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On algebras that are sums of two subalgebras

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Abstract. We study an associative algebra A over an arbitrary field K that is a sum of two subalgebras B and C (i.e. $A = B + C$). Let $\mathcal{M}$ be the class of algebras such that $B, C \in \mathcal{M}$ implies $A \in \mathcal{M}$. We prove, under some natural additional assumptions on $\mathcal{M}$, that if B and C have ideals of finite codimension from $\mathcal{M}$, then A has an ideal of finite codimension from $\mathcal{M}$, too. In particular we show that if B and C have left T-nilpotent ideals (or nil PI ideals) of finite codimension, then A has a left T-nilpotent ideal (or nil PI ideal) of finite codimension.

Introduction

Let R be an associative ring, and let R_1 and R_2 be its subrings such that $R = R_1 + R_2$, i.e., for every $r \in R$ there exist $r_1 \in R_1$ and $r_2 \in R_2$ such that $r = r_1 + r_2$. The question whether, for a given class of rings $\mathcal{M}$, if $R_1, R_2 \in \mathcal{M}$, then $R \in \mathcal{M}$, was studied in many papers (cf. [2, 3, 4, 5, 6, 7, 12, 13, 14]). It was proved that the answer is positive for the following classes: nilpotent rings (in [5]), nil rings of bounded index (in [14]), nil PI rings (i.e. rings with polynomial identities) and PI rings (in [8]), as well as left T-nilpotent rings (in [1]).

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In [9, 10, 11, 16, 17] certain generalizations of some results from the papers cited above were obtained for algebras over an arbitrary field. Let $\mathcal{H}$ be a particular class of algebras. We say that an algebra H is almost in $\mathcal{H}$ (or is almost nilpotent, almost nil of bounded index, etc.) if it has an ideal of finite codimension belonging to $\mathcal{H}$. Assume that A is a K-algebra and B, C are its subalgebras such that $A = B + C$. In the article [11], inspired by Petravchuk's work [17], it was shown that if B and C are almost nilpotent (almost nil of bounded index, respectively), then so is A. In the present work, motivated by methods and results from [11, 17], we approach the issue more generally. Let $\mathcal{M}$ be a class of algebras such that if $B, C \in \mathcal{M}$, then $A \in \mathcal{M}$. Moreover, let $\mathcal{M}$ be homomorphically closed, hereditary, closed under extensions, and contain a class of all nilpotent algebras. We prove that if B and C are almost in $\mathcal{M}$, then so is A. Hence, as a conclusion we obtain that if B and C are almost left T-nilpotent, then so is A. Moreover, we show that if B and C are almost nil PI, then A is almost nil PI, too.

1. The main result

We consider associative algebras over a fixed field K, which are not assumed to have an identity. For an algebra H, the symbol H^* denotes the algebra H with an identity adjoined. If I is an ideal (left ideal, right ideal) of a ring (of an algebra) A, we write $I \lhd A$ ($I <_l A$, $I <_r A$).

Let R be an algebra and R_1, R_2 its subalgebras such that $R = R_1 + R_2$. Moreover, let $\mathcal{N}$ be the class of all nilpotent algebras.

Throughout the paper, by $\mathcal{M}$ we denote an arbitrary homomorphically closed class of algebras that satisfies the following conditions:

1) $\mathcal{N} \subseteq \mathcal{M}$;
2) $\mathcal{M}$ is closed under extensions, i.e., if $I \lhd H$ and $I, H/I \in \mathcal{M}$, then $H \in \mathcal{M}$;
3) $\mathcal{M}$ is hereditary, i.e., if $I \lhd H$ and $H \in \mathcal{M}$, then $I \in \mathcal{M}$;
4) $\mathcal{M}$ satisfies the condition that if $R_1, R_2 \in \mathcal{M}$, then $R \in \mathcal{M}$.

All of the homomorphically closed classes of algebras known so far that satisfy condition 4) also satisfy conditions 1), 2) and 3).

By $\mathcal{F}$, $\mathcal{B}$, $\mathcal{P}$, and $\mathcal{T}$ we denote the class of all finite dimensional algebras, nil of bounded index algebras, nil PI-algebras and left T-nilpotent algebras, respectively. As we mentioned in the introduction, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{P}$ and $\mathcal{T}$

are examples of the class $\mathcal{M}$. Let $\mathcal{MF} = \{A \mid \exists I \lhd A, I \in \mathcal{M}, A/I \in \mathcal{F}\}$. Obviously, $\mathcal{M} \subseteq \mathcal{MF}$ and $\mathcal{F} \subseteq \mathcal{MF}$.

Throughout the paper, A is an algebra over K, and B and C are subalgebras of A such that $A = B + C$. Moreover, let $B_0 \lhd B$ and $C_0 \lhd C$, where $\dim_K B/B_0 < \infty$ and $\dim_K C/C_0 < \infty$.

Using the above notation, we can formulate the main result of this article as follows:

Theorem 1. *If $B \in \mathcal{MF}$ and $C \in \mathcal{MF}$, then $A \in \mathcal{MF}$.*

In particular, the above claim for $\mathcal{M} = \mathcal{N}$ is proven in [11, Theorem 1], and for $\mathcal{M} = \mathcal{B}$ in [11, Theorem 2]. In the present article, using Theorem 1, [8, Corollary 6] and [1, Theorem 2.9], we additionally obtain the following

Corollary 1. *If $B \in \mathcal{TF}$ and $C \in \mathcal{TF}$, then $A \in \mathcal{TF}$.*

Corollary 2. *If $B \in \mathcal{PF}$ and $C \in \mathcal{PF}$, then $A \in \mathcal{PF}$.*

2. Preliminary results

We shall need the following

Lemma 1 ([11, Lemma 13]). *Let R be a K-algebra and let S, T be finite dimensional K-subspaces of R. If M and P are K - subspaces of R such that $\dim(SMT + P)/P < \infty$, then $\dim M/N < \infty$, where $N = \{v \in M | SvT \subseteq P\}$.*

We will also need the following result of Mekey [15, Theorem]. A simple proof of this result, based on Lemma 1, can be found in [10, Lemma 5].

Lemma 2 ([15]). *Let H be an algebra over an arbitrary field, and P a subalgebra of A such that $\dim H/P < \infty$. Then P contains an ideal I of H such that $\dim H/I < \infty$.*

The following modification of Petravchuk's Lemma 7 from [17] (cf. also [18]) will be very useful for further consideration. A simple proof can be found in [11, Lemma 5].

Lemma 3. *Let P_1 and P_2 be subalgebras of an algebra H, and let I be an ideal of H such that $I \subseteq P_1 + P_2$. Then there exist subalgebras $Q_1 \subseteq P_1$ and $Q_2 \subseteq P_2$ of H such that $Q_1 + Q_2$ is subalgebra of H and $I \subseteq Q_1 + Q_2$.*

Now we are ready to obtain an extension of [17, Proposition 1].

Proposition 1. *For the class $\mathcal{MF}$ the following statements hold:*

(i) *every subalgebra and every quotient algebra of an algebra from $\mathcal{MF}$ belongs to $\mathcal{MF}$;*

(ii) *if P, $Q \in \mathcal{MF}$, then the direct product $P \times Q$ belongs to $\mathcal{MF}$;*

(iii) *if $H \in \mathcal{M}$, $I \lhd H$, $H/I \in \mathcal{MF}$ and $I \in \mathcal{MF}$, then $H \in \mathcal{MF}$.*

Proof. In the case of the statements (i) and (ii), the proof is obvious. Now we prove (iii). Let $I \lhd H$, $H/I \in \mathcal{MF}$ and $I \in \mathcal{MF}$. Hence there exists an ideal J of I such that $J \in \mathcal{M}$ and $I/J \in \mathcal{F}$. Let J_H be an ideal of H generated by J. It is easy to see that since $J \lhd I \lhd H$, $(J_H)^3 \subseteq J$. Because the class $\mathcal{M}$ is closed under extensions, hereditary and $\mathcal{N} \subseteq \mathcal{M}$, we have $J_H \in \mathcal{M}$. Clearly, we can assume that $J_H = 0$, which implies $I \in \mathcal{F}$. Let $P/I \in \mathcal{M}$ be an ideal of H/I such that $H/P \in \mathcal{F}$. Consider $G = r_P(I)$, the right annihilator of I in P. Obviously $G \lhd H$. Since $I \in \mathcal{F}$, we have $S/G \in \mathcal{F}$, so $H/G \in \mathcal{F}$. Combining $(G \cap I)^2 = 0$ with $G/(G \cap I) \approx (G + I)/I \in \mathcal{M}$, we infer that $G \in \mathcal{M}$. Hence $H \in \mathcal{MF}$, which ends the proof. $\square$

Let us extend the definition introduced by Petravchuk in [17, Definition 2] to the class $\mathcal{M}$.

Definition 1. *An algebra $A = B + C$ over an arbitrary field K is called an $\mathcal{M}$-counter-example, if A satisfies the following conditions:*

(1) *$A \notin \mathcal{MF}$;*

(2) *the subalgebras B and C have ideals $B_0 \lhd B$ and $C_0 \lhd C$ such that $B_0, C_0 \in \mathcal{M}$ and the number $\dim A/(B_0 + C_0)$ is the smallest possible;*

(3) *the algebra A does not have any nonzero ideal lying in the K-subspace $B_0 + C_0$ from condition (2).*

Remark 1. Suppose that $A = B + C$ is an algebra satisfy conditions (1) and (2) from Definition 1. We show that a certain homomorphic image of A is an $\mathcal{M}$-counter-example. Indeed, let T be the sum of all ideals of A that are contained in $B_0 + C_0$. By Lemma 3, $T \subseteq Q_1 + Q_2$, where $Q = Q_1 + Q_2$ is a subalgebra of A and Q_1, $Q_2 \in \mathcal{M}$. Thus $Q \in \mathcal{M}$ and consequently $T \in \mathcal{M}$. Additionally from Proposition 1, $A/T \notin \mathcal{MF}$. Clearly $A/T = (B+T)/T + (C+T)/T$. Now it is easy to see that A/T is an $\mathcal{M}$-counter-example.

Lemma 4. *Let A be an $\mathcal{M}$-counter-example. Then*

(i) for every $0 \neq I \lhd A$, $A/I \in \mathcal{MF}$;

(ii) the algebra A has no nonzero ideal from $\mathcal{MF}$;

(iii) A is a prime algebra.

Proof. Let A be an $\mathcal{M}$-counter-example and $0 \neq I \lhd A$. By Definition 1, $I \nsubseteq B_0 + C_0$. Denote $\overline{A} = A/I$, $\overline{B} = (B+I)/I$, $\overline{C} = (C+I)/I$. Moreover $\overline{B_0} = (B_0 + I)/I$ and $\overline{C_0} = (C_0 + I)/I$. Clearly $\overline{A} = \overline{B} + \overline{C}$. By Definition 1(i), $\overline{B}, \overline{C} \in \mathcal{MR}$. Additionally $\dim \overline{A}/(\overline{B_0} + \overline{C_0}) < \dim A/(B_0 + C_0)$. So $\overline{A} \in \mathcal{MF}$, which gives (i).

By (i) and Proposition 1, (ii) becomes obvious.

Let us prove statement (iii). Suppose, contrary to our claim, that A is not a prime algebra. Hence there exist nonzero ideals I and J of A such that $IJ = 0$. Since $(I \cap J)^2 = 0$, part (ii) yields $I \cap J = 0$. Hence A can be embeded into the product $A/I \times A/J$. By statements (i) and (ii) of Lemma 3, we obtain that $A \in \mathcal{MF}$, which is a contradiction. □

3. Proof of Theorem 1

Proof of Theorem 1: Suppose the assertion of the theorem is false. It follows that there exists an $\mathcal{M}$-counter-example, and therefore we can assume that A is an $\mathcal{M}$-counter-example. Let S, T be arbitrary finite dimensional K-subspaces of B^*. Clearly, $N = \{v \in A \mid SvT \subseteq B_0 + C_0\} \neq 0$ since $0 \neq B_0 \subseteq N$. Lemma 1 implies that $\dim A/N < \infty$, so by using Lemma 2, we have that N contains an ideal $I_B(S,T)$ of A such that $\dim A/I_B(S,T) < \infty$. Consider $G_B(S,T) = B + I_B(S,T)$. Clearly, $G_B(S,T)$ is a subalgebra of A. Moreover

$$G_B(S,T) = G_B(S,T) \cap A = G_B(S,T) \cap (B + C),$$

so the modularity of the lattice of subgroups of the group A^+ implies $G_B(S,T) = B + C \cap G_B(S,T)$. Note that $G_B(S,T) \cap C_0 \lhd C$, so $C_0 \cap G_B(S,T) \in \mathcal{M}$. If $\dim G_B(S,T)/(A_0 + C_0 \cap G_B(S,T)) < \dim A/(A_0 + C_0)$, then by Definition 1 and Remark 1, we obtain that $G_B(S,T) \in \mathcal{M}$

and consequently $I_B(S,T) \in \mathcal{M}$ which leads to a contradiction with Lemma 4(ii). Hence $\dim G_B(S,T)/(A_0+C_0 \cap G_B(S,T)) = \dim A/(A_0+C_0)$.

Let $Z = \{e_1, e_2, \ldots, e_n\}$ be a basis of the K-linear space $A/(B_0+C_0)$. We can assume that $Z \subseteq A$. Since $\dim G_B(S,T)/(A_0 + G_B(S,T) \cap C_0) = \dim A/(A_0 + C_0)$, it follows that $Z \subseteq G_B(S,T)$. Analogously we can define a subalgebra $G_C(S,T) = C + I_C(S,T)$ of A for arbitrary finite dimensional K-subspaces S, T of C^* and some ideal $I_C(S,T) \lhd A$. Similarly we obtain that $Z \subseteq G_C(S,T)$. Let $\mathcal{B}$ and $\mathcal{C}$ be the sets of all finite dimensional K-subspaces of B^* and C^*, respectively. Consider

$$G = \bigcap_{S,T \in \mathcal{B}} G_B(S,T) \cap \bigcap_{S,T \in \mathcal{C}} G_C(S,T).$$

Now, it is not hard to check that $AGA \subseteq B_0 + C_0$. Moreover $0 \neq Z \subseteq G$. But $AGA \lhd A$. Using Definition 1(3), we have $AGA = 0$. By Lemma 4(iii), A is a prime algebra, so $GA = 0$ and consequently $G = 0$, a contradiction. □

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