{ for } \gamma^\nabla \in L^\nabla.$$

The rational number $p = \mathscr{B}_\Gamma^\vee(\gamma^\nabla)$ is the invariant characterizing the given orbit, since all elements of this orbit have the same value $\mathscr{B}_\Gamma^\vee(\gamma^\nabla)$, see Table 2.1.

2.5. Loctets

A spindle-like subset consisting of 8 linkage label vectors is called a *loctet* (linkage octet).

The D-component of $\mathscr{L}(D_4)$ consists of three orbits, each orbit consists of one loctet, Fig. 4.

The D-component of $\mathscr{L}(D_4(a_1))$ consists of three orbits, two of which consist of one loctet, Fig. 5.

The E-component of $\mathscr{L}(D_5)$ consists of two orbits, each orbit consists of two loctets, Fig. 11.

The E-component of $\mathscr{L}(D_6)$ consists of two orbits, each orbit consists of four loctets, Fig. 12.

The E-component of $\mathscr{L}(D_7)$ consists of two orbits, each orbit consists of eight loctets, see [25, Figs. C.63, C.64].

3.   Quadratic forms associated with Carter diagrams

Let L be the subspace spanned by the vectors $S = \{\tau_1, \ldots, \tau_n\}$. We write this fact as follows:
$$L = [\tau_1, \ldots, \tau_n].$$

The subspace L is said to be the *S-associated subspace*. Let B be the Cartan matrix corresponding to the root system Φ.

Proposition 3.1. *(i) The restriction of the bilinear form associated with the Cartan matrix B on the subspace L coincides with the bilinear form associated with the partial Cartan matrix B_Γ, i.e., for any pair of vectors $v, u \in L$, we have*

$$(v, u)_\Gamma = (v, u), \ \ and \ \mathscr{B}_\Gamma(v) = \mathscr{B}(v). \tag{3.1}$$

(ii) For every Carter diagram, the matrix B_Γ is positive definite.

Proof. (i) From (1.5) we deduce:

$$(v, u)_\Gamma = (\sum_i t_i \tau_i, \sum_j q_j \tau_j)_\Gamma = \sum_{i,j} t_i q_j (\tau_i, \tau_j)_\Gamma = \sum_{i,j} t_i q_j (\tau_i, \tau_j) = (v, u).$$
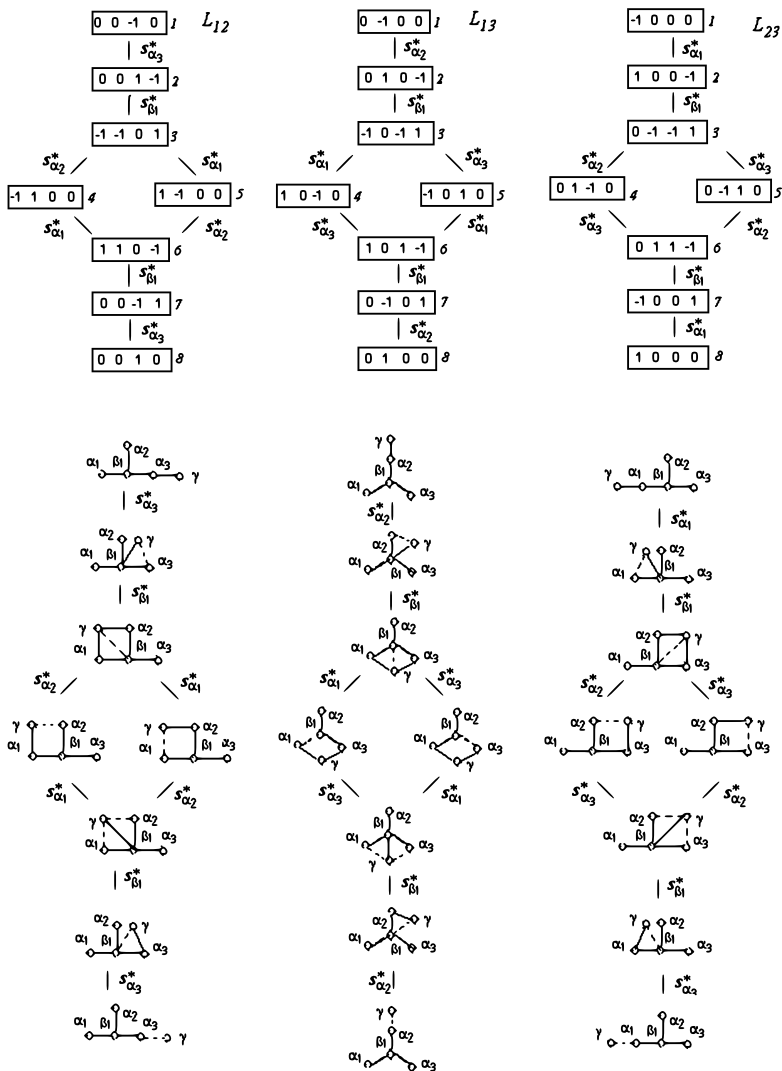
(ii) This follows from (i).

Figure 4: Three components of the linkage system $\mathscr{L}(D_4)$ contain 24 elements. The linkage label vectors are shown above, the corresponding linkage diagrams are below.

If Γ is a Dynkin diagram, the partial Cartan matrix B_Γ is the Cartan matrix associated with Γ. By (3.1) the matrix B_Γ is positive definite. The symmetric bilinear form associated with B_Γ is denoted by $(\cdot, \cdot)_\Gamma$ and the corresponding quadratic form is denoted by $\mathscr{B}_\Gamma$.
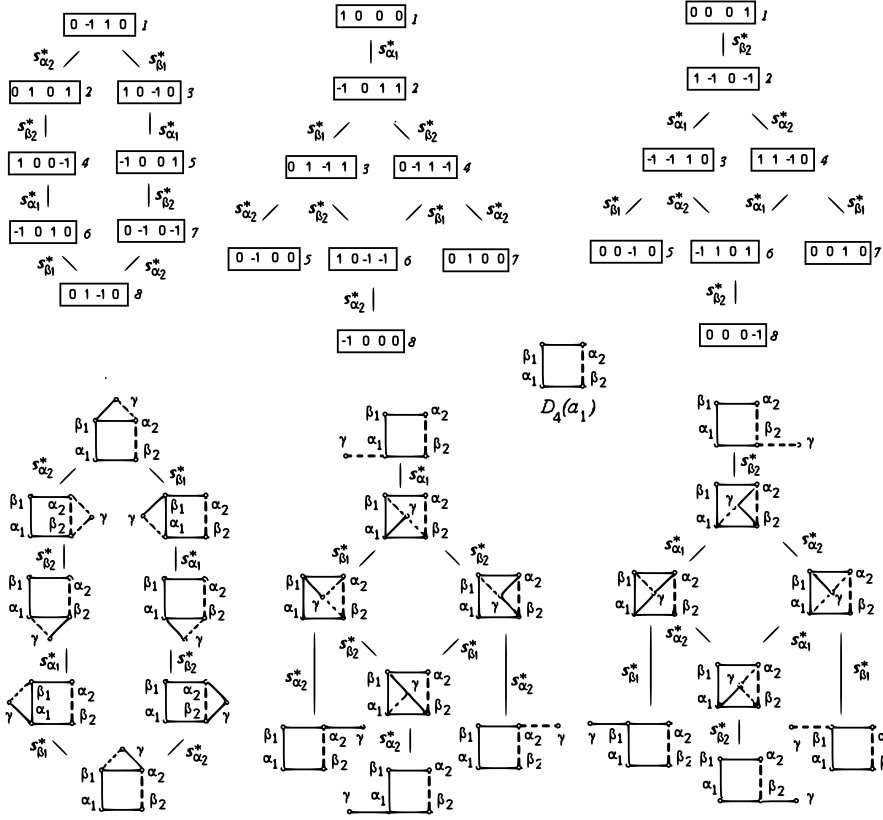
Figure 5: Three components of the linkage system $\mathscr{L}(D_4(a_1))$ contain 24 elements. The linkage label vectors are shown above, the corresponding linkage diagrams are below.

3.1. The inverse quadratic form $\mathscr{B}_\Gamma^\vee$

3.1.1. Action of B_Γ^{-1} on the space L

Let $S = \{\tau_1, \ldots, \tau_l\}$ be a Γ-set, L – the space spanned by roots of S, γ be a root in L:

$$\gamma = t_1\tau_1 + \cdots + t_l\tau_l,$$

where t_i are some rational numbers. Then, for $\gamma \in L$, we have

$$\gamma^\nabla = \begin{pmatrix} (\gamma, \tau_1) \\ \ldots \\ (\gamma, \tau_l) \end{pmatrix} = B_\Gamma \begin{pmatrix} t_1 \\ \ldots \\ t_l \end{pmatrix} = B_\Gamma\gamma, \text{ and } \gamma = B_\Gamma^{-1}\gamma^\nabla. \qquad (3.2)$$

3.1.2. The projection of the linkage root

Let $L^\perp$ be the orthogonal complement of L to $L(\gamma)$ in the sense of the symmetric bilinear form $(\cdot, \cdot)$ associated with the root system Φ:

$$L(\gamma) = L \oplus L^\perp.$$

Let γ_L be the projection of the linkage root γ on L. For any root $\theta \in L(\gamma)$ such that $\theta \notin L$, we have $L(\theta) = L(\gamma)$, and θ is uniquely decomposed into the following sum:

$$\theta = \theta_L + \mu, \quad \text{where} \quad \theta_L \in L, \quad \mu \in L^\perp. \tag{3.3}$$

Given any vector θ by decomposition (3.3), we introduce also the *conjugate vector* $\overline{\theta}$ as follows:

$$\theta = \theta_L + \mu, \quad \overline{\theta} = \theta_L - \mu, \quad \text{where} \quad \theta_L \in L, \quad \mu \in L^\perp.$$
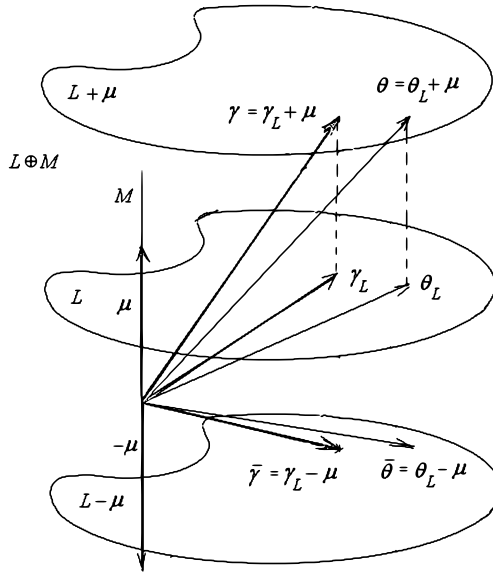

Figure 6:   The roots $\gamma = \gamma_L + \mu$ and $\overline{\gamma} = \gamma_L - \mu$.

The vector μ is said to be the *normal extending vector*, see Fig. 6. The following proposition collects a number of properties of the projection γ_L of the linkage root γ, the linkage label vector γ^∇, and the normal extending vector μ:

Proposition 3.2. *(i) The linkage label vector γ^∇ and the projection γ_L are related as follows:*

$$\gamma^\nabla = \overline{\gamma}^\nabla = \gamma_L^\nabla = B_\Gamma \gamma_L. \tag{3.4}$$

(ii) The vector μ is, up to sign, a fixed vector for any root $\theta \in L(\gamma)\backslash L$.

(iii) The value $\mathscr{B}_\Gamma(\theta_L)$ is constant for any root $\theta \in L(\gamma)\backslash L$.

(iv) The vector $\theta = \theta_L + \mu$ is a root if and only if the conjugate vector $\overline{\theta} = \theta_L - \mu$ is a root. In addition, $\theta \in L(\gamma)\backslash L$ if and only if $\overline{\theta} \in L(\overline{\gamma})\backslash L$.

(v) Let θ is a root, $\theta \notin L$. Then, vector $\theta_L + t\mu$ is a root if and only if $t = \pm 1$.

(vi) If δ is a root in $L(\gamma)\backslash L$ such that $\delta^\nabla = \gamma^\nabla$, then $\delta = \gamma$ or $\delta = \overline{\gamma}$.

(vii) For any root $\theta \in L(\gamma)\backslash L$, we have

$$\mathscr{B}_\Gamma^\vee(\theta^\nabla) = \mathscr{B}_\Gamma(\theta_L),$$

and $\mathscr{B}_\Gamma^\vee(\theta^\nabla)$ is a constant for all roots $\theta \in L(\gamma)\backslash L$.

Proof. (i) Let $\gamma = \gamma_L + \mu$. Since $(\mu, \tau_i) = 0$ for any i, we have

$$(\gamma, \tau_i) = (\gamma_L, \tau_i) \text{ for any } i, \text{ i.e., } \gamma^\nabla = \gamma_L^\nabla.$$

Then, by (3.2) we have

$$\gamma^\nabla = \overline{\gamma}^\nabla = \gamma_L^\nabla = B_\Gamma \gamma_L.$$

(ii) Any root $\theta \in L(\gamma)$ such that $\theta \notin L$ decomposes as $\tau + \gamma$ for some $\tau \in L$. Since $\gamma = \gamma_L + \mu$, where $\gamma_L \in L$, we have $\theta = \tau + \gamma_L + \mu$. Here, $\tau + \gamma_L \in L$. Let us put $\theta_L = \tau + \gamma_L$. Then $\theta = \theta_L + \mu$.

(iii) Let $\mathscr{B}$ be the quadratic form associated with the root system Φ. By (3.3) we have $\theta_L \perp \mu$, and $\mathscr{B}(\theta) = \mathscr{B}(\theta_L) + \mathscr{B}(\mu)$. Here, $\mathscr{B}(\theta) = 2$ since θ is the root, and by (ii), we have $\mathscr{B}(\theta_L) = 2 - \mathscr{B}(\mu)$, i.e, $\mathscr{B}(\theta_L)$ is constant. By (3.1), we have $\mathscr{B}_\Gamma(\theta_L) = \mathscr{B}(\theta_L)$, i.e., $\mathscr{B}_\Gamma(\theta_L)$ is also constant for all $\theta \in L(\gamma)$.

(iv) Let θ be a root, i.e., $\mathscr{B}(\theta) = \mathscr{B}(\theta_L) + \mathscr{B}(\mu) = 2$. Then for $\overline{\theta}$, we have $\mathscr{B}(\overline{\theta}) = \mathscr{B}(\theta_L) + \mathscr{B}(-\mu) = 2$ as well. By (1.3) $\overline{\theta}$ is a root as well. Further, $\theta \in L(\gamma)$ means that $\theta = \gamma + \tau$, where $\tau \in L$. Thus, $\theta = \gamma_L + \mu + \tau$. Put $\theta_L = \gamma_L + \tau$. Then $\theta_L \in L$, and

$$\theta = \theta_L + \mu, \ \ \overline{\theta} = \theta_L - \mu = \gamma_L + \tau - \mu = \overline{\gamma} + \tau \in L(\overline{\gamma}).$$

(v) Consider root $\theta = \theta_L \pm \mu$. Since $\theta_L \perp \mu$, the following relations hold:

$$\mathscr{B}(\theta_L + t\mu) = \mathscr{B}(\theta \pm \mu + t\mu) = \mathscr{B}(\theta + (t \pm 1)\mu) = \mathscr{B}(\theta) + (t \pm 1)^2 \mathscr{B}(\mu).$$

Because θ is a root, then $\mathscr{B}(\theta_L + t\mu) = \mathscr{B}(\theta) = 2$ and $t = \pm 1$.

(vi) By $\delta^\nabla = \gamma^\nabla$ and (3.4), the roots δ and γ have the same projections on L: $\delta_L = \gamma_L$. If $\delta = \delta_L + t\mu = \gamma_L + t\mu$ is a root, then by (v) we have $t = \pm 1$. Thus, $\delta = \gamma$ or $\overline{\gamma}$.

(vii) By heading (i), since $\theta_L \in L$, we have $\theta_L^\nabla = B_\Gamma \theta_L$. Thus,

$$\mathscr{B}_\Gamma^\vee(\theta^\nabla) = \langle B_\Gamma^{-1}\theta^\nabla, \theta^\nabla \rangle = \langle \theta_L, B_\Gamma \theta_L \rangle = \mathscr{B}_\Gamma(\theta_L).$$

3.2. The linkage root theorem

In this section we give a criterion for a given vector to be a linkage root.

Theorem 3.3 (Linkage root criterion)**.** *(i) Let θ^∇ be the linkage label vector corresponding to a certain root $\theta \in L(\gamma)$, i.e., $\theta^\nabla = B_\Gamma \theta_L$. The root θ is a linkage root, (i.e., θ is linearly independent of roots of L) if and only if*

$$\mathscr{B}_\Gamma^\vee(\theta^\nabla) < 2. \tag{3.5}$$

(ii) Let $\theta \in L(\gamma)$ be a root connected only with one $\tau_i \in S$. The root θ is a linkage root if and only if

$$b_{ii}^\vee < 2, \tag{3.6}$$

where $b_{ii}^\vee$ is the ith diagonal element of B_Γ^{-1}.

Proof. (i) By Proposition 3.2(vi) and Proposition 3.1 we have

$$\mathscr{B}_\Gamma^\vee(\theta^\nabla) = \mathscr{B}_\Gamma(\theta_L) = \mathscr{B}(\theta_L) = \mathscr{B}(\theta) - \mathscr{B}(\mu) \leq 2. \tag{3.7}$$

Then, $\mathscr{B}_\Gamma^\vee(\theta^\nabla) = 2$ if and only if $\mathscr{B}(\mu) = 0$. If $\mathscr{B}_\Gamma^\vee(\theta^\nabla) = 2$ then $\mathscr{B}(\mu) = 0$, $\mu = 0$ and $\theta = \theta_L$. Therefore, θ is linearly depends on vectors of L. If $\mathscr{B}_\Gamma^\vee(\theta^\nabla) < 2$ then $\mathscr{B}(\mu) > 0$, i.e., $\mu \neq 0$ and θ is linearly independent of roots of L.

(ii) We have

$$\theta^\nabla = \begin{pmatrix} (\theta, \tau_1) \\ \dots \\ (\theta, \tau_i) \\ \dots \\ (\theta, \tau_l) \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ \pm 1 \\ \dots \\ 0 \end{pmatrix},$$

and $\mathscr{B}_\Gamma^\vee(\theta^\nabla) = b_{ii}^\vee$. Thus, statement (ii) follows from (i).

3.3.   Quadratic forms for homogeneous Carter diagrams

Proposition 3.4. *Let $\{\Gamma', \Gamma\}$ be a pair of Carter diagrams from the adjacency list* (1.8), *M be the transition matrix transforming Γ'-set S' to Γ-set S as in § 1.6. The vector γ^{∇} is the linkage label vector for Γ if and only if $\,{}^{t}M\gamma^{\nabla}$ is the linkage label vector for Γ'.*

Proof. By Theorem 1.1, the transition matrix M maps the Γ'-set S' to the Γ-set S:

$$M\tau_i' = \tau_i, \ \ \text{where} \ \ \tau_i' \in S', \tau_i \in S.$$

Let $B_{\Gamma'}$ and B_{Γ} be the partial Cartan matrices for Γ' and Γ. The transition matrix M transforms S' to S and relates the partial Cartan matrices $B_{\Gamma'}$ and B_{Γ} as follows:

$$ {}^{t}M \cdot B_{\Gamma'} \cdot M = B_{\Gamma}, \tag{3.8}$$

see [27, § 4.7]. Since M is an involution, we get

$$M \cdot B_{\Gamma'}^{-1} \cdot {}^{t}M = B_{\Gamma}^{-1}.$$

Further,

$$\mathscr{B}_{\Gamma}^{\vee}(\gamma^{\nabla}) = \langle B_{\Gamma}^{-1}\gamma^{\nabla}, \gamma^{\nabla}\rangle = \langle B_{\Gamma'}^{-1} \cdot {}^{t}M\gamma^{\nabla}, {}^{t}M\gamma^{\nabla}\rangle = \mathscr{B}_{\Gamma'}^{\vee}({}^{t}M\gamma^{\nabla}).$$

Then, $\mathscr{B}_{\Gamma}^{\vee}(\gamma^{\nabla}) < 2$ if and only if $\mathscr{B}_{\Gamma'}^{\vee}({}^{t}M\gamma^{\nabla}) < 2$. By Theorem 3.3, we get what we need.

3.4.   The chain of homogeneous pairs

Consider any homogeneous pair of Carter diagrams $\{\Gamma', \Gamma\}$. There exists the chain of homogeneous pairs from the adjacency list (1.8) which connect Γ' and Γ as follows:

$$\{\Gamma_0, \Gamma_1\}, \{\Gamma_1, \Gamma_2\}, \ldots, \{\Gamma_{k-1}, \Gamma_k\}, \ \text{where} \ \Gamma_0 = \Gamma' \ \text{and} \ \Gamma_k = \Gamma. \tag{3.9}$$

Let $S_j = \{\tau_1^j, \ldots, \tau_n^j\}$ be the associated Γ_j-set, $M^{(j)}$ be the transition matrices connecting jth pair in (3.9), where $j = 0, \ldots, k-1$, see Theorem 1.1. Then,

$$\begin{cases} M^{(j)}(\tau_i^j) = \tau_i^{j+1} \text{ for some } i \in \{1, \ldots, n\}, \\ M^{(j)}(\tau_k^j) = \tau_k^j \ \ \text{for } k \neq i. \end{cases}$$

Two Carter diagrams obtained from each other by a sequence of reflections (1.9), are called *similar Carter diagrams*, see [27, Fig. 4]. In fact, mappings L_{τ_i} can be considered as a degenerated case of transition matrices M of Theorem 1.1. The Carter diagrams in the adjacency list (1.8) are given up to similarity, so using the transition matrices $M^{(j)}$ together with mappings L_{τ_i} we can construct the map F transforming the partial Cartan matrices $B_{\Gamma'}$ and B_Γ to each other, also the linkage roots for Γ' and Γ to each other, see [27, § 4.2.1]:

$$F\tau_i' = \tau_i, \quad {}^tF \cdot B_{\Gamma'} \cdot F = B_\Gamma, \quad \mathscr{B}_\Gamma^\vee(\gamma^\nabla) = \mathscr{B}_{\Gamma'}^\vee({}^tF\gamma^\nabla). \qquad (3.10)$$

3.5. The Weyl group of the quadratic form

3.5.1. Dual reflections

Let Γ be a Carter diagram, $S = \{\tau_1, \ldots, \tau_l\}$ be some Γ-set, and L be S-associated subspace:

$$L = [\tau_1, \ldots, \tau_l].$$

For any $\tau_i \in S$, we define vectors τ_i^∇ as follows:

$$\tau_i^\nabla := B_\Gamma \tau_i. \qquad (3.11)$$

Let L^∇ be the linear space spanned by the vectors τ_i^∇, where $\tau_i \in S$. The map $\tau_i \longrightarrow \tau_i^\nabla$ given by (3.11) is expanded to the linear mapping $L \longrightarrow L^\nabla$, and

$$u^\nabla = B_\Gamma u = \begin{pmatrix} (u, \tau_1) \\ \ldots \\ (u, \tau_l) \end{pmatrix} \text{ for any } u \in L, \qquad (3.12)$$

see (3.2). Consider the restriction of the reflection s_{τ_i} on the subspace L. For any $v \in L$, by Proposition 3.1 we have:

$$s_{\tau_i}v = v - 2\frac{(\tau_i, v)}{(\tau_i, \tau_i)}\tau_i = v - (\tau_i, v)_\Gamma\tau_i$$
$$= v - \langle B_\Gamma\tau_i, v\rangle\tau_i = v - \langle\tau_i^\nabla, v\rangle\tau_i. \qquad (3.13)$$

We define the *dual reflection* $s_{\tau_i}^*$ acting on a vector $u^\nabla \in L^\nabla$ as follows:

$$s_{\tau_i}^* u^\nabla := u^\nabla - \langle u^\nabla, \tau_i\rangle\tau_i^\nabla. \qquad (3.14)$$

Let W_S (resp. $W_S^\vee$) be the group generated by reflections $\{s_{\tau_i} \mid \tau_i \in S\}$ (resp. $\{s_{\tau_i}^* \mid \tau_i \in S\}$), where $S = \{\tau_1, \ldots, \tau_l\}$. The group W_S is uniquely determined by the quadratic form $\mathscr{B}_\Gamma$. In Proposition 3.8 it will be shown that W_S (resp. $W_S^\vee$) preserves $\mathscr{B}_\Gamma$ (resp. $\mathscr{B}_\Gamma^\vee$), so W_S (resp. $W_S^\vee$) is called the *Weyl group* of $\mathscr{B}_\Gamma$ (resp. $\mathscr{B}_\Gamma^\vee$)[1].

Proposition 3.5. *(i) For any $\tau_i \in S$, we have*

$$s_{\tau_i}^* = {}^t s_{\tau_i} = {}^t s_{\tau_i}^{-1}. \tag{3.15}$$

(ii) The mapping

$$\pi : w \to {}^t w^{-1}$$

determines an isomorphism between W_S and $W_S^\vee$.

Proof. (i) By (3.13) and (3.14), for any $v \in L, u^\nabla \in L^\nabla$, we have:

$$\langle s_{\tau_i}^* u^\nabla, v \rangle = \langle u^\nabla - \langle u^\nabla, \tau_i \rangle \tau_i^\nabla, v \rangle = \langle u^\nabla, v \rangle - \langle u^\nabla, \tau_i \rangle \langle v, \tau_i^\nabla \rangle,$$
$$\langle u^\nabla, s_{\tau_i} v \rangle = \langle u^\nabla, v - \langle \tau_i^\nabla, v \rangle \tau_i \rangle = \langle u^\nabla, v \rangle - \langle \tau_i^\nabla, v \rangle \langle u^\nabla, \tau_i \rangle.$$

Thus,

$$\langle s_{\tau_i}^* u^\nabla, v \rangle = \langle u^\nabla, s_{\tau_i} v \rangle, \quad \text{for any } v \in L, u^\nabla \in L^\nabla,$$

and (3.15) holds.

(ii) Let $\pi(w_i) = {}^t w_i^{-1}$ for $i = 1, 2$. Then,

$$\pi(w_1)\,\pi(w_2) = {}^t w_1^{-1}\, {}^t w_2^{-1} = {}^t(w_2^{-1}\, w_1^{-1}) = {}^t(w_1\, w_2)^{-1} = \pi(w_1\, w_2).$$

Let us put

$$w^* := {}^t w^{-1}.$$

Then, $\pi(w) = w^*$, and

$$w_1^*\, w_2^* = (w_1 w_2)^*. \tag{3.16}$$

---

[1] The Weyl group of the integral quadratic form and their roots were studied by A. Roiter in [23], and then this study was continued in [9, 18, 20, 24] and others.

3.5.2. Action of the Weyl group $W_S^\vee$

By (3.11), (3.12), and (3.14), for any $u^\nabla \in L^\nabla$, we have

$$
\begin{aligned}
s_{\tau_i}^* u^\nabla &= u^\nabla - \langle u^\nabla, \tau_i \rangle \tau_i^\nabla = u^\nabla - u_{\tau_i}^\nabla(\tau_i^\nabla) = u^\nabla - u_{\tau_i}^\nabla(B_\Gamma \tau_i), \\
(s_{\tau_i}^* u^\nabla)_{\tau_k} &= u_{\tau_k}^\nabla - u_{\tau_i}^\nabla(B_\Gamma \tau_i)_{\tau_k} = u_{\tau_k}^\nabla - u_{\tau_i}^\nabla(\tau_i, \tau_k).
\end{aligned}
\tag{3.17}
$$

Let u be a linkage root, then vector u^∇ is the linkage label vector, the coordinates of u^∇ belong to the set $\{-1, 0, 1\}$. By (3.17) we get

$$
(s_{\tau_i}^* u^\nabla)_{\tau_k} = \begin{cases}
-u_{\tau_i}^\nabla, & \text{if } k = i, \\
u_{\tau_k}^\nabla + u_{\tau_i}^\nabla, & \text{if } \{\tau_k, \tau_i\} \text{ is solid, i.e., } (\tau_k, \tau_i) = -1, \\
u_{\tau_k}^\nabla - u_{\tau_i}^\nabla, & \text{if } \{\tau_k, \tau_i\} \text{ is dotted, i.e., } (\tau_k, \tau_i) = 1, \\
u_{\tau_i}^\nabla, & \text{if } \tau_k \text{ and } \tau_i \text{ are disconnected, i.e., } !(\tau_k, \tau_i) = 0.
\end{cases}
\tag{3.18}
$$

Proposition 3.6. *If u^∇ is a linkage label vector, then values $(s_{\tau_i}^* u^\nabla)_{\tau_k}$ in (3.18) belong to the set $\{-1, 0, 1\}$, i.e., we obtain a new linkage label vector $s_{\tau_i}^* u^\nabla$.*

Proof. Let $\{\tau_k, \tau_i\}$ be a solid edge. If $u_{\tau_k}^\nabla = u_{\tau_i}^\nabla = 1$ (resp. $u_{\tau_k}^\nabla = u_{\tau_i}^\nabla = -1$) then roots $\{-u, \tau_k, \tau_i\}$ (resp. $\{u, \tau_k, \tau_i\}$) constitute the root system corresponding to the extended Dynkin diagram $\widetilde{A}_2$, which is impossible, see [26, Lemma A.1]. For remaining pairs $\{u_{\tau_k}^\nabla, u_{\tau_i}^\nabla\}$, we have $-1 \leq u_{\tau_k}^\nabla + u_{\tau_i}^\nabla \leq 1$. Now, let $\{\tau_k, \tau_i\}$ be a dotted edge. If $u_{\tau_k}^\nabla = 1$ and $u_{\tau_i}^\nabla = -1$ (resp. $u_{\tau_k}^\nabla = -1$ and $u_{\tau_i}^\nabla = 1$) then roots $\{u, -\tau_k, \tau_i\}$ (resp. $\{u, \tau_k, -\tau_i\}$) constitute the root system $\widetilde{A}_2$, which is impossible. For remaining pairs $\{u_{\tau_k}^\nabla, u_{\tau_i}^\nabla\}$, we have $-1 \leq u_{\tau_k}^\nabla - u_{\tau_i}^\nabla \leq 1$.

Remark 3.7. Eq. (3.18) is very useful for calculating linkage systems. Note that in (3.18) the action of dual reflections $s_{\tau_i}^*$ on the linkage label vector u^∇ is non-trivial if and only if $u_i^\nabla \neq 0$.

Proposition 3.8. *(i) For reflections $s_{\tau_i}^*$, the following relations hold:*

$$
B_\Gamma s_{\tau_i} = s_{\tau_i}^* B_\Gamma,
\tag{3.19}
$$

$$
(s_{\tau_i} \gamma)^\nabla = s_{\tau_i}^* B_\Gamma \gamma_L = s_{\tau_i}^* \gamma^\nabla.
\tag{3.20}
$$

(ii) For $w^* \in W_S^\vee$, we have

$$
(w\gamma)^\nabla = w^* \gamma^\nabla.
$$

(iii) The following relations hold:

$$
\begin{aligned}
\mathscr{B}_\Gamma(s_{\tau_i} v) &= \mathscr{B}_\Gamma(v) \text{ for any } v \in L, \\
\mathscr{B}_\Gamma^\vee(s_{\tau_i}^* u^\nabla) &= \mathscr{B}_\Gamma^\vee(u^\nabla) \text{ for any } u^\nabla \in L^\nabla.
\end{aligned}
\tag{3.21}
$$

Proof. (i) The equality (3.19) is true since for any $u, v \in L$ the following is true:

$$(s_{\tau_i}u, v)_\Gamma = (u, s_{\tau_i}v)_\Gamma, \text{ i.e.,}$$
$$\langle B_\Gamma s_{\tau_i}u, v \rangle = \langle B_\Gamma u, s_{\tau_i}v \rangle = \langle s_{\tau_i}^* B_\Gamma u, v \rangle, \text{ and}$$
$$\langle (B_\Gamma s_{\tau_i} - s_{\tau_i}^* B_\Gamma)u, v \rangle = 0.$$

Let us consider eq. (3.20). Since $(\tau_i, \mu) = 0$ for any $\tau_i \in S$, and $s_{\tau_i}\mu = \mu$, we have

$$(s_{\tau_i}\gamma)^\nabla = \begin{pmatrix} (s_{\tau_i}\gamma_L + \mu, \tau_1) \\ \dots \\ (s_{\tau_i}\gamma_L + \mu, \tau_l) \end{pmatrix} = \begin{pmatrix} (s_{\tau_i}\gamma_L, \tau_1) \\ \dots \\ (s_{\tau_i}\gamma_L, \tau_l) \end{pmatrix} = (s_{\tau_i}\gamma_L)^\nabla.$$

Then, by (3.19) and (3.4)

$$(s_{\tau_i}\gamma)^\nabla = (s_{\tau_i}\gamma_L)^\nabla = B_\Gamma s_{\tau_i}\gamma_L = s_{\tau_i}^* B_\Gamma \gamma_L = s_{\tau_i}^* \gamma^\nabla.$$

(ii) Let $w = s_{\tau_1}s_{\tau_2}\dots s_{\tau_m}$ be a decomposition of $w \in W$. Since $s_\tau^* = {}^t s_\tau^{-1} = {}^t s_\tau$, we deduce from (3.16) and (3.20) the following:

$$(w\gamma)^\nabla = (s_{\tau_1}s_{\tau_2}\dots s_{\tau_m}\gamma)^\nabla = s_{\tau_1}^*(s_{\tau_2}\dots s_{\tau_m}\gamma)^\nabla$$
$$= s_{\tau_1}^* s_{\tau_2}^*(s_{\tau_3}\dots s_{\tau_m}\gamma)^\nabla = \dots = s_{\tau_1}^* s_{\tau_2}^* \dots s_{\tau_m}^* \gamma^\nabla = w^*\gamma^\nabla.$$

(iii) Further, by (3.19) we have

$$\mathscr{B}_\Gamma(s_{\tau_i}v) = \langle B_\Gamma s_{\tau_i}v, s_{\tau_i}v \rangle = \langle s_{\tau_i}^* B_\Gamma v, s_{\tau_i}v \rangle = \langle B_\Gamma v, v \rangle = \mathscr{B}_\Gamma(v).$$
$$\mathscr{B}_\Gamma^\vee(s_{\tau_i}^* u^\nabla) = \langle B_\Gamma^\vee s_{\tau_i}^* u^\nabla, s_{\tau_i}^* u^\nabla \rangle = \langle s_{\tau_i} B_\Gamma^\vee u^\nabla, s_{\tau_i}^* u^\nabla \rangle$$
$$= \langle B_\Gamma^\vee u^\nabla, u^\nabla \rangle = \mathscr{B}_\Gamma^\vee(u^\nabla).$$

so (3.21) holds.

By Propositions 3.6 each linkage label vector belongs to some orbit of the group $W_S^\vee$ action. By Proposition 3.8(iii) all elements of some orbit have the same value $\mathscr{B}_\Gamma^\vee(u^\nabla)$. The rational number $p = \mathscr{B}_\Gamma^\vee(u^\nabla)$ is the invariant characterizing the given orbit.

To find the preimage root γ_1 for some label vector γ_1^∇ is suffices to know the preimage for one element γ on the orbit. By Proposition 3.8(ii) if $\gamma_1^\nabla = w^*\gamma^\nabla$ for some w, then $\gamma_1 = w\gamma$.

4.   Extensions

4.1.   The vertex extension

Let Γ be one of simply-laced Dynkin diagrams, and let Γ' be one of the Carter diagram out of the homogeneous class $C(\Gamma)$. Each Carter diagram Γ' from $C(\Gamma)$ has the same rank as rank(Γ). Consider any Γ-set $S = \{\tau_1, \ldots, \tau_n\}$ and any Γ'-set, $S' = \{\tau'_1, \ldots, \tau'_n\}$ obtained by the transformation constructed according to Theorem 1.1. Then the linear spans for S and S' coincide:

$$L(S) := \text{Span}(S), \quad L(S') := \text{Span}(S'), \quad L(S) = L(S'). \tag{4.1}$$

In this chapter, we will use definition $L := L(S) = L(S')$. Let $\widetilde{\Gamma}$ be a Dynkin diagram with the root system $\Phi(\widetilde{\Gamma})$ such that

(1) $\text{rank}(\Phi(\widetilde{\Gamma})) = \text{rank}(\Phi(\Gamma)) + 1$,

(2) $L \subset \text{Span}(\Phi(\widetilde{\Gamma}))$.

The choose of the root system $\widetilde{\Gamma}$ is ambiguous. The pair $\{\Gamma', \widetilde{\Gamma}\}$, where $\Gamma' \in C(\Gamma)$, is said to be the *vertex extension* of Γ' and is denoted by $\Gamma' \prec \widetilde{\Gamma}$.

Figure 7: Two vertex extensions of D_5: $D_5 \prec E_6$ and $D_5 \prec D_6$.

Remark 4.1. It may happen that a given root subset L can be extended to two different subsets $L(\gamma) \neq L(\delta)$ such that $\mathscr{B}_\Gamma^\vee(\gamma^\nabla) \neq \mathscr{B}_\Gamma^\vee(\delta^\nabla)$. For example, there are two extensions for D_5 with different values of $\mathscr{B}_\Gamma^\vee(\theta^\nabla)$:

$$\mathscr{B}_\Gamma^\vee(\gamma^\nabla) = \frac{5}{4} \quad \text{for} \quad D_5 \prec E_6, \qquad \mathscr{B}_\Gamma^\vee(\delta^\nabla) = 1 \quad \text{for} \quad D_5 \prec D_6,$$

see Fig. 7. The matrices B_Γ and B_Γ^{-1} for $\Gamma = D_5$ are as follows:

$$B_\Gamma = \begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ 0 & 0 & 2 & -1 & 0 \\ -1 & -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{bmatrix} \begin{matrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \end{matrix}$$

$$B_{\Gamma}^{-1} = \frac{1}{4} \begin{bmatrix} 5 & 4 & 3 & 6 & 2 \\ 4 & 8 & 4 & 8 & 4 \\ 3 & 4 & 5 & 6 & 2 \\ 6 & 8 & 6 & 12 & 4 \\ 2 & 4 & 2 & 4 & 4 \end{bmatrix} \begin{matrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \end{matrix}$$

4.2. Partial linkage systems

Let $\widetilde{\Gamma}$ be a Dynkin diagram such that the pair $\{\widetilde{\Gamma}, \Gamma\}$ is the vertex extension $\Gamma' \prec \widetilde{\Gamma}$, see § 4.1. Denote by $\mathscr{L}_{\widetilde{\Gamma}}(\Gamma', S')$ the set of linkage diagrams

$$\mathscr{L}_{\widetilde{\Gamma}}(\Gamma', S') = \{\gamma^\nabla \mid \gamma \in \Phi(\widetilde{\Gamma}), \gamma \notin L(S')\}, \tag{4.2}$$

where $\Phi(\widetilde{\Gamma})$ is the root system associated with $\widetilde{\Gamma}$, S' some Γ'-set and the linear space $L = L(S')$ is given by (4.1).

Proposition 4.2. *The set $\mathscr{L}_{\widetilde{\Gamma}}(\Gamma', S')$ does not depend on choosing Γ'-set S'.*

Proof. Let S'_1 and S'_2 be two different Γ'-sets in $\Phi(\widetilde{\Gamma})$, and let $S'_i = \{\tau^i_1, \ldots, \tau^i_l\}$ for $i = 1, 2$. Consider vector $\gamma_1 \in \Phi(\widetilde{\Gamma})$ obtained by the vertex extension of Γ' using the Γ'-set S'_1. Let $\gamma_{1,L}$ be the projection of γ onto $L = L(S'_1)$. By (3.3) we have

$$\gamma_1 = \mu_1 + \sum_{j=1}^{l} t_j \tau^1_j, \tag{4.3}$$

where $\{t_j \mid j = 1 \ldots l\}$ are coefficients of decomposition of $\gamma_{1,L}$ by the basis $\{\tau^i_1, \ldots, \tau^i_l\}$. From γ_1 we construct $\gamma_2 \in \Phi(\widetilde{\Gamma})$ by the same vertex extension using Γ'-set S'_2:

$$\gamma_2 = \mu_2 + \sum_{j=1}^{l} t_j \tau^2_j, \tag{4.4}$$

where basis vectors τ^1_j and τ^2_j have the same coefficient t_j. Here, μ_1, μ_2 are normal extending vectors from § 3.1.2. Coordinates of the linkage label vectors γ_1^∇ and γ_2^∇ are as follows:

$$(\gamma_1^\nabla)_k = \sum_{j=1}^{l} t_j (\tau^1_j, \tau^1_k), \quad (\gamma_2^\nabla)_k = \sum_{j=1}^{l} t_j (\tau^2_j, \tau^2_k), \quad k = 1, \ldots, l. \tag{4.5}$$

Since inner products (τ_j^1, τ_k^1) and (τ_j^1, τ_k^1) correspond to the same edge of Γ' (or both are equal to 0) we get $\gamma_1^\nabla = \gamma_2^\nabla$. Thus, we get

$$\mathscr{L}_{\widetilde{\Gamma}}(\Gamma', S_1') = \mathscr{L}_{\widetilde{\Gamma}}(\Gamma', S_2'). \tag{4.6}$$

Thus, the set of linkage diagrams (4.2) can be denoted by $\mathscr{L}_{\widetilde{\Gamma}}(\Gamma')$. The set of linkage diagrams $\mathscr{L}_{\widetilde{\Gamma}}(\Gamma')$ is said to be *linkage system Γ' over* Γ or *partial linkage system.*

4.3. Full linkage systems

The union of all partial systems by all possible vertex extensions is said to be the *linkage system* or *full linkage system* of Γ:

$$\mathscr{L}(\Gamma') = \bigcup_{\Gamma' \prec \widetilde{\Gamma}} \mathscr{L}_{\widetilde{\Gamma}}(\Gamma').$$

Proposition 4.3. *(i) For the homogeneous Carter diagrams Γ and Γ' the sizes of the full linkage systems are the same:*

$$|\mathscr{L}(\Gamma)| = |\mathscr{L}(\Gamma')|. \tag{4.7}$$

(ii) For the full linkage systems (4.7), the estimate of the number of linkage diagrams is as follows:

$$|\mathscr{L}(\Gamma')| = |\mathscr{L}(\Gamma)| \leq \bigcup \big(|\Phi(\widetilde{\Gamma}_i)| - |\Phi(\Gamma)|\big),$$

where the union is taken by all vertex extensions $\Gamma' \prec \widetilde{\Gamma}_i$.

Proof. (i) This fact follows from Proposition 3.4 and § 3.4.
(ii) By (4.7), for any Γ' out of the homogeneous class $C(\Gamma)$ we have

$$|\mathscr{L}_{\widetilde{\Gamma}}(\Gamma')| = |\mathscr{L}_{\widetilde{\Gamma}}(\Gamma)| \leq |\Phi(\widetilde{\Gamma})| - |\Phi(\Gamma)|. \tag{4.8}$$

Then, the number of linkage diagrams in the full linkage systems can be estimated as follows:

$$|\mathscr{L}(\Gamma')| = |\mathscr{L}(\Gamma)| \leq \bigcup \big(|\Phi(\widetilde{\Gamma}_i)| - |\Phi(\Gamma)|\big),$$

where the union is taken by all vertex extensions $\Gamma' \prec \widetilde{\Gamma}_i$.

5. The linkage systems $\mathscr{L}(D_l)$ and $\mathscr{L}(D_l(a_k))$

In this section we assume that Γ is one of the Carter diagrams D_l or $D_l(a_k)$.

5.1. D-components $\mathscr{L}_{D_{l+1}}(D_l)$ and $\mathscr{L}_{D_{l+1}}(D_l(a_k))$

Lemma 5.1. *Let $D_l \prec D_{l+1}$ be some vertex extension with root systems*

$$\Phi(D_l) = \{\tau_1, \ldots, \tau_l\}, \quad \Phi(D_{l+1}) = \{\tau_1, \ldots, \tau_l, \tau_{l+1}\},$$

where τ_{l+1} is a simple positive root in $\Phi(D_{l+1}) \backslash \Phi(D_l)$. Let φ be some positive root in $\Phi(D_{l+1}) \backslash \Phi(D_l)$, and μ_{max} be maximal root in $\Phi(D_{l+1})$.
  (i) The vector

$$\delta = \mu_{max} - \varphi + \tau_{l+1} \tag{5.1}$$

is also a root in $\Phi(D_{l+1}) \backslash \Phi(D_l)$.
  (ii) The linkage label vectors φ^{∇} and $-\delta^{\nabla}$ coincide.
  (iii) For any $l \geq 4$, the linkage systems $\mathscr{L}_{D_{l+1}}(D_l)$ and $\mathscr{L}_{D_{l+1}}(D_l(a_k))$, where $1 \leq k \leq \left[\frac{l-2}{2}\right]$, have $2l$ elements:

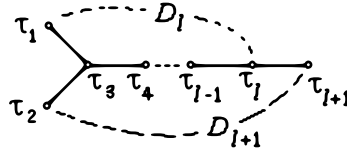$$|\mathscr{L}_{D_{l+1}}(D_l)| = |\mathscr{L}_{D_{l+1}}(D_l(a_k))| = 2l. \tag{5.2}$$

Figure 8:   The vertex extension $D_l \prec D_{l+1}$.

Proof. Let $\mathscr{B}$ be the quadratic Tits form associated with D_{l+1}.
  (i) By (1.3) it suffices to prove that $\mathscr{B}(\delta) = 2$. If $\varphi = \mu_{max}$ (resp. τ_{l+1}), then $\delta = \tau_{l+1}$ (resp. μ_{max}). In both cases, δ is the root in $\Phi(D_{l+1}) \backslash \Phi(S)$. Suppose $\varphi \neq \mu_{max}, \tau_{l+1}$. We need to prove that

$$\mathscr{B}(\mu_{max} - \varphi + \tau_{l+1}) = 2, \text{ i.e.,}$$

$$\mathscr{B}(\mu_{max}) + \mathscr{B}(\varphi) + \mathscr{B}(\tau_{l+1}) + 2(\mu_{max}, \tau_{l+1}) - 2(\varphi, \mu_{max} + \tau_{l+1}) = 2. \tag{5.3}$$

Since $\mathscr{B}(\varphi) = \mathscr{B}(\tau_{l+1}) = \mathscr{B}(\mu_{max}) = 2$, eq. (5.3) is equivalent to the following:

$$(\varphi, \mu_{max} + \tau_{l+1}) - (\mu_{max}, \tau_{l+1}) = 2.$$

Further, $(\mu_{max}, \tau_i) = 0$ for any $i \neq l$. In particular, $(\mu_{max}, \tau_{l+1}) = 0$ and it suffices to prove that

$$(\varphi, \mu_{max} + \tau_{l+1}) = 2 \text{ for any } \varphi \in \Phi(D_{l+1}) \backslash \Phi(D_l). \qquad (5.4)$$

We have

$$(\gamma, \mu_{max} + \tau_{l+1}) = \begin{cases} 2 \text{ for } \gamma = \tau_{l+1}, \text{ since } \mu_{max} \perp \tau_{l+1}, \\ 0 \text{ for } \gamma = \tau_l, \text{ since } (\tau_l, \mu_{max}) = 1, (\tau_l, \tau_{l+1}) = -1, \\ 0 \text{ for } \gamma = \tau_i, \text{ where } i < l. \end{cases} \qquad (5.5)$$

Since φ is the positive root in $\Phi(D_{l+1}) \backslash \Phi(D_l)$, then τ_{l+1} appears with a unit coefficient in the decomposition of φ in the basis $\{\tau_1, \ldots, \tau_{l+1}\}$. Then, by (5.5) the relation (5.4) holds for any root $\varphi \in \Phi(D_{l+1}) \backslash \Phi(D_l)$. Therefore, $\mathscr{B}(\delta) = 2$, and δ is also a root.

(ii) By (5.1) we have

$$(\delta, \tau_i) = (\mu_{max} + \tau_{l+1}, \tau_i) - (\varphi, \tau_i), \text{ where } 1 \leq i \leq l.$$

By (5.5) for $i \leq l$, we have

$$\delta_i^\nabla = (\delta, \tau_i) = -(\varphi, \tau_i) = -\varphi_i^\nabla, \text{ where } 1 \leq i \leq l.$$

This means that $\delta^\nabla = -\varphi^\nabla$.

(iii) The number of roots of $\Phi(D_l)$ is $2l(l-1)$, see [3, Table IV]. Then by (4.7) and (4.8) we have

$$\mid \mathscr{L}_{D_{l+1}}(D_l(a_k)) \mid = \mid \mathscr{L}_{D_{l+1}}(D_l) \mid \leq 2(l+1)l - 2l(l-1) = 4l.$$

By (ii), this size is two times smaller:

$$\mid \mathscr{L}_{D_{l+1}}(D_l(a_k)) \mid = \mid \mathscr{L}_{D_{l+1}}(D_l) \mid \leq 2l.$$

In Fig. 9, we present exactly $2l$ linkage label vectors for the Carter diagram D_l. To construct the partial linkage system $\mathscr{L}_{D_{l+1}}(D_l)$ (resp. $\mathscr{L}_{D_{l+1}}(D_l(a_k))$), we use the action of the Weyl group $W_S^\vee$, see § 3.5.2, and Remark 3.7, see Fig. 10.

So, we conclude that there are exactly $2l$ linkage label vectors in the D-component $\mathscr{L}_{D_{l+1}}(D_l)$. Then, by (4.7) this is also true for the D-component $\mathscr{L}_{D_{l+1}}(D_l(a_k))$.

5.2. E-component $\mathscr{L}_{E_8}(D_7)$

By (5.2) the linkage system $\mathscr{L}_{D_8}(D_7)$ contains 14 elements. We will show that all of them lie in $\mathscr{L}_{E_8}(D_7)$.

Lemma 5.2. *There are 14 pairs of roots $\eta, \lambda \in \Phi(E_8)$ such that*

$$\eta^{\nabla} = -\lambda^{\nabla}.$$

Proof. The 7 pairs of roots $\eta, \lambda \in \Phi(E_8)$ such that $\eta^{\nabla} = -\lambda^{\nabla}$ listed in Table 5.2. The opposite roots form 7 pairs of negative roots $\eta, \lambda \in \Phi(E_8)$ such that $\eta^{\nabla} = -\lambda^{\nabla}$. For $1 \leq i \leq 7$, the sum $\eta_i + \lambda_i$ is the same vector:

$$\eta_i + \lambda_i \;\; = \;\; \begin{matrix} 4 & 7 & 10 & 8 & 6 & 4 & 2 \\ & & 5 & & & & \end{matrix} \qquad (5.6)$$

The order of coordinates is as follows:

$$\begin{matrix} \tau_1 & \tau_3 & \tau_4 & \tau_5 & \tau_6 & \tau_7 & \tau_8 \\ & & \tau_2 & & & & \end{matrix}$$

Vectors $\eta_i + \lambda_i$ from (5.6) are orthogonal to any τ_i, $2 \leq i \leq 8$, so $\eta_i^{\nabla} = -\lambda_i^{\nabla}$. Note that there is no need orthogonality for τ_1, see (1.15). Further, we calculate the label vectors η_i^{∇} (or λ_i^{∇}). They are listed in the last column of Table 5.2.

5.3.   The size and structure of $\mathscr{L}(D_l)$ and $\mathscr{L}(D_l(a_k))$

Theorem 5.3. *(i) The full linkage system $\mathscr{L}(D_4)$ (resp. $\mathscr{L}(D_4(a_1))$) consists only of D-component, see Figs. 4, 5. Each of these D-components consists of three orbits of size 8 each.*

(ii) For $l > 7$, the full linkage system $\mathscr{L}(D_l)$ (resp. $\mathscr{L}(D_l(a_1))$) consists only of D-component for any k, see Figs. 9, 10. Each of these D-components consists of one orbit of size $2l$.

(iii) For $l = 5$, the full linkage system $\mathscr{L}(D_5)$ (resp. $\mathscr{L}(D_5(a_1))$) consists of the D-component (10 elements) and the E-component consisting of two orbits (2×16 elements). In total, $\mathscr{L}(D_5)$ contains 42 elements. One of two orbits of the E-component of the $\mathscr{L}(D_5)$ is shown in Fig. 11.

(iv) For $l = 6$, the full linkage system $\mathscr{L}(D_6)$ (resp. $\mathscr{L}(D_6(a_1))$, $\mathscr{L}(D_6(a_2))$) consists of the D-component (12 elements) and the E-com-

	$\eta \in \Phi(E_8)\backslash\Phi(D_7)$	$\lambda \in \Phi(E_8)\backslash\Phi(D_7)$	$\eta^\nabla = -\lambda^\nabla \in \mathscr{L}_{E_8}(D_7)$
1	2343210 2	2465432 3	0 0 0 0 0 −1 0
2	2343211 2	2465431 3	0 0 0 0 −1 1 0
3	2343221 2	2465421 3	0 0 0 −1 1 0 0
4	2343321 2	2465321 3	0 0 −1 1 0 0 0
5	2344321 2	2464321 3	0 −1 1 0 0 0 0
6	2354321 2	2454321 3	−1 1 0 0 0 0 −1
7	2354321 3	2454321 2	−1 0 0 0 0 0 1

Table 5.2: The 7 pairs of positive roots η, λ such that $\eta^\nabla = -\lambda^\nabla$.

ponent consisting of two orbits (2×32 elements). In total, $\mathscr{L}(D_6)$ contains 76 elements. Two orbits of the E-component of the linkage system $\mathscr{L}(D_6)$ are shown in Fig. 12.

(v) For $l = 7$, the full linkage system $\mathscr{L}(D_7)$ (resp. $\mathscr{L}(D_7(a_1))$, $\mathscr{L}(D_7(a_2))$) consists of one D-component (14 elements) and two E-components consisting of two orbits (2×64 elements). In total, $\mathscr{L}(D_7)$ contains 142 elements.

Proof. (i) The linkage system $\mathscr{L}_{D_5}(D_4)$ (resp. $\mathscr{L}_{D_5}(D_4(a_1))$) contains three orbits, see Fig. 4 (resp. Fig. 5).

(ii) For $l > 7$, the D-component $\mathscr{L}_{D_{l+1}}(D_l)$ and $\mathscr{L}_{D_{l+1}}(D_l(a_k))$ are depicted in Figs. 9 and 10. By Lemma 5.1(iii) there are $2l$ elements in each of these components.

(iii) By (4.7) and (4.8) for $l = 5$,

$$| \mathscr{L}_{E_6}(D_5(a_1)) | = | \mathscr{L}_{E_6}(D_6) | \le | \Phi(E_6) | - | \Phi(D_5) | = 72 - 40 = 32.$$

The full linkage system $\mathscr{L}(D_5)$ consists of 10-element linkage system $\mathscr{L}_{D_6}(D_5)$ (Lemma 5.1(iii)) and two orbits of the E-component, each of

which contains 16 linkages, i.e.,

$$| \, \mathscr{L}(D_5) \, | = | \, \mathscr{L}(D_5(a_1)) \, | \, \leq \, 32 + 10 = 42. \qquad (5.7)$$

Both E-components are depicted in Fig. 11. They can be obtained from each other as follows: for any linkage diagram the solid edges become dotted, and vice versa, the corresponding label vector becomes just opposite. Thus, in (5.7) both $\mathscr{L}(D_5)$ and $\mathscr{L}(D_5(a_1))$ contain exactly 42 elements.

(iv) Similarly, for $l = 6$,

$$| \, \mathscr{L}_{E_7}(D_6(a_2)) \, | = | \, \mathscr{L}_{E_7}(D_6(a_1)) \, | = | \, \mathscr{L}_{E_7}(D_6) \, | \leq$$
$$| \, \Phi(E_7) \, | - | \, \Phi(D_6) \, | \leq \, 126 - 60 = 66.$$

For the full linkage system, we have the following estimate:

$$| \, \mathscr{L}(D_6) \, | = | \, \mathscr{L}(D_6(a_2)) \, | = | \, \mathscr{L}(D_6(a_1)) \, | \, \leq \, 66 + 12 = 78.$$

Let the coordinates of roots of E_7 be as follows

$$\begin{array}{cccccc} \tau_6 & \tau_1 & \tau_2 & \tau_3 & \tau_4 & \tau_5 \\ & & \beta_2 & & & \end{array}$$

Consider maximal and minimal roots $\pm\mu$ in E_7. Here,

$$\pm\mu \in \Phi(E_7) \backslash \Phi(D_6) \text{ and } \pm\mu_{max} \notin \Phi(D_6).$$

Note that $\pm\mu$ are orthogonal to any simple root except for τ_6. Then vectors $\pm\mu^{\nabla}$ are zero, i.e.,

$$| \, \mathscr{L}(D_6) \, | = | \, \mathscr{L}(D_6(a_1)) \, | = | \, \mathscr{L}(D_6(a_2)) \, | \, \leq \, 78 - 2 = 76. \qquad (5.8)$$

The full linkage system for $\mathscr{L}(D_6)$ consists of the 12-element linkage system $\mathscr{L}_{D_7}(D_6)$ (Lemma 5.1(iii)) and two orbits of the E-component, each of which contains 32 linkages. These E-components are depicted in Fig. 12, they contain $2 \times 32 = 64$ elements. Thus, D-component and E-components together contain $12 + 64 = 76$ elements in $\mathscr{L}(D_6)$. So, in (5.8) all full linkage systems $\mathscr{L}(D_6)$, $\mathscr{L}(D_6(a_1))$ and $\mathscr{L}(D_6(a_1))$ contain exactly 76 elements.

(v) For $l = 7$, we have

$$| \, \mathscr{L}_{E_8}(D_7(a_2)) \, | = | \, \mathscr{L}_{E_8}(D_7(a_1)) \, | = | \, \mathscr{L}_{E_8}(D_7) \, | \leq$$
$$| \, \Phi(E_8) \, | - | \, \Phi(D_7) \, | \leq \, 240 - 84 = 156.$$

By Lemma 5.2(i), in $\mathscr{L}_{E_8}(D_7)$ there are 14 pairs having the same linkage label vectors, then

$$| \mathscr{L}_{E_8}(D_7(a_2)) | = | \mathscr{L}_{E_8}(D_7(a_1)) | = | \mathscr{L}_{E_8}(D_7) | \le 156 - 14 = 142.$$

In fact, $| \mathscr{L}_{E_8}(D_7) |$ contains only 128 elements in two 64-element orbits which can be found in [25, Figs. C.63, C.64]. The full linkage system for $\mathscr{L}(D_7)$ consists of the 14-element linkage system $\mathscr{L}_{D_8}(D_7)$ (Lemma 5.1(iii)) and two orbits of the E-component. The full linkage systems $\mathscr{L}(D_7)$, $\mathscr{L}(D_7(a_1))$ and $\mathscr{L}(D_7(a_2))$ contain exactly 142 elements.
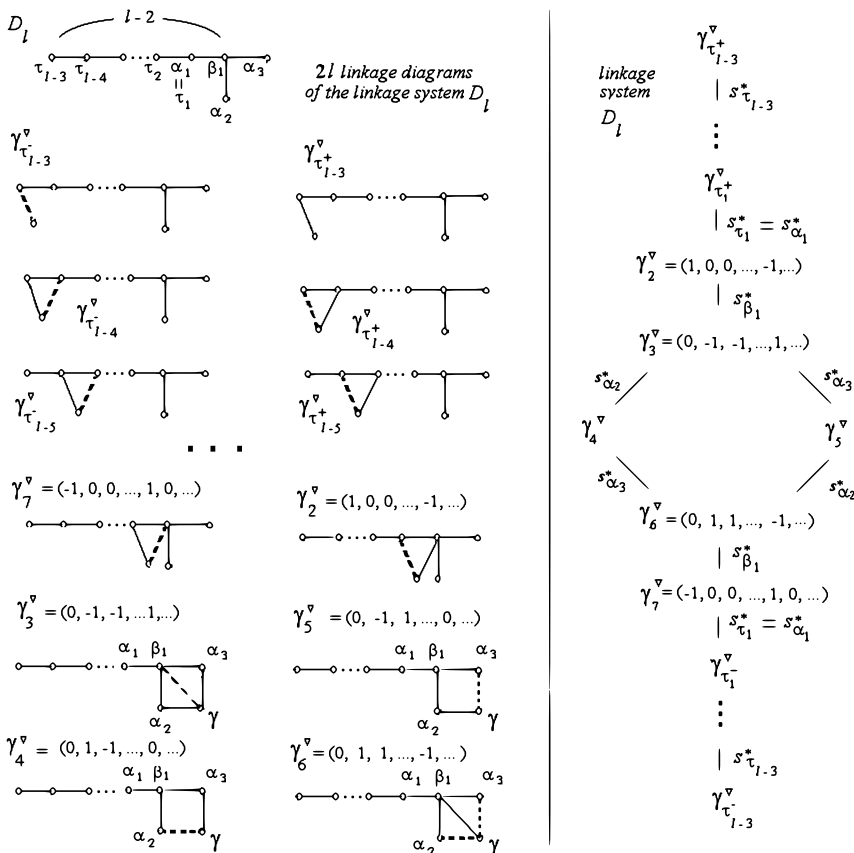

Figure 9: For $l > 4$, the partial linkage system $\mathscr{L}_{D_{l+1}}(D_l)$ contains $2l$ linkages. For $l > 7$, $\mathscr{L}_{D_{l+1}}(D_l)$ coincides with the full linkage system $\mathscr{L}(D_l)$.

Figure 10: For $l > 4$, $\mathscr{L}_{D_{l+1}}(D_l(a_k))$ contains $2l$ linkages. For $l > 7$, $\mathscr{L}(D_l(a_k)) = \mathscr{L}_{D_{l+1}}(D_l(a_k))$.
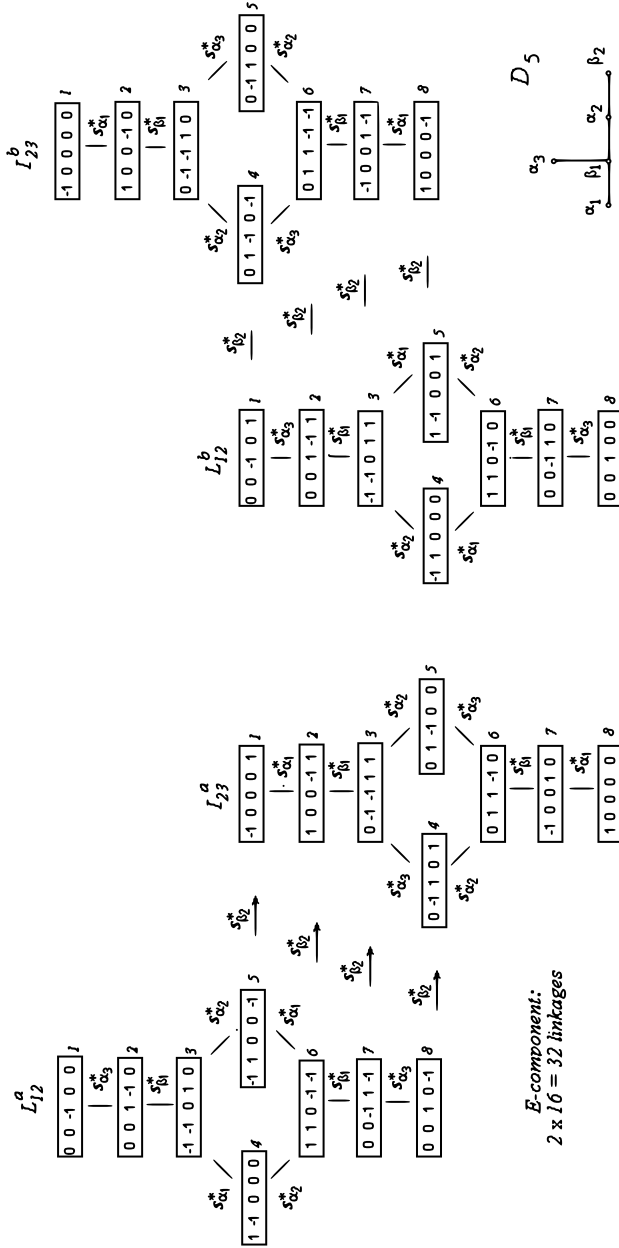
Figure 11: Two orbits of the E-component of the linkage system $\mathscr{L}(D_5)$. Each orbit consists of two loctets.
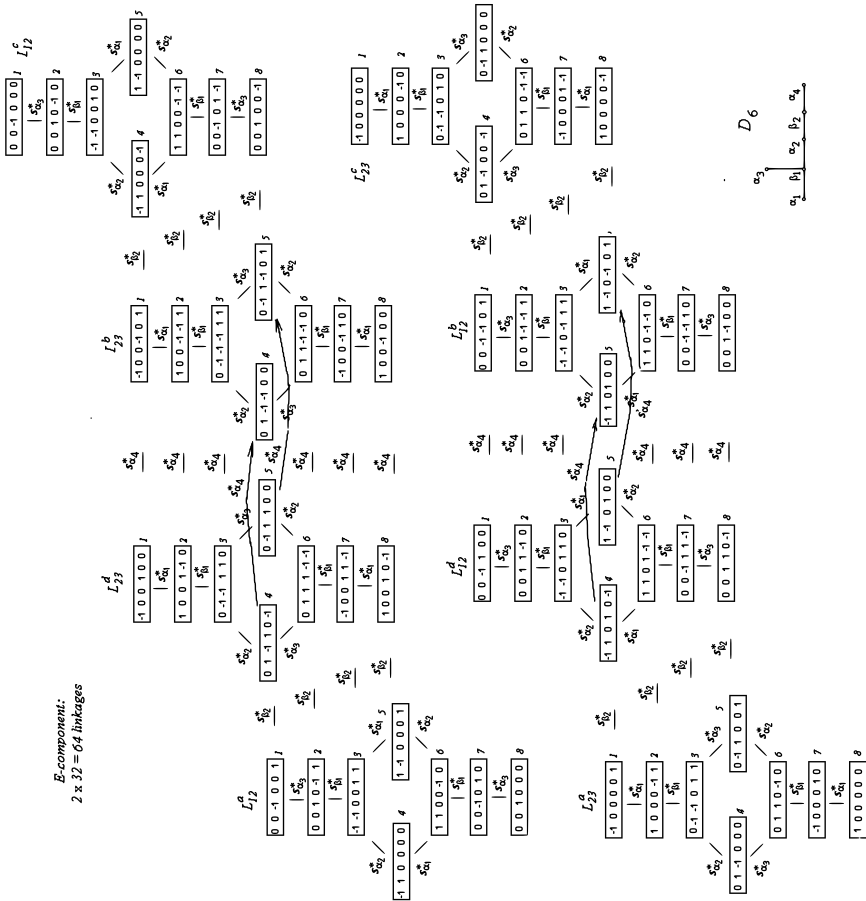
Figure 12: Two orbits of the E-component of $\mathscr{L}(D_6)$. Each orbit consists of four loctets.

A.   Gabrielov transformations and the Ovsienko theorem

A.1.   Gabrielov transformations

In 1973, A. Gabrielov introduced mappings changing quadratic forms associated with singularities, [10, § 6]. Gabrielov's example corresponding to the singularity $x^3 + y^3 + z^2$ is shown in Fig. 13. Essentially, Gabrielov transformations in Fig. 13 similar to transitions of § 1.6. The sequence of transformation $T^{(i)}$, $i = 1, 2, 3$ modify the underlying diagram to the Dynkin diagram D_4, change the basis (vanishing cycles) and the matrix of the quadratic form (intersection matrix) which is similar to a certain Cartan matrix, [7, 11]. Gabrielov transformations are used for classification of some positive definite forms in [5].
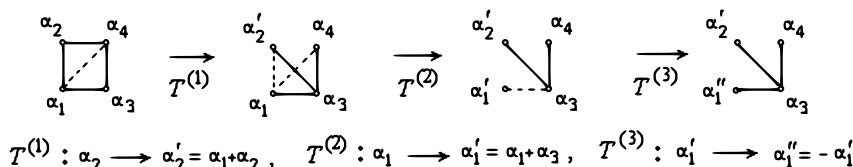

$$T^{(1)} : \alpha_2 \longrightarrow \alpha'_2 = \alpha_1 + \alpha_2 , \quad T^{(2)} : \alpha_1 \longrightarrow \alpha'_1 = \alpha_1 + \alpha_3 , \quad T^{(3)} : \alpha'_1 \longrightarrow \alpha''_1 = -\alpha'_1$$

Figure 13: Gabrielov's example of changing the basis corresponding to a singularity $x^3 + y^3 + z^2$. Transitions $T^{(i)}, i = 1, 2, 3$ act only on one basis vector and fix all others.

Another example corresponds to the singularity $x^4 + y^3 + z^2$, see Fig. 14. The sequence of Gabrielov transformations $T^{(i)}, 1 \le i \le 7$ in Fig. 14 modify the underlying diagram to the Dynkin diagram E_6.

Let $B = B_\Gamma$ be the (partial) Cartan matrix associated with the diagram Γ, E_{ij} be the elementary $n \times n$ matrix with unique nonzero entry a 1 at slot (i, j), and G^B_{ij} – the linear transformation given by matrix $G^B_{ij} = I - b_{ij} E_{ij}$. For each $i \ne j$ in $\{1, \ldots, n\}$, the Gabrielov transformation $\mathscr{G}_{ij}$ is defined as the following composition:

$$B \longrightarrow \mathscr{G}_{ij}(B) = BG^B_{ij}, \tag{A.1}$$

see [2, Proposition 2.17].

A.2.   Inflations and Ovsienko's theorem

Let ε be a sign $\varepsilon \in \{+, -\}$, and for $1 \le i, j \le n$ define the linear transformation $T^{\varepsilon_{ij}} : \mathbb{Z}^n \longmapsto \mathbb{Z}^n$ by

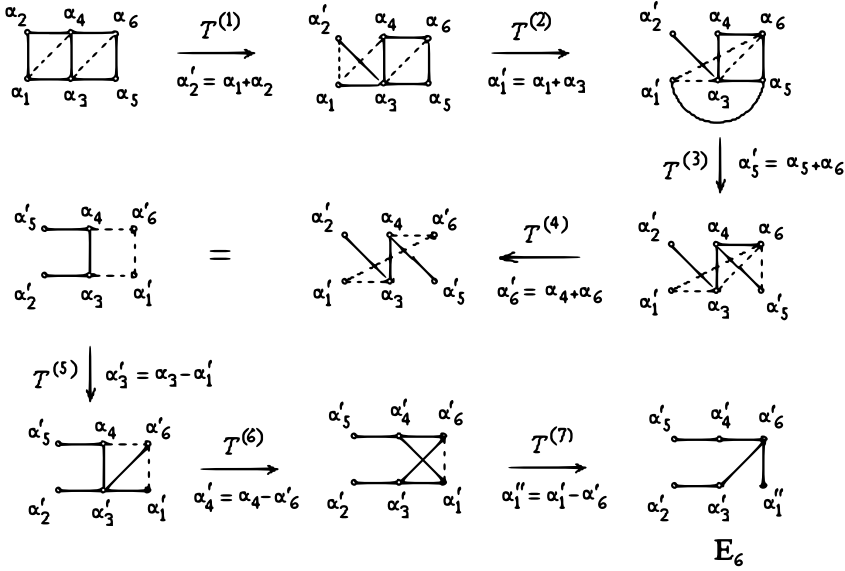$$T^{\varepsilon}_{ij} : v \longmapsto v - \varepsilon v_i \alpha_j .$$

Figure 14: The Gabrielov example of changing the basis corresponding
to a singularity $x^4 + y^3 + z^2$. Transitions $T^{(i)}, 1 \le i \le 7$ act
only on one basis vector and fix all others.

Note that T_{ij}^+ is the inverse of T_{ij}^-. The transformation T_{ij}^- is called
a *deflation* for $B_{\Gamma'}$ if $b_{ij} < 0$, and T_{ij}^+ is called an *inflation* for $B_{\Gamma'}$ if
$b_{ij} > 0$. Inflations and deflations are called *flations*. A finite composition
of *flations* is an *iterated flation*:

$$T = T_{i_1 j_1}^{\varepsilon_1} \ldots T_{i_r j_r}^{\varepsilon_r}. \tag{A.2}$$

Composition (A.2) is inductively defined, see [2, § 2.4].

The inflations technique introduced by S. Ovsienko in [22] in the
context of weakly positive unit quadratic forms, see [16, Remark 4.4] and
[18, Example 5.2], is similar to Gabrielov's transformations. If $|b_{ij}| = 1$
and $\varepsilon \in \{+, -\}$ is such that $|b_{ij}| = \varepsilon b_{ij}$ then the quadratic form BT_{ij}^ε
coincides with the Gabrielov transformation $\mathscr{G}_{ij}(B)$ from (A.1).

Theorem (Ovsienko)**.** *Let B be a positive definite unit form. Then
there exists an iterated inflation T and a unique (up to permutation of
components) disjoint union of Dynkin diagrams Γ such that BT coincides
with the Cartan matrix B_Γ.*

For the proof, see [2, Theorem 2.20]. For the further development of
the inflation mappings, see [1, 14, 16, 18, 19].

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